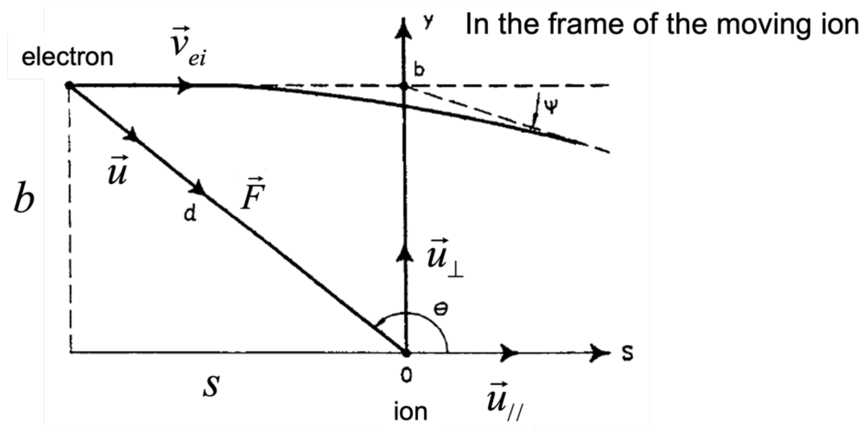
Problem 2: Ion energy loss in the electron beam and cooling forces for electron cooling

Solution:



1. In lecture, we derived that for a single electron passing a single ion close by, the coulomb force (in the ion's moving frame) is

$$\vec{F} = \frac{Ze^2}{4\pi\varepsilon_0 d^2}\vec{u}$$

And the change of momentum is therefore the impact of duration

$$\Delta\vec{p} = \int_{-\infty}^{\infty} \vec{F}dt \approx \frac{1}{v_{ei}}\int_{-\infty}^{\infty} \vec{F}ds$$

we showed that the longitudinal direction's (along s) component of this impact is zero. Thus the net change in momentum (for small deflection angle) is along the transverse direction

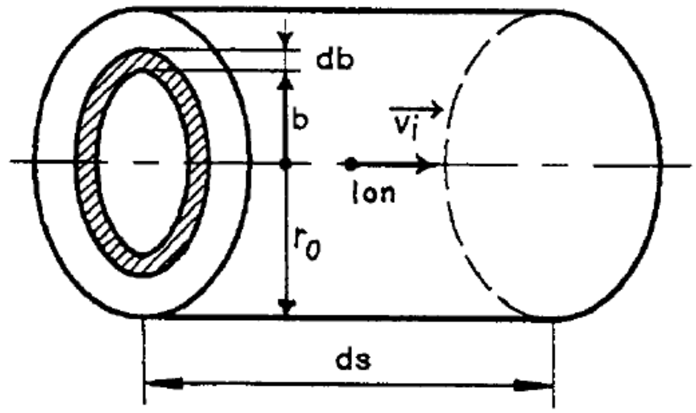
$$\Delta p_\perp \approx -\frac{1}{v_{ei}}\int_{-\infty}^{\infty} F\sin\theta ds = -\frac{1}{v_{ei}}\int_{-\infty}^{\infty}\frac{Ze^2 b}{4\pi\varepsilon_0\left(s^2+b^2\right)^{3/2}}ds = -\frac{1}{v_{ei}}\frac{2Ze^2 b}{4\pi\varepsilon_0 b^2}\int_0^{\infty} d\frac{s}{\left(s^2+b^2\right)^{1/2}} = -\frac{1}{v_{ei}}\frac{2Ze^2}{4\pi\varepsilon_0 b}$$

Thus the energy exchanged is

$$\Delta E(b) \approx \frac{\Delta p_\perp^2}{2m_e} = \frac{2Z^2e^4}{m_e v_{ei}^{\ 2}\left(4\pi\varepsilon_0\right)^2 b^2}$$

This is the energy gain by the electron and due to the conservation of energy, the ion is losing the same amount of the energy.

Now we can calculate the total amount of energy loss of an ion passing through a slice of the electron beam (with length of Δs) by integrating all the energy losses from single collisions



$$\Delta E_{slice} = 2\pi \Delta s n_e \int_{b_{\min}}^{b_{\max}} b \Delta E_{loss}(b) db$$

Thus the energy loss rate (measured by longitudinal distance s) is $\Delta E_{slice}/\Delta s$ can be written as

$$\frac{dE}{ds} = 2\pi n_e \int_{b_{\min}}^{b_{\max}} b \Delta E_{loss}(b) db = \frac{4\pi n_e Z^2 e^4}{m_e v_{ei}{}^2 (4\pi\varepsilon_0)^2} \ln\left(\frac{b_{\max}}{b_{\min}}\right)$$

2. The integral can be carried out to find out the asymptotic behavior of the cooling force. One can use the vector forms for $\overrightarrow{v_{\mathrm{i}}}$ and $\overrightarrow{v_{\mathrm{e}}}$

$$\overrightarrow{v_{\mathrm{i}}} = v_{\mathrm{i}\parallel} \cdot \overrightarrow{u_\parallel} + v_{\mathrm{i}\perp} \cdot \overrightarrow{u_\perp}$$
$$\overrightarrow{v_{\mathrm{e}}} = v_{\mathrm{e}\parallel} \cdot \overrightarrow{u_\parallel} + v_{\mathrm{e}\perp} \cdot \overrightarrow{u_\perp}$$

With cooling force in format of

$$\vec{F} = \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \int_{-\infty}^{\infty} \frac{\vec{v}_e - \vec{v}_i}{|\vec{v}_e - \vec{v}_i|^3} f(\vec{v}_e) d^3 v_e$$

Thus in both transverse and longitudinal directions, for large ion velocities where $\sigma_{ve\parallel} \ll \sigma_{ve\perp} \ll v_{\mathrm{i}}$, the cooling force becomes

$$F_{\parallel,\perp} \cong \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \int_{-\infty}^{\infty} \frac{1}{v_{i\parallel,\perp}{}^2} f(\overrightarrow{v_e}) d^3 v_e$$
$$= \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \frac{1}{v_{i\parallel,\perp}{}^2}$$

3. For small ion velocities where $v_{\mathrm{i}} \ll \sigma_{ve\parallel} \ll \sigma_{ve\perp}$, the cooling force can be expressed as

$$F_{\parallel} = \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \frac{v_{\mathrm{i}\parallel}}{(2\pi)^{3/2} \sigma_{ve\perp}^2 \sigma_{ve\parallel}}$$

And

$$F_{\perp} = \frac{4\pi n_e Z^2 e^4 L_c}{m_e (4\pi\varepsilon_0)^2} \frac{\sqrt{\pi} v_{\mathrm{i}\perp}}{8\sigma_{ve\perp}^3}$$