Home Work PHY 554 #9. **Due November 14, 2016**

HW 1 (4 points): For 3 GeV electron storage ring with circulating current of 500 mA and a bending radius of ρ=8 meters calculate the energy loss per turn, the critical photon energy, total synchrotron radiation power and the photon beam spectral brightness at critical photon energy. Assume horizontal geometrical emittance of 1 nm rad (1e-9 m rad), vertical emittance of 20 pm rad (20e-12 m rad, at the radiation point βx=0.5 m; βy=1.5 m.

Solution: E=3 GeV, γ=5,871.
Losses per turn are (in CGS system) are:

$$\Delta E_{SR} = \frac{4\pi}{3}\frac{e^2}{3\rho}\gamma^4\beta^3$$

Practical formula for the energy loss of ultra-relativistic electron is

$$\Delta E_{SR}[keV] = \frac{88.5 \cdot E^4[GeV]}{\rho[m]}$$

gives particle losses of 0.896 MeV per turn. Total power of synchrotron radiation is equal to product of energy lost by electron by number of electrons per second:

$$P_{SR} = \Delta E_{SR} \cdot \dot{N}_e = \Delta E_{SR}\frac{I_e}{e} \Rightarrow P_{SR}[kW] = \frac{88.5 \cdot E^4[GeV]}{\rho[m]} I_e[A]$$

which give us total power of 448 kW.
Critical photon energy is

$$\varepsilon_c = \hbar\omega_c;\ \omega_c = \frac{3}{2}\frac{c}{\rho}\gamma^3$$

or in practical units

$$\varepsilon_c[keV] \cong 0.665 \cdot B[T] \cdot E^2[GeV] \cong 2.22 \cdot \frac{E^3[GeV]}{\rho[m]}$$

giving ε_c of 7.5 KeV.
To define brightness, we first find angular density of the photons. We could use formula from lecture 17:

$$\frac{dW}{d\omega} = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^2 I(\omega)}{d\omega d\Omega}\cos\theta\, d\theta = \frac{2\pi}{\gamma}\int_{-\gamma\frac{\pi}{2}}^{\gamma\frac{\pi}{2}} \frac{d^2 I(\omega)}{d\omega d\Omega} d(\gamma\theta)$$

$$\approx \frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma}{2\pi c}\frac{\omega^2}{\omega_c^2}\int_{-\infty}^{\infty}\left(1+y^2\right)^2\left\{\frac{y^2}{\left(1+y^2\right)}K_{\frac{1}{3}}^{2}\left(\frac{\omega}{2\omega_c}\left(1+y^2\right)^{\frac{3}{2}}\right) + K_{\frac{2}{3}}^{2}\left(\frac{\omega}{2\omega_c}\left(1+y^2\right)^{\frac{3}{2}}\right)\right\}dy$$

or much easier, practical formula (5.6) (slide 20, lecture 18)

$$\left.\frac{d^3 F_B}{d\theta\, d\psi\, d\omega/\omega}\right|_{\psi=0} = 1.33\times 10^{13} E_e^2(\mathrm{GeV}) I(\mathrm{A}) H_2(E/E_c)\frac{\mathrm{photons/s}}{\mathrm{mrad}^2\cdot(0.1\%\ \mathrm{BW})} \qquad (5.6)$$

and tabulated values for H_2 function

y	$G_1(y)$	$H_2(y)$
0.0010	2.131×10^{-1}	2.910×10^{-2}
0.0100	4.450×10^{-1}	1.348×10^{-1}
0.1000	8.182×10^{-1}	6.025×10^{-1}
0.3000	9.177×10^{-1}	1.111×10^{0}
0.5000	8.708×10^{-1}	1.356×10^{0}
0.7000	7.879×10^{-1}	1.458×10^{0}
1.000	6.514×10^{-1}	1.454×10^{0}
3.000	1.286×10^{-1}	5.195×10^{-1}
5.000	2.125×10^{-2}	1.131×10^{-1}
7.000	3.308×10^{-3}	2.107×10^{-2}
10.00	1.922×10^{-4}	1.478×10^{-3}

for $y = \varepsilon / \varepsilon_c = 1$: H_2=1.454, to find

$$\frac{d^3F}{d\theta d\psi d\omega / \omega} \simeq 8.7 \cdot 10^{13} \frac{ph/s}{mrad^2 (0.1\% BW)}$$

We need now to determine what is the radiation area to define brightness of the photon beam. First, let's find area occupied by electron beam:

$$A_e = 2\pi \sigma_x \sigma_y = 2\pi \sqrt{\beta_x \varepsilon_x} \sqrt{\beta_y \varepsilon_y} \approx 7.7\ 10^{-10}\ m^2 = 7.7\ 10^{-4}\ \ mm^2$$

$$\sigma_x = \sqrt{\beta_x \varepsilon_x} \approx 22 \mu m; \sigma_y = \sqrt{\beta_y \varepsilon_y} \approx 5.5 \mu m$$

Thus, without taking diffraction of the photons, brightness of the this beam will be

$$B = \frac{1}{A_2} \frac{d^3F}{d\theta d\psi d\omega / \omega} \simeq 1.1 \cdot 10^{17} \frac{ph/s}{mm^2 mrad^2 (0.1\% BW)}$$

Extras: In order to prove that diffraction does not play role in this case we should compare minimum radiation spot size caused by diffraction with the electron beam sizes. To estimate the spot size caused by diffraction, we should use uncertainty principle for photon intensity:

$$\sigma_{x,y} \sigma_{\theta x, \theta y} \geq \frac{\lambda}{4\pi}; \ \sigma_{x,y} \geq \frac{\lambda}{4\pi \sigma_{\theta x, \theta y}}$$

with $\lambda_c = 1.65$ Å (1.65E-10 m). What is important to remember here is that maximum radiation/observation angle is limited to $\sigma_{\theta x, \theta y} \leq 1/\gamma$, which yields

$$\sigma_{x,y} \geq \sigma_{\min} = \frac{\gamma \lambda}{4\pi} \sim 8\ nm$$

Hence, for all practical reasons, hard X-ray brightness would not be affected by diffraction. It would not be true for visible or IR radiation, where diffraction spot size could exceed that of electron beam.

HW 2 (6 points): For the 3 GeV storage ring described above, consider an undulator with N=40 periods and with K=1 installed in the straight section. Assume horizontal geometrical emittance of 1 nm rad (1e-9 m rad), vertical emittance of 20 pm rad (20e-12 m rad, at the radiation point βx=βy=2.5 m.

(a) Find undulator period that fundamental wavelength will be λ=0.5 nm (5 Å)
(b) What will be spectral brightness at the fundamental wavelength?

Solution:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right) \to \lambda_u = \frac{2\gamma^2\lambda}{1+\frac{K^2}{2}} = 2.3\ cm$$

The beam area with equal β-functions 2.5 m is

$$\sigma_x = \sqrt{\beta_x\varepsilon_x} \approx 50\,\mu m; \sigma_y = \sqrt{\beta_y\varepsilon_y} \approx 7\,\mu m$$

Diffraction spot for 1/γ angular opening and 5 Å radiation is ~ 25 nm, but from the undulator we are interested in the central cone

$$\theta_c = \frac{\sqrt{1+\frac{K^2}{2}}}{\gamma\sqrt{N}} \cong 40\,\mu rad$$

which is 4 times smaller. It means that spot diffraction limited size is ~ 0.1 μm – still much smaller than the beam sizes. Using formula (5.65) we can calculate the brightness

$$\bar{B}_{\Delta\omega/\omega}(0) = \frac{7.25 \times 10^6 \gamma^2 N^2 I(\mathrm{A})}{\sigma_x(\mathrm{mm})\sigma_y(\mathrm{mm})\left(1+\frac{\sigma_x'^2}{\theta_{\mathrm{cen}}^2}\right)^{1/2}\left(1+\frac{\sigma_y'^2}{\theta_{\mathrm{cen}}^2}\right)^{1/2}} \cdot \frac{K^2 f(K)}{\left(1+K^2/2\right)^2}\ \frac{\mathrm{photons/s}}{\mathrm{mm^2mrad^2(0.1\%BW)}}$$

$$\text{with } f(K) = \left(J_o\left(\frac{K^2}{4(1+K^2/2)}\right) - J_1\left(\frac{K^2}{4(1+K^2/2)}\right)\right)^2;$$

For K=1 f(K)= 0.828. We also need to calcualte angular spread of the beam

$$\sigma_{x'} = \sqrt{\varepsilon_x/\beta_x} \approx 20\ \mu rad; \sigma_{y'} = \sqrt{\varepsilon_y/\beta_y} \approx 2.8\,\mu rad$$

$$\left(1+\frac{\sigma_{x'}^2}{\theta_c^2}\right)\left(1+\frac{\sigma_{y'}^2}{\theta_c^2}\right) = 1.38$$

Putting all together gives

$$B = 1.77\cdot 10^{20}\frac{ph/s}{mm^2 mrad^2(0.1\%BW)}$$

Note: These are brightness estimations for parameters are relevant for modern light sources such as NSLS II or MAX IV. Accurate calculations of the brightness usually requires simulations, but they are not very different from what we get here.