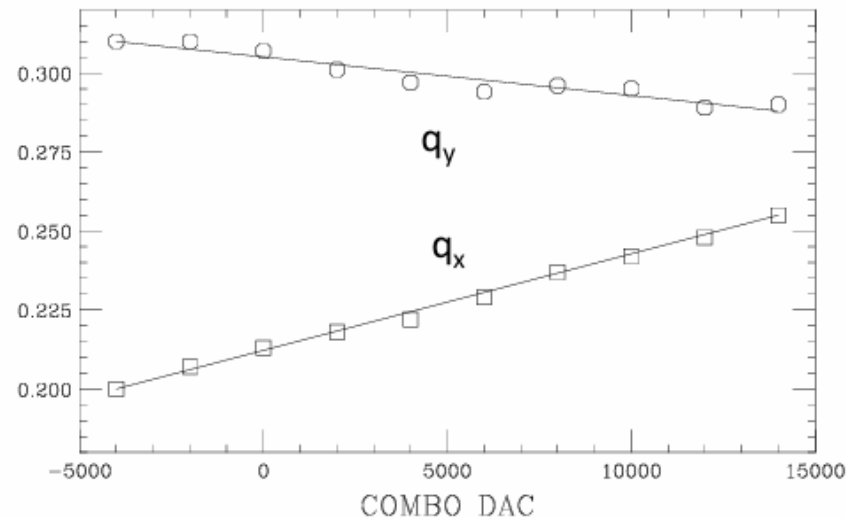


PHY 554

Fundamentals of Accelerator Physics

Lecture 8: Quadrupole Errors and Beta Beat



From quadrupole errors to tune shift, beta beating and correction

What we learned:

For a distributed dipole field error:

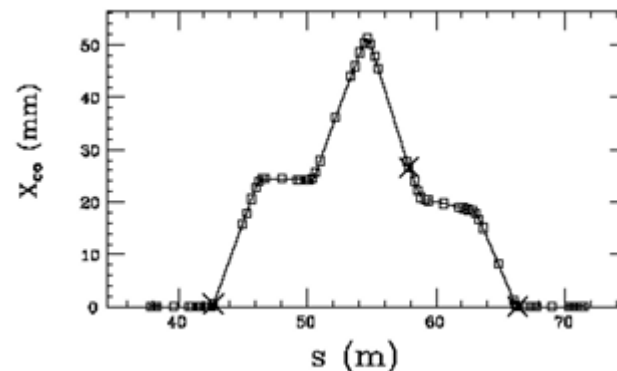
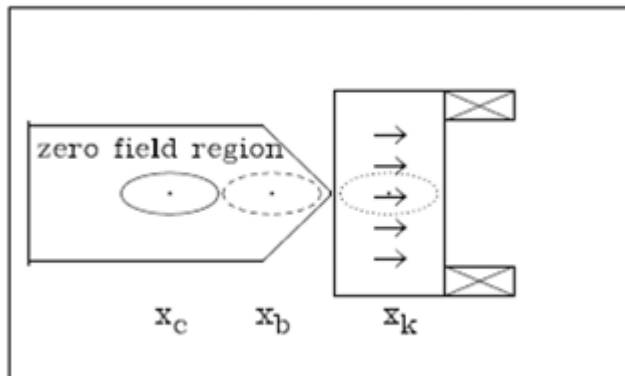
$$X_{co}(s) = \frac{\sqrt{\beta(s)}}{2 \sin \pi \nu} \int_s^{s+C} ds_0 \sqrt{\beta(s_0)} \cos[\pi \nu - |\psi(s) - \psi(s_0)|] \theta \delta(s - s_0)$$

$$= \frac{\sqrt{\beta(s)}}{2 \sin \pi \nu} \sum_i \sqrt{\beta(s_i)} \theta_i \cos[\pi \nu - |\psi(s) - \psi(s_i)|]$$

Closed orbit conditions (start from $x=x'=0$):

$$X_{co}(s_f) = 0, X'_{co}(s_f) = 0$$

Injection/extraction channel:
unperturbed closed orbit x_c /bumped
orbit x_b /incoming beam orbit x_k



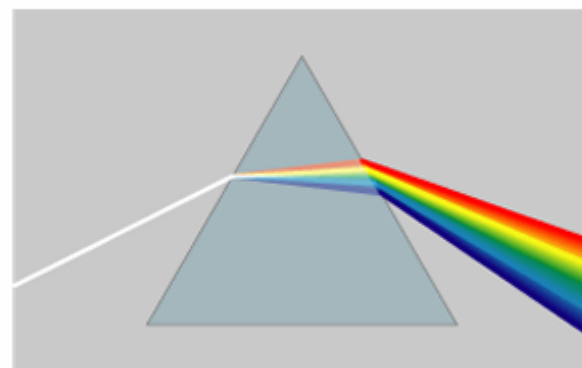
Off-momentum and dispersion

For different particle energy $\delta = \frac{p - p_0}{p_0}$

$$x = x_\beta + D\delta \quad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s)x_\beta = 0, \quad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s)D = \frac{1}{\rho}$$



$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

Extend the matrix representation to 3 by 3

For a pure dipole ($K=0$):

$$M = \begin{pmatrix} \cos\theta & \rho \sin\theta & \rho(1 - \cos\theta) \\ -\frac{1}{\rho} \sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

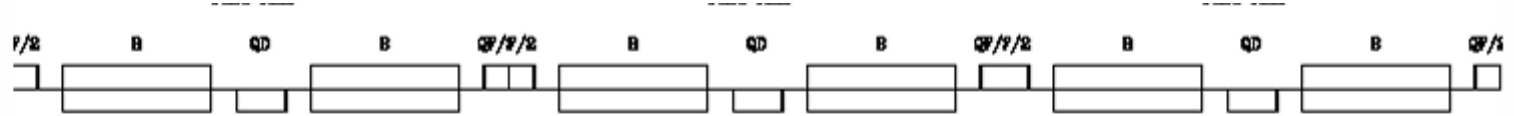
$$\theta \ll 1 \quad \text{i.e.} \quad L \ll \rho$$

For quadrupoles:

$$M(s, s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}} \sin\sqrt{K}\ell & 0 \\ -\sqrt{K} \sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Defocusing
change $K \rightarrow -K$

FODO cell



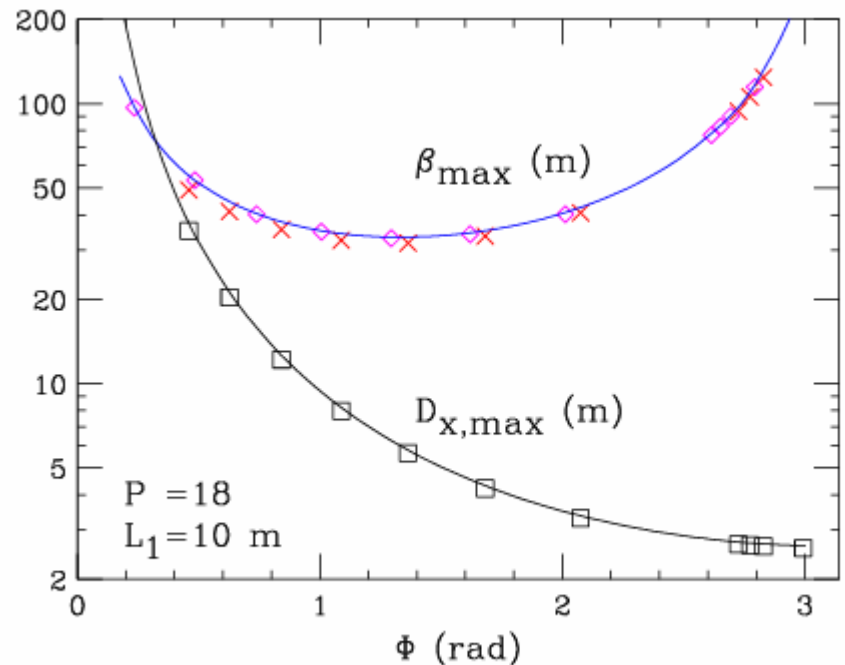
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) & 2L\theta(1 + \frac{L}{4f}) \\ -\frac{L}{2f^2} + \frac{L^2}{4f^3} & 1 - \frac{L^2}{2f^2} & 2\theta(1 - \frac{L}{4f} - \frac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix}$$

$$D_F = \frac{L\theta(1 + \frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}, \quad D'_F = 0$$

$$\beta_{\max} = \frac{2L_1(1 + \frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1 + \sin\frac{\Phi}{2})}{\sin\Phi}$$



Effect of quadrupole field error:

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho} \quad \rightarrow \quad X'' + (K_0 + k)X = 0$$

We assume that the transfer matrix of the unperturbed betatron can be described by

$$M_0(s) = I \cos \Phi_0 + J \sin \Phi_0, \quad \Phi_0 = 2\pi\nu_0 \quad J(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

The perturbation of a thin quadrupole can be described by

$$m(s_1) = \begin{pmatrix} 1 & 0 \\ -k(s_1)ds_1 & 1 \end{pmatrix}$$

Sources of quadrupole field error:

(1) quadrupole and (2) closed orbit in sextupoles

Dipole error changes orbit; quadrupole error changes the optics.

Betatron tune shift

The transfer matrix of the one-turn map is $\mathbf{M}(s_1)=\mathbf{M}_0(s_1)m(s_1)$

$$M(s_1) = \begin{pmatrix} \cos \Phi_0 + \alpha_1 \sin \Phi_0 - \beta_1 \sin \Phi_0 k(s_1) ds_1 & \beta_1 \sin \Phi_0 \\ -\gamma_1 \sin \Phi_0 - [\cos \Phi_0 + \alpha_1 \sin \Phi_0] k(s_1) ds_1 & \cos \Phi_0 - \alpha_1 \sin \Phi_0 \end{pmatrix}$$

$$\cos \Phi = \cos \Phi_0 - \frac{1}{2} \beta_1 k(s_1) ds_1 \sin \Phi_0$$

$$\Delta \Phi \approx \frac{1}{2} \beta_1 k(s_1) ds_1, \quad \Delta \nu \approx \frac{1}{4\pi} \beta_1 k(s_1) ds_1, \quad \Delta \nu \approx \frac{1}{4\pi} \oint \beta_1 k(s_1) ds_1$$

The betatron tune of the accelerator is changed by the quadrupole field error.

In reality, this is how beta functions are measured -- tune response.

The betatron amplitude function is also changed by the quadrupole field error. The betatron amplitude function can be obtained by a one-turn map, i.e.

$$M(s_2) = M(s_2 + C, s_1) m(s_1) M(s_1, s_2)$$

$$\begin{aligned} \Delta[M(s_2)]_2 &= -k(s_1) ds_1 \beta_1 \beta_2 \sin \nu_0 (\phi_1 - \phi_2) \sin \nu_0 (2\pi + \phi_2 - \phi_1) \\ &= -\frac{1}{2} k(s_1) ds_1 \beta_1 \beta_2 (\cos 2\nu_0 (\pi + \phi_2 - \phi_1) - \cos 2\pi \nu_0) \end{aligned}$$

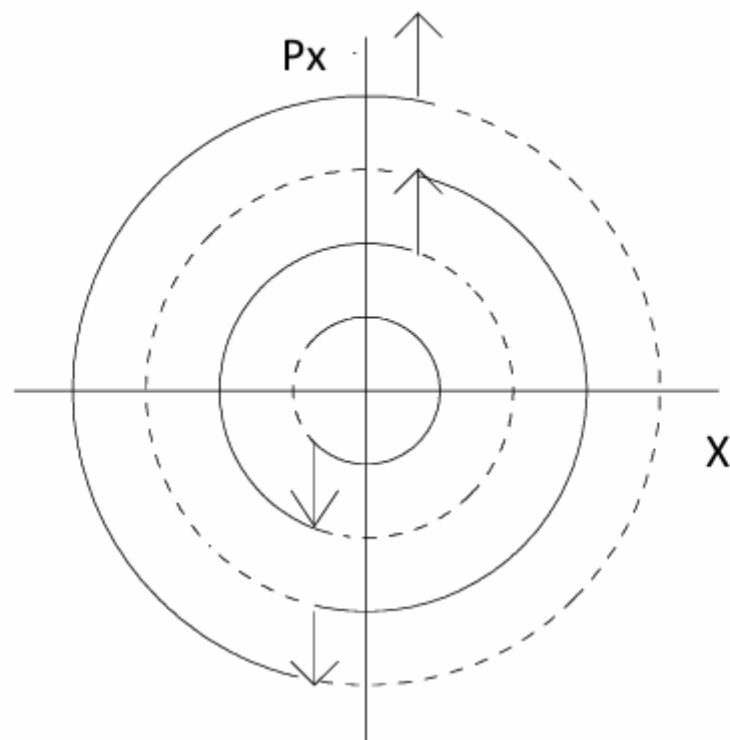
$$\begin{aligned} \Delta[M(s_2)]_2 &= \Delta[\beta_2 \sin \Phi] \cong \Delta\beta_2 \sin \Phi_0 + \beta_2 \cos \Phi_0 \Delta\Phi \\ &= \Delta\beta_2 \sin \Phi_0 + \frac{1}{2} \beta_1 k(s_1) ds_1 \beta_2 \cos \Phi_0 \end{aligned}$$

$$\frac{\Delta\beta_2}{\beta_2} = -\frac{1}{2 \sin \Phi_0} \beta_1 k(s_1) ds_1 \cos 2\nu_0 (\pi + \phi_2 - \phi_1)$$

For a distributed quadrupole field error, the perturbation to the betatron amplitude function becomes

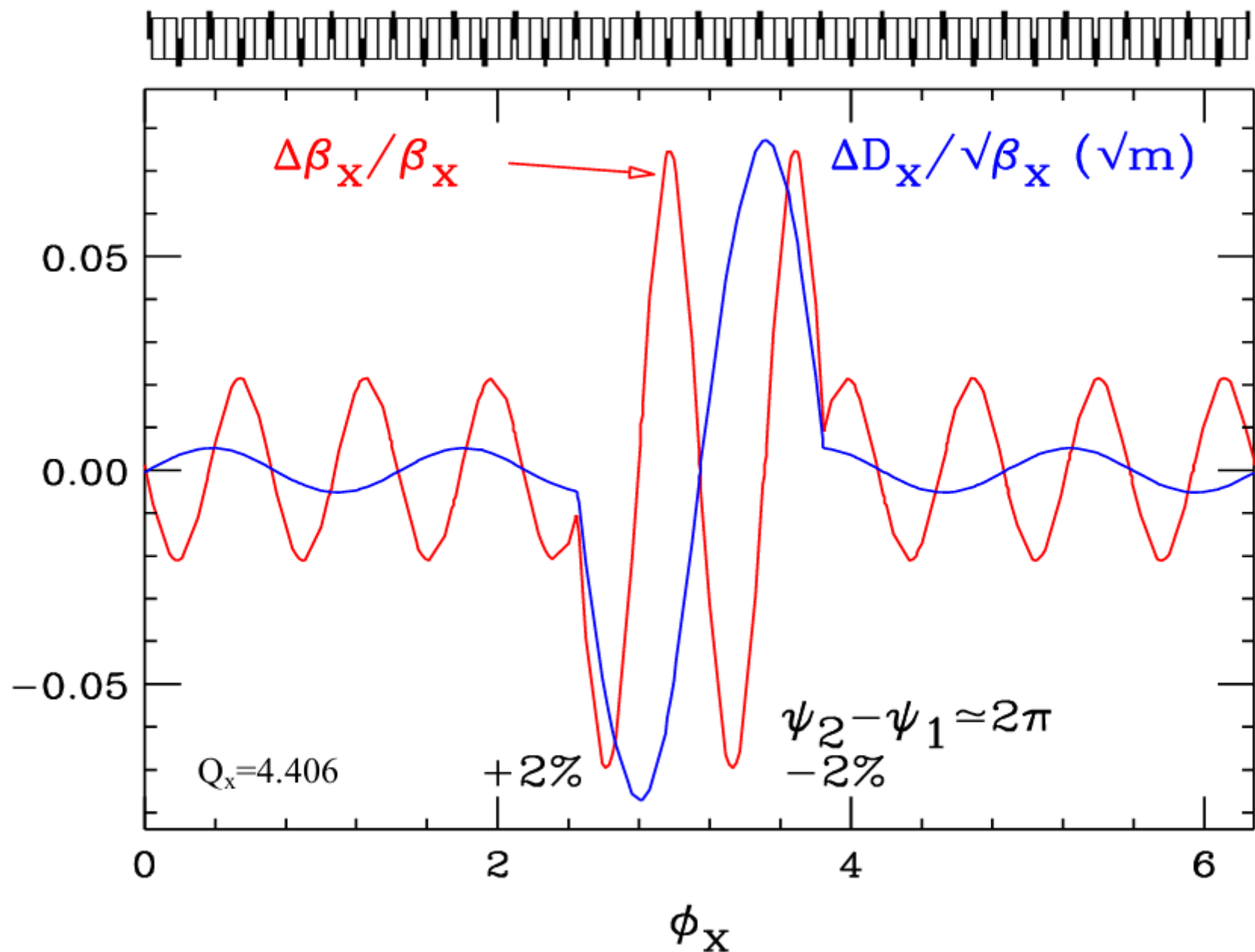
$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{1}{2 \sin \Phi_0} \int_s^{s+C} ds_1 \beta_1 k(s_1) \cos 2\nu_0 (\pi + \phi_2 - \phi_1)$$

Beta beating means the lattice from model is not the lattice the beam sees with errors.

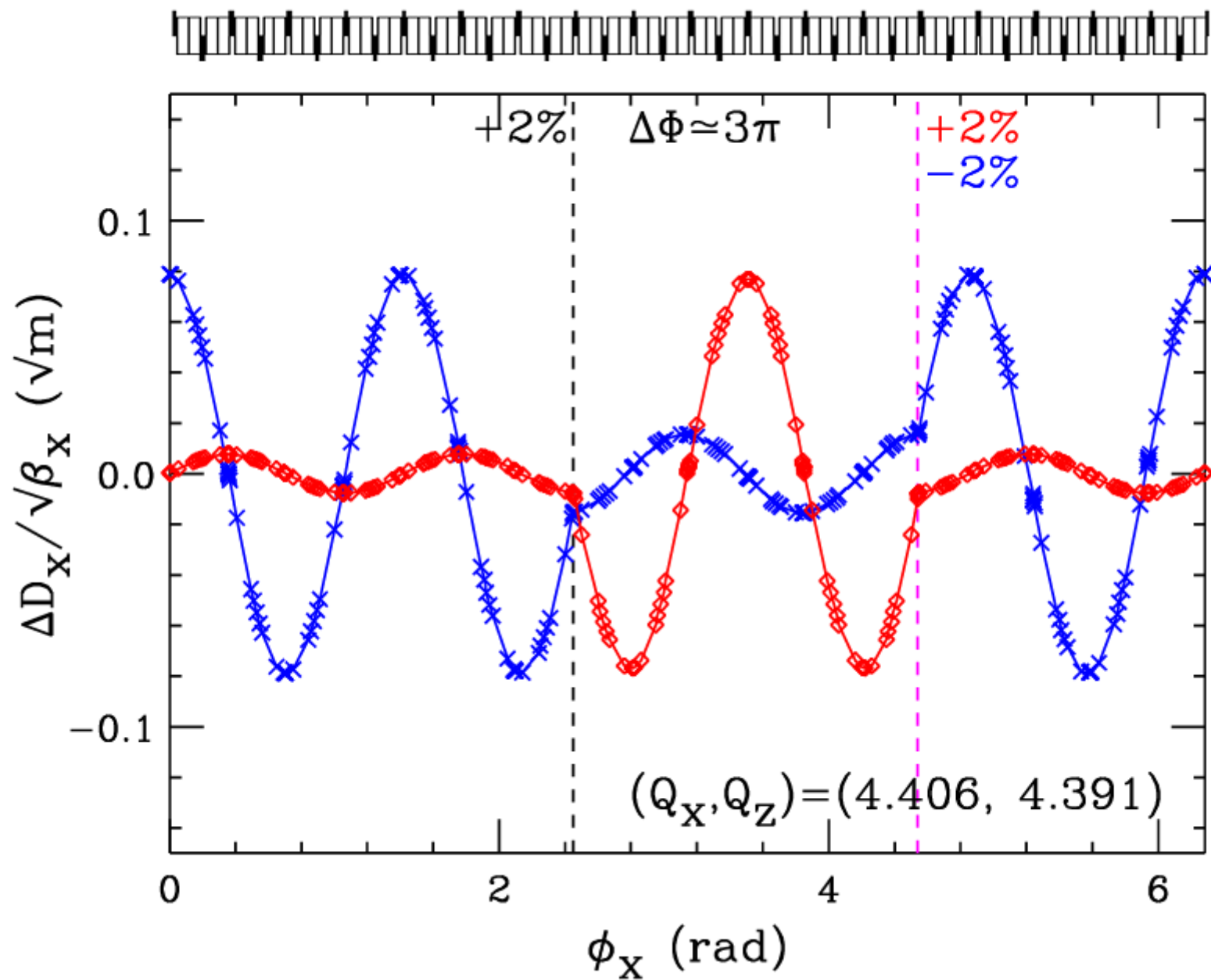


Schematic plot of a particle trajectory at a half-integer betatron tune resulting from an error quadrupole kick $p_x = \beta_x \Delta X' = -\beta_x X/f$, where f is the focal length, X is the displacement from the quadrupole center, and β_x is the betatron amplitude function at the quadrupole. The quadrupole kick is proportional to the displacement X . At a half-integer betatron tune, the betatron coordinate changes sign in each consecutive revolution and the kick angles coherently add in each revolution to produce unstable particle motion.

What can the beta beat pattern tell us?

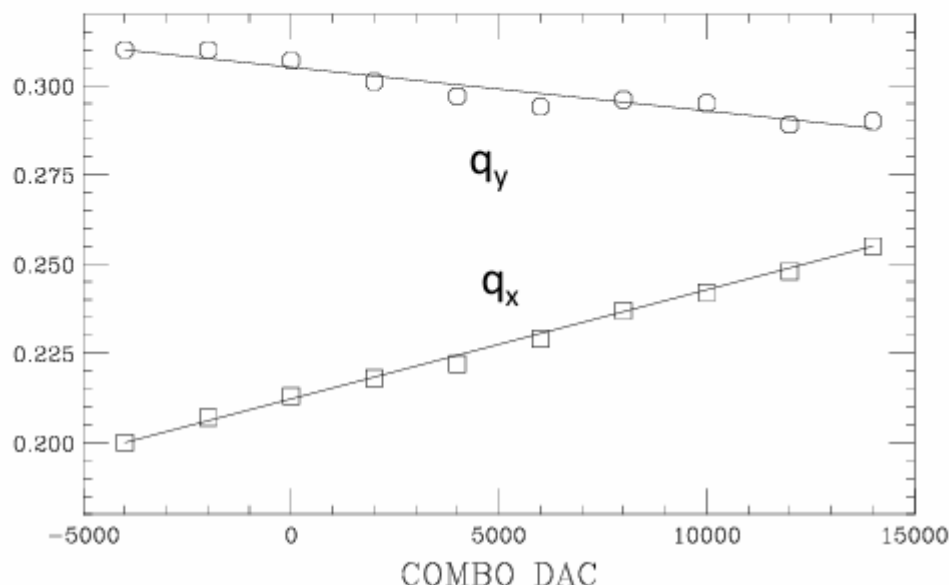


What happens if the -2% is changed to $+2\%$?



Applications of quadrupole error

1. Betatron amplitude function measurement



$$\Delta\nu \approx \frac{1}{4\pi} \oint \beta_1 k(s_1) ds_1$$

$$\langle \beta_{x,y} \rangle = 4\pi \frac{\Delta\nu_{x,y}}{\Delta Kl}$$

The horizontal and vertical tunes, determined by the FFT spectrum of the betatron oscillations, vs quadrupole field strength. The slope can be used to determine the **average** betatron amplitude function in a quadrupole.

The fractional parts of betatron tunes were $q_x = 4 - \nu_x$ and $q_y = 5 - \nu_y$. The experimental result of fractional horizontal tune appeared to “increase” with the strength of the quadrupole.

Q: Is the quadrupole focusing or defocusing? At this location, what can you say about the betatron amplitude functions?

2. Tune jump

$$\Delta\nu = \frac{1}{4\pi} \oint \beta_1 \frac{\Delta B_1}{B\rho} ds_1$$

Homework #3: Lab report III, due next Wednesday 9/30/26

Use the Elegant files from Lab I. Answer following questions (30 pts, each 6):

1. In .ele file, modify it to include energy spread (use $1e-3$), calculate beam size again. Compare the beam size with your expectation with formula in class, do they agree (any single location will work)?
2. Install closed_orbit module in .ele file, re-run Elegant and check what are the closed orbit (x, y columns) for base lattice. Introduce a 0.01 rad dipole error (element "HKICK") at the beginning of the lattice, now check what are the closed orbit again.
3. Do the closed orbit agree with your expectation (comparing at the kick location is sufficient)? Why? Try again with 0.001 rad error, how about now? Can you explain why you see the difference?
4. Now return to base lattice. Change first quad's $k1$ by 0.05 unit: $1/m^2$. What are the tune changes? Do they agree with your estimation?
5. Now try change $k1$ by 0.001 unit from base value. Compare again what are the tune changes, do they agree better? Why?