Nonlinear dynamics

Action-angle variables

The action angle variable (J, Φ) is defined as:

$$2J_z = \gamma_z z^2 + 2\alpha_z z z' + \beta_z z'^2,$$

$$\tan\phi_z = -\alpha_z - \beta_z \frac{z'}{z}$$

where (α,β,γ) are Twiss parameters.
The action angle variable is very important for linear beam dynamics. As we all know, for linear dynamics, it has properties

$$\frac{dJ_z}{ds} = 0, \quad \frac{d\phi_z}{s} = \frac{1}{\beta_z}$$

using a generating function $F_1(z,\phi_z) = -\frac{z^2}{2\beta_z}(\tan\phi_x + \alpha_x)$

and the Hamiltonian reduces to $H = \frac{J_z}{\beta_z}$ note this H is s dependent!

Treatments of nonlinearities

A number of powerful tools for analysis of nonlinear systems can be developed from Hamiltonian mechanics to describe the motion for a particle moving through a component in an accelerator beamline:(truncated) power series; Lie transform; (implicit) generating function.

Hamiltonian is usually not integrable. However, if the Hamiltonian can be written as a sum of integrable terms, an explicit symplectic integrator that is accurate to some specified order can be constructed to solve the system.

For a storage ring, We mainly discuss two approaches to analyze nonlinear dynamics:

1. Canonical perturbation method where nonlinear terms are treated as perturbation to the linear Hamiltonian (may not give correct pictures when nonlinear magnets are strong)
2. Normal form analysis, based on Lie transformation of the one-turn map (especially useful when dealing with resonance driving terms and dynamic aperture problems)

Perturbation treatment

The Hamiltonian for a linear system in action angle variable (J, Φ):

$$H = \nu J$$

the nonlinear elements' contribution can be written as

$$H = \nu J + \varepsilon V(\phi, J, s) = H_0 + \varepsilon V(\phi, J, s)$$

where ε is a small parameter. Please note that the perturbation V from nonlinear element is also a periodic function of the circumference L. Thus it is usually convenient to express it in terms of a sum over different orders:

$$V(\phi, J, s) = \sum_m V_m(J, s) e^{im\phi}$$

and treat them order by order (m being the order of nonlinear term).

Perturbation treatment for quadrupole error

Lets first apply it to the linear case (taking a quadrupole error as an example). Assume we have a small quadrupole field error k(s), the Hamiltonian (for horizontal motion) reads:

$$H = \frac{1}{2}\left(x'^2 + K_x x^2\right) + \frac{k(s)x^2}{2}$$

If transformed into action angle variables, it reads: $x = \sqrt{2\beta(s)J}\cos\Phi$

$$H = \frac{J}{\beta(s)} + \frac{1}{2}k(s)\beta(s)J(1+\cos 2\Phi) = H_0 + \frac{1}{2}k(s)\beta(s)J\cos 2\Phi$$

thus the term H_0 (independent of Φ) is $H_0 = \frac{J}{\beta(s)} + \frac{1}{2}k(s)\beta(s)J$

and the tune becomes $\nu = \frac{1}{2\pi}\int \frac{dH}{dJ}ds = \frac{1}{2\pi}\int\left(\frac{1}{\beta(s)} + \frac{1}{2}k(s)\beta(s)\right)ds$

The change of tune $\Delta\nu = \frac{1}{4\pi}\int k(s)\beta(s)\,ds$ ✔

Perturbation treatment for sextupole

The Hamiltonian (in orbit angle θ) can be written as

$$H = \nu_x J_x + \nu_y J_y + \sum_l G_{3,0,l} J_x^{3/2} \cos(3\phi_x - l\theta)$$

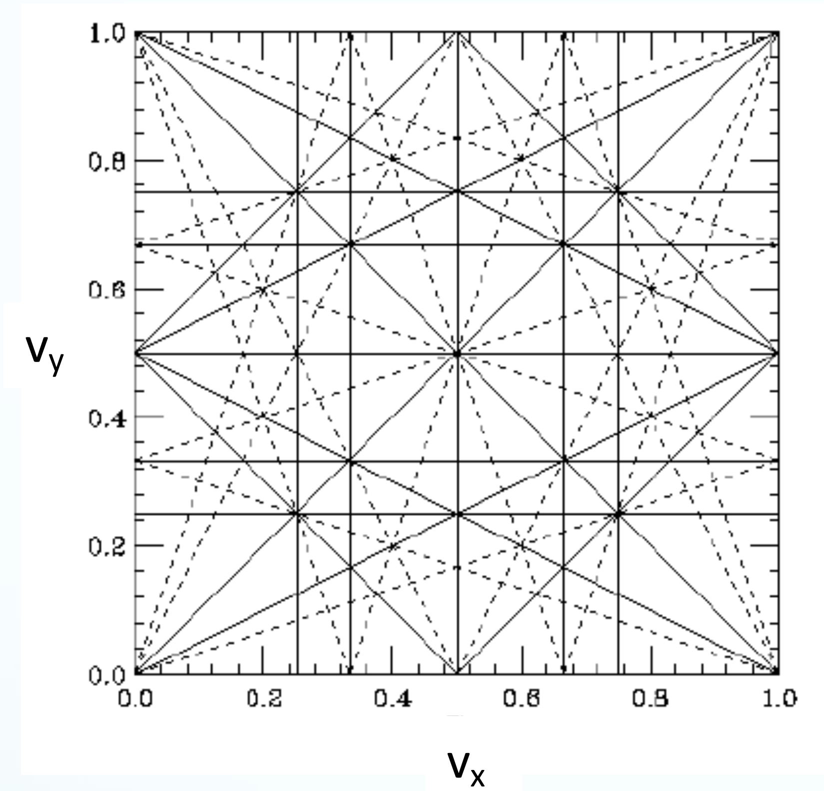
$$+ \sum_l G_{1,2,l} J_x^{1/2} J_y \cos(\phi_x + 2\phi_y - l\theta) + \sum_l G_{1,-2,l} J_x J_y^{1/2} \cos(\phi_x - 2\phi_y - l\theta) + \ldots$$

where G's drive the correspondent resonances and ... drives parametric resonance $\nu_x = l$

Table 2.3: Resonances due to sextupoles and their driving terms

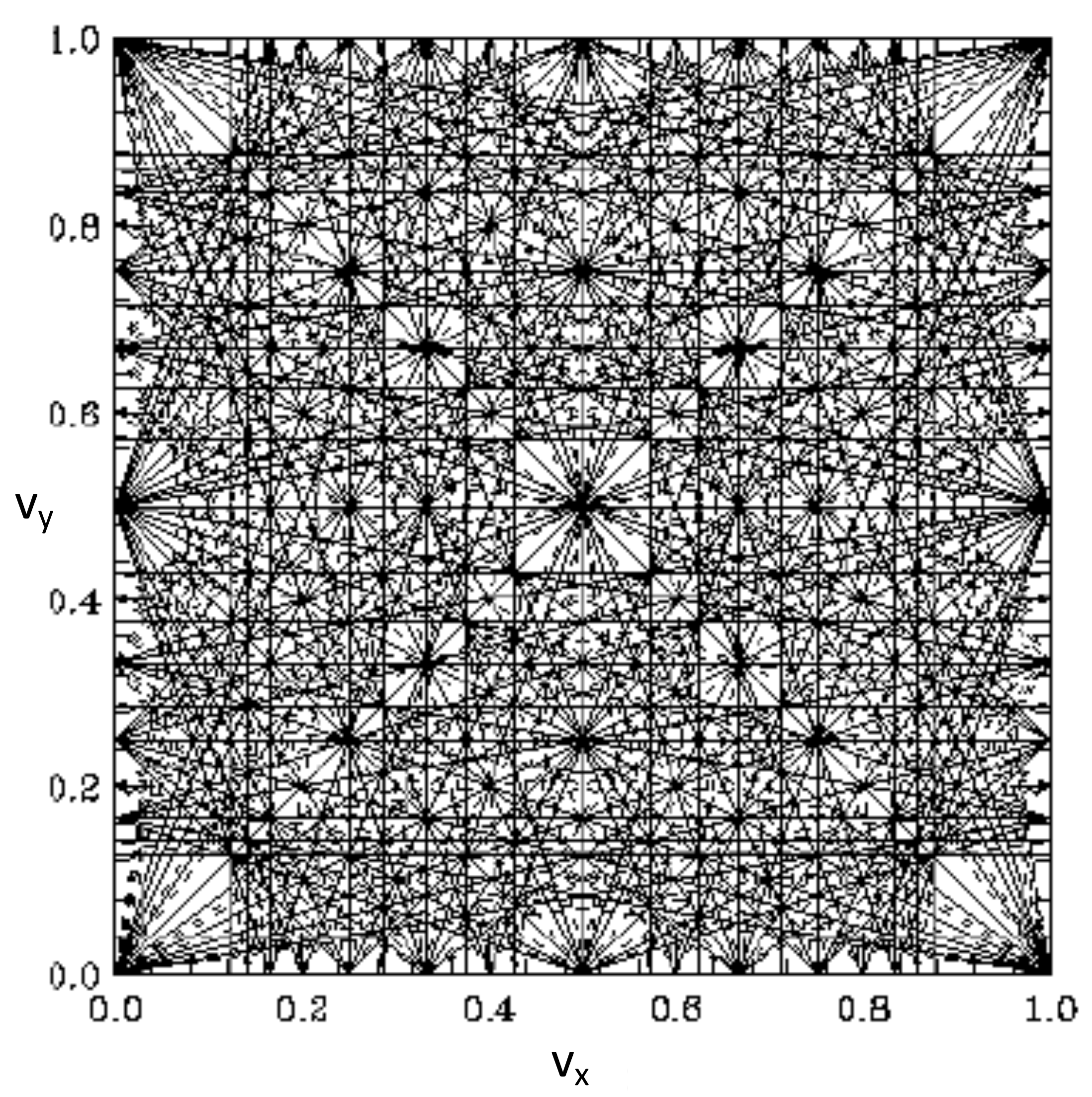
Resonance	Driving term	Lattice	Amplitude	Classification
$\nu_x + 2\nu_z = \ell$	$\cos(\Phi_x + 2\Phi_z)$	$\beta_x^{1/2}\beta_z$	$J_x^{1/2}J_z$	sum resonance
$\nu_x - 2\nu_z = \ell$	$\cos(\Phi_x - 2\Phi_z)$	$\beta_x^{1/2}\beta_z$	$J_x^{1/2}J_z$	difference resonance
$\nu_x = \ell$	$\cos\Phi_x$	$\beta_x^{1/2}\beta_z;\ \beta_x^{3/2}$	$J_x^{1/2}J_z,\ J_x^{3/2}$	parametric resonance
$3\nu_x = \ell$	$\cos 3\Phi_x$	$\beta_x^{3/2}$	$J_x^{3/2}$	parametric resonance

Resonance lines in tune space



Up to 4th order

Up to 8th order



Fixed points and separatrix

Stable and unstable fixed points are the points in phase space where particle can stay there indefinitely (without any perturbation). Considering the mode $3\nu_x = l$, with generating function

$$F_2 = (\phi_x - \frac{l}{3}\theta)J \qquad \phi = \phi_x - \frac{l}{3}\theta, \quad J = J_x$$

The Hamiltonian becomes

$$H = \delta J + G_{3,0,l} J^{3/2} \cos 3\phi, \quad \delta = \nu_x - \frac{l}{3}$$

proximity

Solve for unstable fixed points

$$\frac{dJ}{d\theta} = \frac{d\phi}{d\theta} = 0$$

Gives 3 solutions

$$J_{UFP}^{1/2} = \left|\frac{2\delta}{3G}\right| \qquad \phi_{UFP} = 0, \pm 2\pi/3, \quad if \quad \delta/G < 0$$

$$\phi_{UFP} = \pm\pi/3, \pi \quad if \quad \delta/G > 0$$

UFPs define separatrix (the boundary of stable region)

Triangle changes direction w
at different sides of resonan

Tracking of sextupole

If sextupole can be treated as thin length (usually true with large radius R), the tracking of a particle dynamics in existence of sextupole magnets can be treated as a one turn map and an instantaneous kick. Starting from Hill's equation

$$x'' + K_x(s)x = \frac{1}{2}S(s)(x^2 - y^2), \qquad y'' + K_y(s)y = -S(s)xy$$

The change in the derivatives of coordinates can be written as

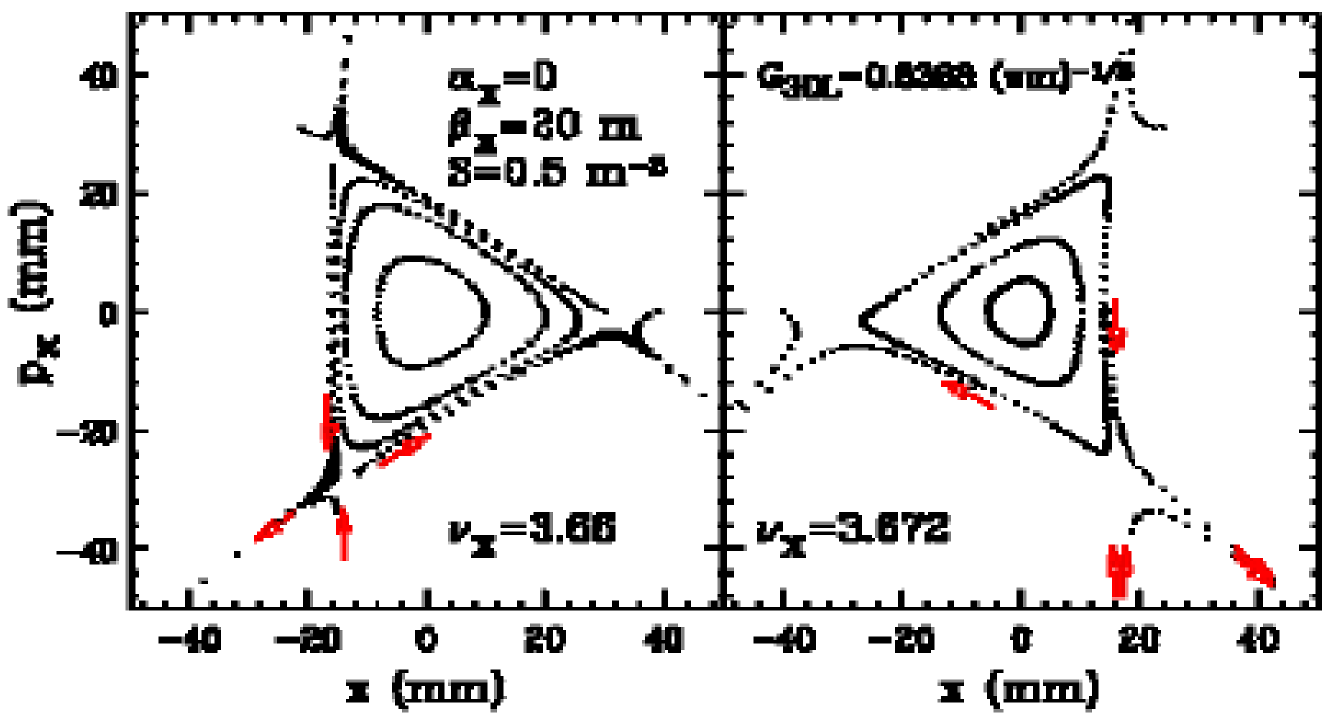
$$\Delta x' = \frac{1}{2}\int S(s)(x^2 - y^2)ds = \frac{1}{2}\bar{S}(x^2 - y^2), \qquad \Delta y' = -\int S(s)xyds = -\bar{S}xy$$

Given the initial particle distribution, the Poincare maps can be obtained by long term tracking applying the one turn map and instant kick in x',y'.

$$x'' + K_x(s)x = \frac{1}{2}S(s)(x^2 - y^2), \qquad y'' + K_y(s)y = -S(s)xy$$

$$\Delta x' = \frac{1}{2}\int S(s)(x^2 - y^2)ds = \frac{1}{2}\bar{S}(x^2 - y^2), \qquad \Delta y' = -\int S(s)xyds = -\bar{S}xy$$

Thus particle motion in existence of sextupole fields can be tracked thru a combination of linear transfer map $M(s_1,s_2)$ and a local kick in the x' which is proportional to the integrated sextupole field strength.



Normalized phase space plots at a tune below (left) and above (right) a third order resonance driven by a single sextupole magnet. Four particles with various initial actions were used in the tracking. The integrated sextupole strength is S = 0.5 m^{-2} with lattice parameters β_x = 20 m and α_x = 0.

Symplectic integration

Outline

- Hamiltonian & symplecticness
- Numerical integrator and symplectic integration
- Application to accelerator beam dynamics
- Accuracy and integration order

Hamiltonian dynamics

In accelerator, particles' motion is predicted by Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \qquad \text{or} \qquad \dot{q} = \nabla_p H(p,q), \quad \dot{p} = -\nabla_q H(p,q)$$

or it can be written in a compact form $\dot{z} = J\nabla_z H(z) \qquad z \equiv (p,q)$

$$J \equiv \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

The solution is a transformation mapping (flow)

$$(p,q) = A_{t,H}(p_0,q_0)$$

or for simplicity $z = A(z_0)$

in matrix representation, the map A is a 2n by 2n matrix.

Symplecticness

A. Hamilton's equations predict the evolution of phase space.

B. Canonical transformation A preserves the form of Hamilton's equations.

C. Transformation A is canonical if and only if it satisfies the relation

$$A^T JA = J \qquad \det A = 1$$

and we call this transformation A symplectic

Proof. Hamilton's equation can be expressed as $\dot{x} = J\frac{\partial H}{\partial x}$

if we have transformation $y = y(x)$

$$\dot{y} = AJA^T\frac{\partial H}{\partial y} = J\frac{\partial H}{\partial y} \quad \text{if} \quad A^T JA = J \quad \text{i.e. symplectic}$$

Preservation of area

Symplecticness is equivalent to the preservation of area.

In a 2d(d=1) space, the area of a parallelogram is defined as the magnitude of the wedge product $dp \wedge dq$

While for a transformation $z = A(z_0)$

we have
$$dp = \frac{\partial p}{\partial p_0} dp_0 + \frac{\partial p}{\partial q_0} dq_0, \quad dq = \frac{\partial q}{\partial p_0} dp_0 + \frac{\partial q}{\partial q_0} dq_0$$

$$dp \wedge dq = \frac{\partial p}{\partial p_0} \frac{\partial q}{\partial q_0} dp_0 \wedge dq_0 + \frac{\partial p}{\partial q_0} \frac{\partial q}{\partial p_0} dq_0 \wedge dp_0$$

wedge products are anticommutative $dp \wedge dq = -dq \wedge dp$

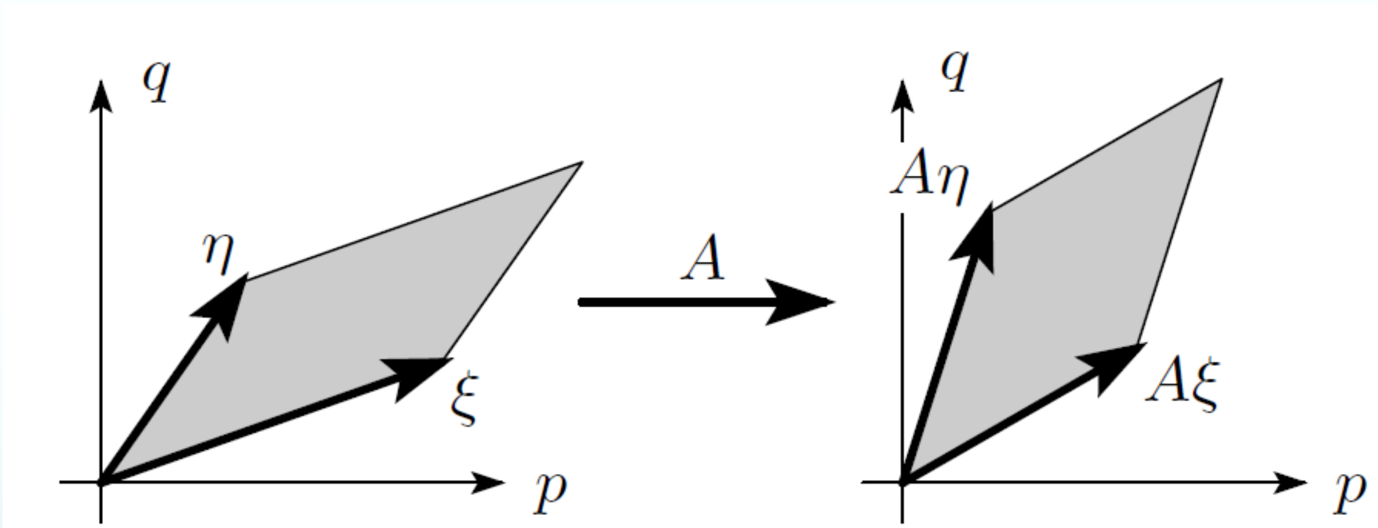
$$dp \wedge dq = \frac{\partial p}{\partial p_0} \frac{\partial q}{\partial q_0} dp_0 \wedge dq_0 - \frac{\partial p}{\partial q_0} \frac{\partial q}{\partial p_0} dp_0 \wedge dq_0 = \det A * dp_0 \wedge dq_0 = dp_0 \wedge dq_0$$

Preservation of area

The area of a parallelogram (with sides η and ξ) is given by $\eta^T J \xi$.

The area of a transformed parallelogram (with sides Aη and Aξ) is

$\eta^T A^T J A \xi = \eta^T J \xi$, if and only if A is symplectic



The symplecticness for a more general case (with d>1) can be written as

Conservation of volumn (Liouville's theorem)

Numerical integrators

A system with differential equations

$$\dot{x} = f(t,x) \qquad x = (p,q)$$

can usually be solved using integration method with infinitesimal integration steps Δt=h in each iteration. For Hamilton's equations,

Euler(nonsymplectic)

$$x_{n+1} = x_n + hJ\nabla H(x_n), \quad x_{n+1} = x_n + hJ\nabla H(x_{n+1})$$

explicit implicit

Euler(symplectic, 1st)

$$p_{n+1} = p_n - h\nabla_q H(p_n, q_{n+1}), \quad q_{n+1} = q_n + h\nabla_q H(p_n, q_{n+1})$$

Implicit midpoint(symplectic, 2nd)

$$x_{n+1} = x_n + hJ\nabla H\left(\frac{x_{n+1} + x_n}{2}\right)$$

Numerical integrators

Störmer-Verlet(symplectic,2nd)

$$p_{n+\frac{1}{2}} = p_n - \frac{h}{2}\nabla_q H(p_{n+\frac{1}{2}}, q_n)$$

$$q_{n+1} = q_n + \frac{h}{2}\left(\nabla_p H(p_{n+\frac{1}{2}}, q_n) + \nabla_p H(p_{n+\frac{1}{2}}, q_{n+1})\right)$$

$$p_{n+1} = p_{n+\frac{1}{2}} - \frac{h}{2}\nabla_q H(p_{n+\frac{1}{2}}, q_{n+1})$$

It is simply the symmetric composition (2nd order) of the two symplectic Euler methods with step size h/2.

For a 2nd order differential equation $\ddot{q} = -\nabla U(q), \quad H(p,q) = \frac{1}{2}p^T p + U(q)$

It can be simplified as $q_{n+1} - 2q_n + q_{n-1} = -h^2 \nabla U(q_n), \quad p_n = \frac{q_{n+1} - q_{n-1}}{2h}$

Runge-Kutta methods

s-stage Runge-Kutta

$$k_i = f(t + c_i h, x_n + h\sum_{j=1}^{s} a_{ij} k_j), \quad i = 1, \ldots, s$$

$$x_{n+1} = x_n + h\sum_{i=1}^{s} b_i k_i$$

where $c_i = \sum_{j=1}^{s} a_{ij}, \quad \sum_{i=1}^{s} b_i = 1$. For a case where

$$s = 4, \quad c_1 = 0, \quad c_2 = c_3 = \frac{1}{2}, \quad c_4 = 1,$$

$$a_{21} = a_{32} = \frac{1}{2}, \quad a_{43} = 1$$

$$b_1 = b_4 = \frac{1}{6}, \quad b_2 = b_3 = \frac{2}{6}$$

it simplifies to the famous 4th order Runge-Kutta integrator.

Runge-Kutta methods

Runge… …an
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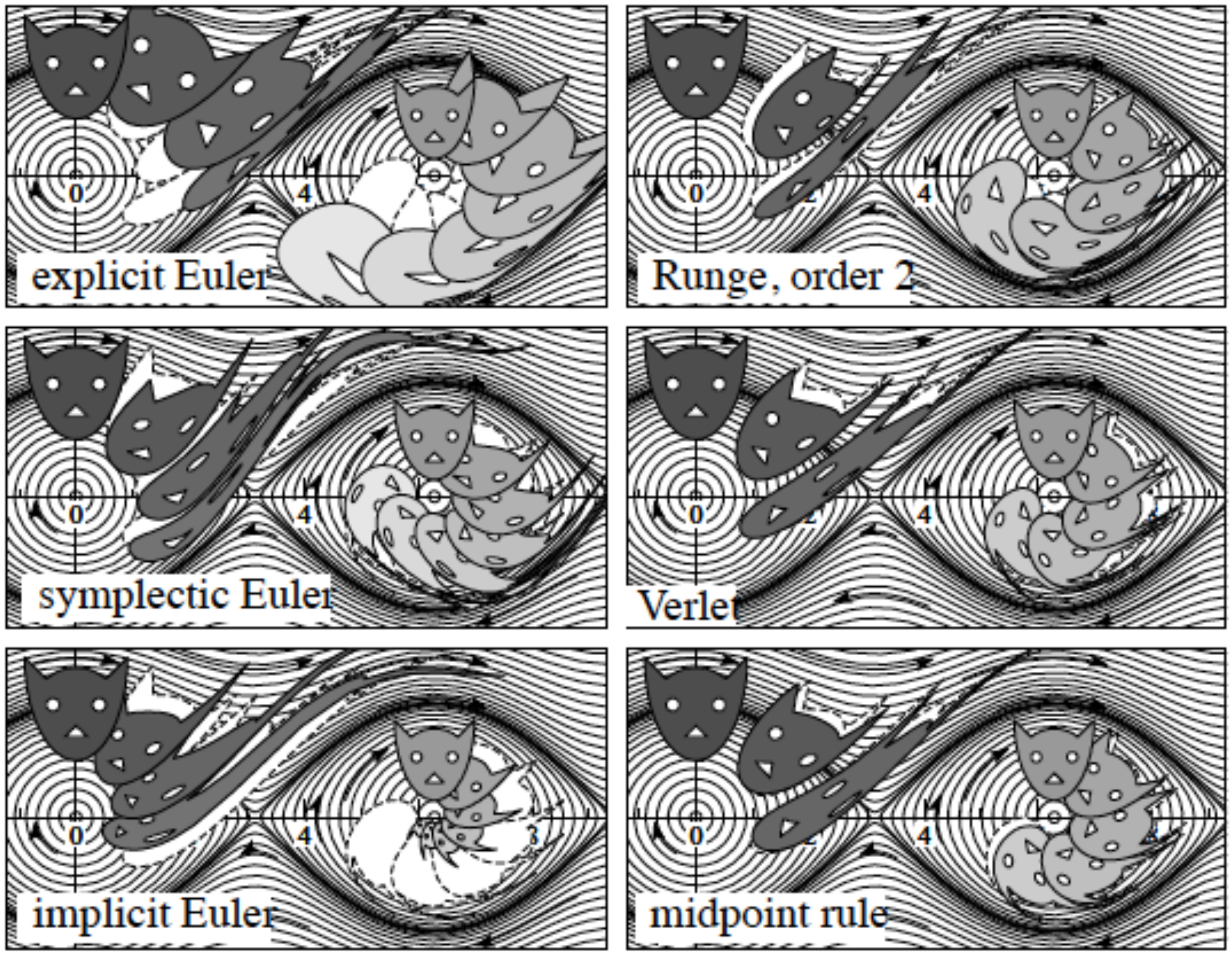


Symplectic mapping

In accelerator, we usually use transfer map to calculate lattice properties. For example, matrix for a quadrupole is

$$M = \begin{bmatrix} \cos kL & \frac{1}{k}\sin kL & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cosh kL & \frac{1}{k}\sinh kL \\ 0 & 0 & k\sinh kL & \cosh kL \end{bmatrix}$$

$$k = \sqrt{K}$$

What a simulation code does is it slices the element into pieces and apply the kicks.

Symplectic mapping(cont'd)

Thus the transfer matrix becomes

$$M_{s\to s+\Delta s} = \begin{bmatrix} \cos k\Delta s & \frac{1}{k}\sin k\Delta s \\ -k\sin k\Delta s & \cos k\Delta s \end{bmatrix}$$

And then Taylor expansion gives

$$M_{s\to s+\Delta s} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \Delta s \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} + \Delta s^2 \begin{bmatrix} -\frac{k^2}{2} & 0 \\ 0 & -\frac{k^2}{2} \end{bmatrix} + \cdots$$

Truncation is required and up to 1st order

$$M_{s\to s+\Delta s} \approx \begin{bmatrix} 1 & \Delta s \\ -k^2\Delta s & 1 \end{bmatrix}$$

While the determinant of it is not unity– not symplectic.

Symplectic mapping(cont'd)

One trick to make the determinant 1 is to artificially add in one 2nd order term

$$M_{s\rightarrow s+\Delta s} \approx \begin{bmatrix} 1 & \Delta s \\ -k^2\Delta s & 1-k^2\Delta s^2 \end{bmatrix}$$

Which makes the transfer map not as accurate as if we simply keep it up to 2nd order

$$M_{s\rightarrow s+\Delta s} \approx \begin{bmatrix} 1-\frac{1}{2}k^2\Delta s^2 & \Delta s \\ -k^2\Delta s & 1-\frac{1}{2}k^2\Delta s^2 \end{bmatrix}$$

Which is not symplectic!

Symplecticity is not equal to accuracy!!

Symplectic mapping(cont’d)

1. Classical theories of numerical integration give information about how well different methods approximate the trajectories for fixed times as step sizes tend to zero. Dynamical systems theory asks questions about asymptotic, i.e. infinite time, behavior.
2. Geometric integrators are methods that exactly conserve qualitative properties associated to the solutions of the dynamical system under study.
3. The difference between symplectic integrators and other methods become most evident when performing long time integrations (or large step size).
4. Symplectic integrators do not usually preserve energy either, but the fluctuations in H from its original value remain small.

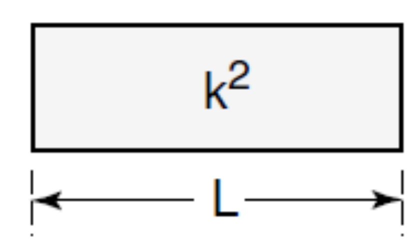
Symplectic mapping(cont'd)

One way of thinking is to use thin lens approximation, divide the quadrupole into drifts and thin lens which all have transfer matrices with unity determinant.

Transfer matrices for drift and sudden kick

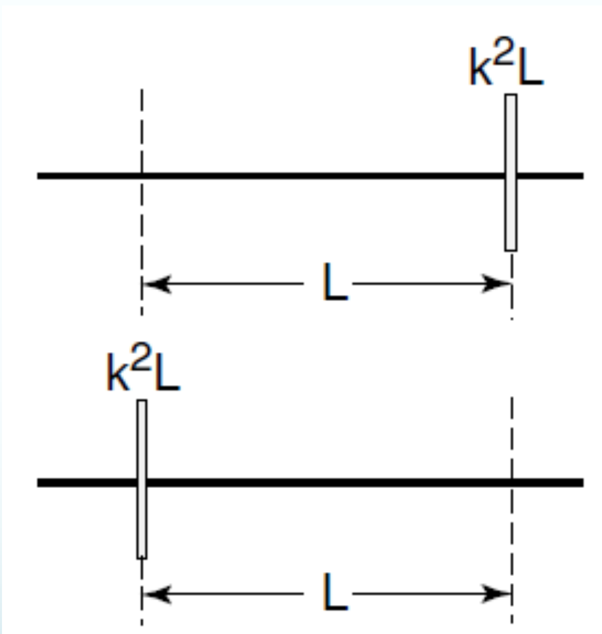
$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -k^2 L & 1 \end{bmatrix}$$

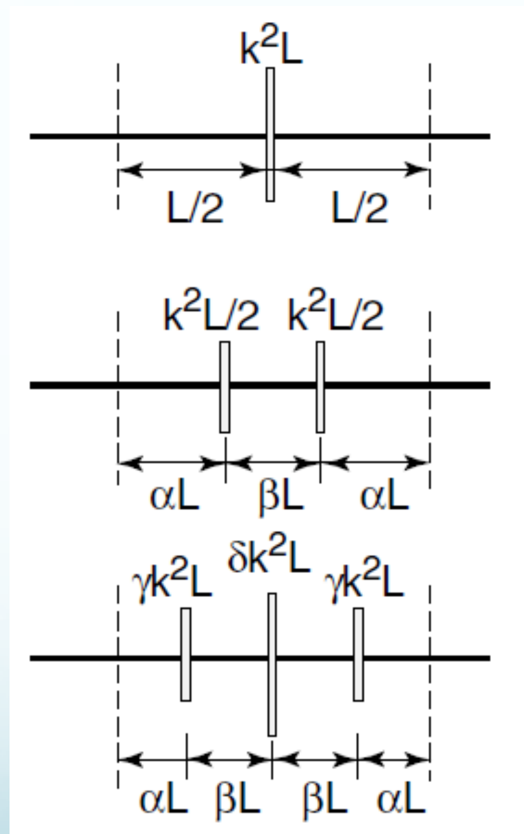
With a quadrupole at length L



Symplectic mapping(cont'd)

So we have various ways of dividing the quadrupole which result into different order of symplicticity.





Symplectic mapping(cont'd)

After splitting the magnets, we need to solve for the parameters(symplicticity is automatically preserved). Take the 2nd on the right as an example. Total transfer map is

$$M = \begin{bmatrix} 1 & \alpha L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}k^2 L & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}k^2 L & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha L \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{2}k^2L^2 + \frac{1}{4}\alpha\beta k^4 L^4 & L - \alpha(\alpha+\beta)k^2L^3 + \frac{1}{4}\alpha^2\beta k^4 L^5 \\ -k^2 L + \frac{1}{4}\beta k^4 L^3 & 1 - \frac{1}{2}k^2L^2 + \frac{1}{4}\alpha\beta k^4 L^4 \end{bmatrix}$$

Comparing with

$$M = \begin{bmatrix} \cos kL & \frac{1}{k}\sin kL & 0 & 0 \\ -k\sin kL & \cos kL & 0 & 0 \\ 0 & 0 & \cosh kL & \frac{1}{k}\sinh kL \\ 0 & 0 & k\sinh kL & \cosh kL \end{bmatrix}$$

Symplectic mapping(cont'd)

Keeping them equal up to 4^{th} order then gives

$$\alpha(\alpha+\beta)=\frac{1}{6}$$
$$\frac{1}{4}\beta=\frac{1}{6}$$
$$2\alpha+\beta=1$$

Last one arises from geometry condition.

Unfortunately, this does not have a solution—symplicticify failure.
But the 3^{rd} one on the right has a solution

$$\beta = \frac{1-2^{1/3}}{2(2-2^{1/3})} \approx -0.1756$$

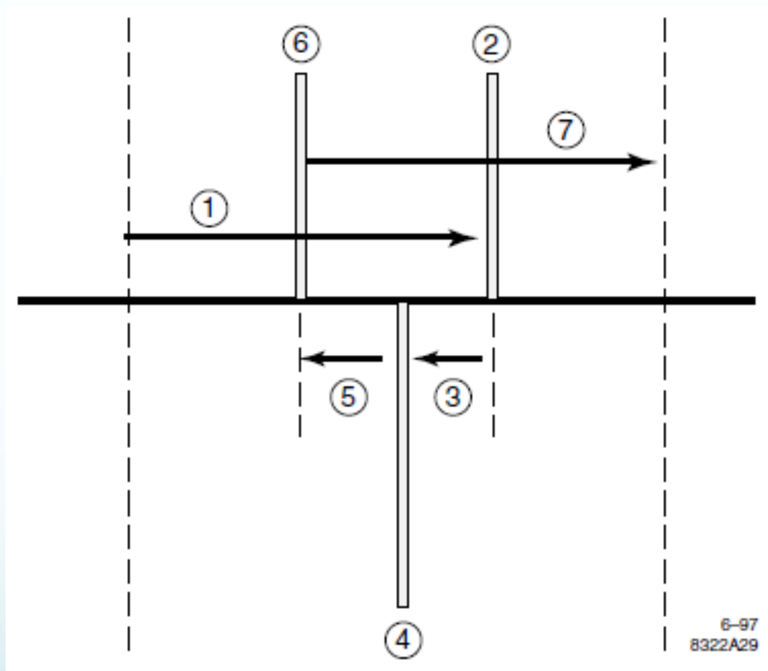
$$\gamma = \frac{1}{24\beta^2} = \frac{1}{2-2^{1/3}} \approx 1.3512$$

$$\alpha = \frac{1}{2}-\beta = \frac{1}{2(2-2^{1/3})} \approx 0.6756$$

$$\delta = 1-2\gamma = -\frac{2^{1/3}}{2-2^{1/3}} \approx -1.7024$$

Symplectic mapping(cont'd)

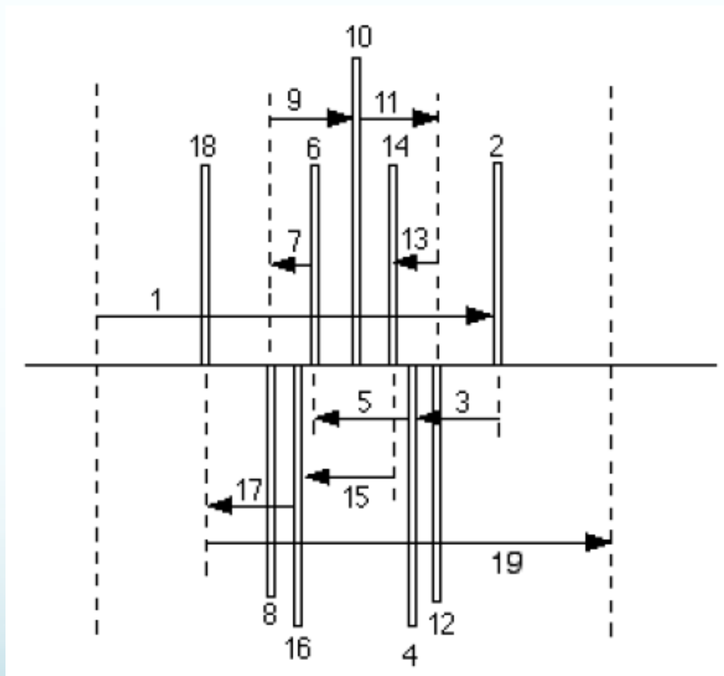
Notice that both β and δ are negative. This means we need to go through 7 steps for the symplectic integration shown as follows.



This results in a 4th order symplectic integration.

Symplectic mapping(cont'd)

Higher order of symplectic integration can be achieved simply by dividing the magnet into more pieces and solving much more complicated set of equations. A 6th order integration is done in 19 steps.



Accuracy vs order

Order does bring up complicity but does it provide higher accuracy? Considering the amplitude of phase space given by

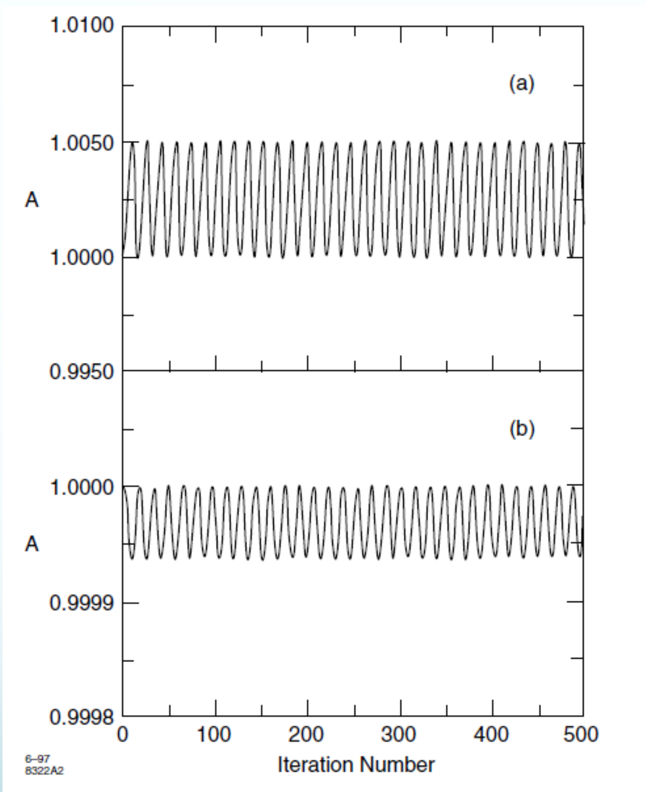
$$A = \sqrt{x^2 + (x'/k)^2}$$

With initial A to be normalized to 1.

Exact tracking should always A while if we use symplectic mapping it's not the case.

Accuracy vs order(cont'd)

Comparison of 2^{nd} order and 4^{th} order of symplectic integration is given as



With the top one as 2^{nd} order and bottom one the 4^{th}. Stability is always preserved but the accuracy is greatly improved by using higher order integration.

Notice that the deviation from 1 tells us the deviation from a pure circle– distortion. Higher order also improves the shape distortion introduced by this symplecticify process.

Accuracy vs order(cont'd)

A list shown all the integrators from 2^{nd} order to 5^{th} order is shown here with the error information and the model needed to achieve it.

Integrator	Model	Error
1st order	$(L)(SL)$	$\mathcal{O}(L^2)$
1st order	$(SL)(L)$	$\mathcal{O}(L^2)$
Ray tracing	$(\frac{L}{n})(\frac{SL}{n}) \cdots$ repeat n times	$\mathcal{O}(\frac{L^2}{n})$
2nd order(thin-lens)	$(\frac{L}{2})(SL)(\frac{L}{2})$	$\mathcal{O}(L^3)$
Ray tracing	$(\frac{L}{2n})(\frac{SL}{n})(\frac{L}{2n}) \cdots$ repeat n times	$\mathcal{O}(\frac{L^3}{n^2})$
4th order	$(\alpha L)(\gamma SL)(\beta L)(\delta SL)(\beta L)(\gamma SL)(\alpha L)$	$\mathcal{O}(L^5)$
Ray tracing	$(\frac{\alpha L}{n})(\frac{\gamma SL}{n})(\frac{\beta L}{n})(\frac{\delta SL}{n})(\frac{\beta L}{n})(\frac{\gamma SL}{n})(\frac{\alpha L}{n}) \cdots$ repeat n times	$\mathcal{O}(\frac{L^5}{n^4})$

Notic

Have to change way of slicing.

4^{th} order Runge-Kutta is not symplectic

Considering DE $x'' = f(x, x', s)$, with given initial x & x'. A 4^{th} order Runge-Kutta solves it at x=L

$$
\begin{aligned}
x(L) &\approx x(0) + Lx'(0) + \frac{1}{6}L(t_1 + t_2 + t_3) \\
x'(L) &\approx x'(0) + \frac{1}{6}(t_1 + 2t_2 + 2t_3 + t_4)
\end{aligned}
$$

With

$$
\begin{aligned}
t_1 &= Lf[x(0), x'(0), 0] \\
t_2 &= Lf[x(0) + \frac{1}{2}Lx'(0), x'(0) + \frac{1}{2}t_1, \frac{1}{2}L] \\
t_3 &= Lf[x(0) + \frac{1}{2}Lx'(0) + \frac{1}{4}Lt_1, x'(0) + \frac{1}{2}t_2, \frac{1}{2}L] \\
t_4 &= Lf[x(0) + Lx'(0) + \frac{1}{2}Lt_2, x'(0) + t_3, L]
\end{aligned}
$$

4^{th} order Runge-Kutta is not symplectic

For a quadrupole, it gives

$$x(L) \approx x(0)\left[1-\frac{1}{2}k^2L^2+\frac{1}{24}k^4L^4\right]+\frac{1}{k}x'(0)\left[kL-\frac{1}{6}k^3L^3\right]$$
$$x'(L) \approx -kx(0)\left[kL-\frac{1}{6}k^3L^3\right]+x'(0)\left[1-\frac{1}{2}k^2L^2+\frac{1}{24}k^4L^4\right]$$

with sextupole, it becomes

$$\begin{aligned} x(L) &\approx x_0+x_0'L+\frac{1}{2}Sx_0^2L^2+\frac{1}{3}Sx_0x_0'L^3+\frac{S}{12}(x_0'^2+Sx_0^3)L^4 \\ &+\frac{1}{24}S^2x_0^2x_0'L^5+\frac{1}{96}S^3x_0^4L^6 \\ x'(L) &\approx x_0'+Sx_0^2L+Sx_0x_0'L^2+\frac{S}{3}(x_0'^2+Sx_0^3)L^3+\frac{5}{12}S^2x_0^2x_0'L^4 \\ &+S^2x_0(\frac{5}{24}x_0'^2+\frac{1}{16}Sx_0^3)L^5+\frac{1}{12}S^2x_0'(\frac{1}{2}x_0'^2+x_0^3)L^6 \\ &+\frac{1}{16}S^3x_0^2x_0'^2L^7+\frac{1}{48}S^3x_0x_0'^3L^8+\frac{1}{384}S^3x_0'^4L^9 \end{aligned}$$

4^{th} order Runge-Kutta is not symplectic

For quadrupole, the determinant is

$$1-\frac{k^6L^6}{72}+\frac{k^8L^8}{576}$$

For sextupole, the determinant is

$$\begin{aligned}1-&\frac{1}{72}(2x_0'^2-9Sx_0^3)S^2L^6+\frac{7}{36}x_0^2x_0'S^3L^7+\frac{1}{144}x_0(7x_0'^2+15Sx_0^3)S^3L^8\\&+\frac{1}{288}x_0'(-x_0'^2+46Sx_0^3)S^3L^9+\frac{1}{576}x_p^2(45x_0'^2+16Sx_0^3)S^4L^{10}\\&+\frac{1}{288}x_0x_o'(4x_o'^2+13Sx_0^3)S^4L^{11}+\frac{1}{576}x_0^3(15x_0'^2+2Sx_0^3)S^5L^{12}\\&+\frac{1}{576}x_0^2x_0'(4x_0'^2+3Sx_0^3)S^5L^{13}+\frac{1}{1152}x_0x_0'^2(x_0'^2+3Sx_0^3)S^5L^{14}\\&+\frac{1}{2304}x_0^3x_0'^3S^6L^{15}\end{aligned}$$

Both of them are not 1– not symplectic!!

Normal form treatment

Instead of describing the dynamics in a beam line using an s-dependent Hamiltonian, we can construct a map, for example, in the form of a Lie transformation. Such a map may be constructed by concatenating the maps for individual elements. The beam dynamics (for example, the strengths of different resonances) may then be extracted from the transformation.

To better understand the concept of map (transformation), we take a look at the well-known linear transport matrix for a periodic accelerator (say, a storage ring)

$$M = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}, \ \beta\gamma = 1 + \alpha^2$$

the matrix is symplectic.

Normal form analysis of a linear system involves finding a transformation to variables in which the map appears as a pure rotation.

Normal form treatment

Consider matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

We find that

$$NMN^{-1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} = R$$

Becomes a pure rotation in phase space.

Normal form treatment

The coordinates are "normalized" $\vec{x}_N = N\vec{x}$

And the normalized coordinates transform in one revolution as

$$\vec{x}_N \rightarrow NM\vec{x} = NMN^{-1}N\vec{x} = RN\vec{x} = R\vec{x}_N$$

Is simply a rotation in phase space.

Note that since the transformation N is symplectic, the normalized variables are canonical variables.

Normal form treatment

The treatment of nonlinear dynamics follows the same procedure however more complicated.

We can assume the map can be represented by a Lie transformation and factorized as

$$\mathrm{M} = R e^{:f_3:} e^{:f_4:} \cdots$$

Where f3 is a homogeneous polynomial of order 3 of the phase space coordinates and f4 is a homogeneous polynomial of order 4. The detailed order depends on the truncation.

The linear part of the map can be written in action angle variables as

$$R = e^{:-\mu J:}$$

Normal form treatment

To simplify this map, i.e., separate the contribution from different orders, we can construct a map M3

$$U = e^{:F_3:} M e^{:-F_3:}$$

Where F3 is a generator that removes resonance driving terms from $e^{:f_3:}$

So we have

$$U = e^{:F_3:} R e^{:f_3:} e^{:f_4:} e^{:-F_3:} = R R^{-1} e^{:F_3:} R e^{:f_3:} e^{:-F_3:} e^{:F_3:} e^{:f_4:} e^{:-F_3:}$$

Using relation $e^{:h:} e^{:g:} e^{:-h:} = e^{:e^{:h:} g:}$

$$U = R e^{:R^{-1} F_3:} e^{:f_3:} e^{:-F_3:} e^{:e^{:F_3:} f_4:}$$

Normal form treatment

Using Baker-Campbell-Hausdorff formula

$$e^{:A:}e^{:B:} = e^{:C:}, \quad where \quad C = A + B + \frac{1}{2}[A,B] + \cdots$$

The map now becomes

$$U = \mathcal{R}e^{:R^{-1}F_3 + f_3 - F_3 + O(4):}e^{:e^{:F_3:}f_4:}$$

We can further reduce it to (non-trivial)

$$U = \mathcal{R}e^{:f_3^{(1)}:}e^{:f_4^{(1)}:} = \mathcal{R}e^{:R^{-1}F_3 + f_3 - F_3:}e^{:f_4^{(1)}:}$$

Where $f_3^{(1)} = R^{-1}F_3 + f_3 - F_3$ contains all the 3rd order contribution.

Normal form treatment

Thus the solution is

$$F_3 = \frac{f_3 - f_3^{(1)}}{I - R^{-1}}$$

Since f3 is periodic in the angle variable Φ, we can write

$$f_3 = \sum_m \bar{f}_{3,m}(J)e^{im\phi}$$

We can construct a f3(1) that does not have phase dependence, i.e., we can write it as

$$f_3^{(1)} = \bar{f}_{3,0}(J)$$

Thus now the generation function F3 reads

$$F_3 = \sum_{m\neq 0} \frac{\bar{f}_{3,m}(J)e^{im\phi}}{1 - e^{-im\mu}}$$

Normal form treatment

Taking Octupole as an example (assume it is the only nonlinear element in the beam line), we can write the map as

$$\mathrm{M} = Re^{:f_4:}$$

where f4 is

$$f_4 = -\frac{1}{24} k_3 l x^4$$

Rewrite it in action-angle variables $x = \sqrt{2\beta J} \cos\Phi$

$$f_4 = -\frac{1}{6} k_3 l \beta^2 J^2 \cos^4 \Phi = -\frac{1}{48} k_3 l \beta^2 J^2 (3 + 4\cos 2\Phi + \cos 4\Phi)$$

Thus the generation function for normalized map $f_{4,0}$ reads $f_{4,0} = -\frac{1}{16} k_3 l \beta^2 J^2$

And the normalized map becomes (with BCH theorem)

$$\mathrm{M}_4 = Re^{:f_{4,0}:} = e^{:-\mu J - \frac{1}{16} k_3 l \beta^2 J^2:}$$

Normal form treatment

Thus the mapping of action-angle variables becomes

$$J \to J$$

$$\Phi \to \Phi + \mu + \frac{1}{8} k_3 l \beta^2 J$$

In other words, we see the tune shift with amplitude right away.

Similar to previous case for sextupole, we have

$$\mathbf{M}_4 = Re^{:f_{4,0}:} = e^{:-\mu J - \frac{1}{16} k_3 l \beta^2 J^2:} \doteq e^{:F_4:} M e^{:-F_4:}$$

Last equation is valid if we keep the normalization up to 4th order.

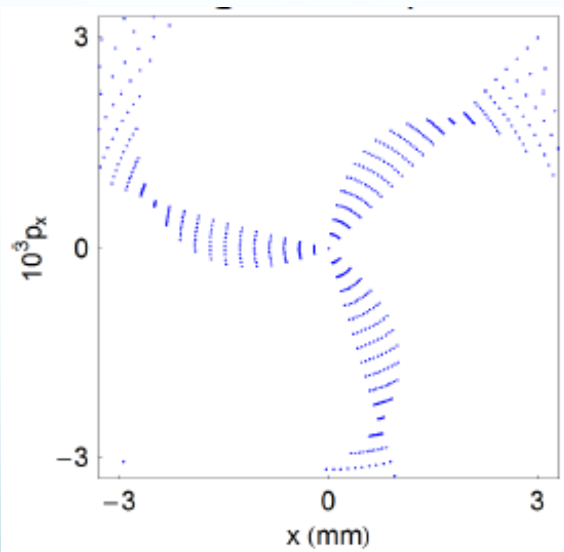
We can obtain the normalization generator F_4 easily $$F_4 = \sum_{m \neq 0} \frac{f_{4,m}(J) e^{im\phi}}{1 - e^{-im\mu}}$$

$$F_4 = -\frac{1}{96} k_3 l \beta^2 J^2 \left(\frac{4[\cos 2\Phi - \cos 2(\Phi + \mu)]}{1 - \cos 2\mu} + \frac{\cos 4\Phi - \cos 4(\Phi + \mu)}{1 - \cos 4\mu} \right)$$

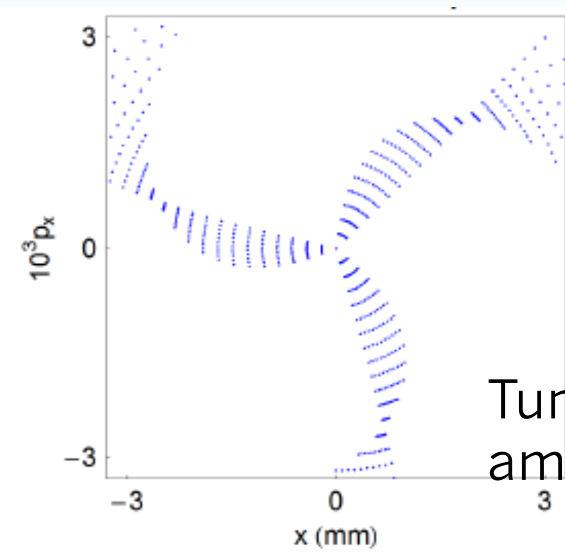
Normal form treatment

The normalized map now contains only action variable (easy to integrate) while all the phase information has been pushed to higher order.

From the generator F_4, we see the octupole drives half integer and quarter integer resonances. We can track the Poincare map using exact map and the normalized map respectively (assum k_3l= 4800 m^{-3} and β=1 m). Assuming the tune μ is $0.33\times2\pi$ far from resonances



exact map



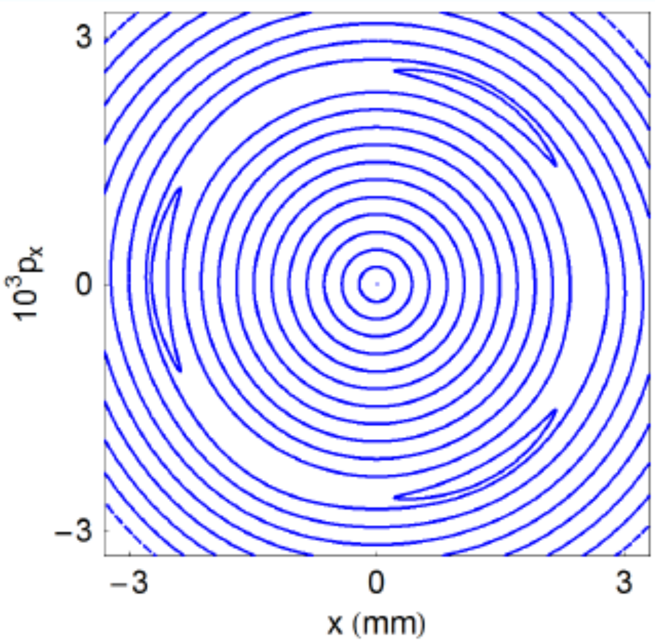
normalized map

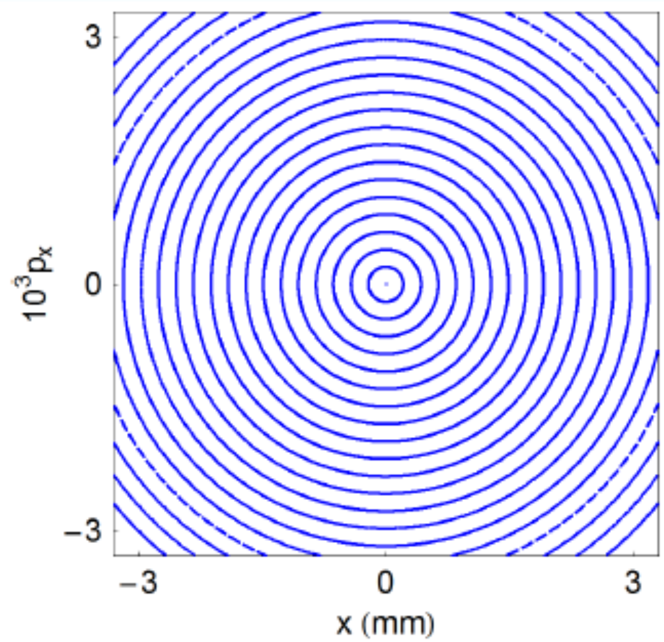
30 turns

Tune shift with amplitude!!

Normal form treatment

Tracking for longer turns results in different feature where we pay the price of the simplified (normalized) map. Some of the phase information (3rd order resonance island) is lost during this process.





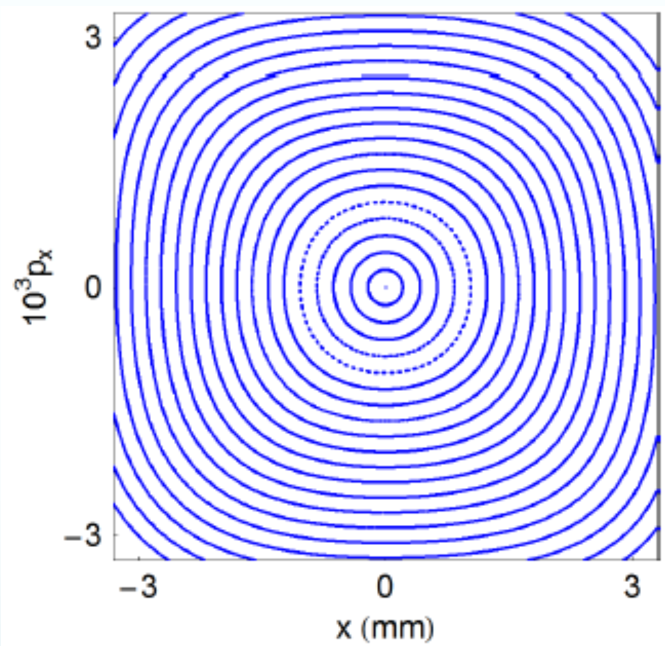
2500 turns

exact map

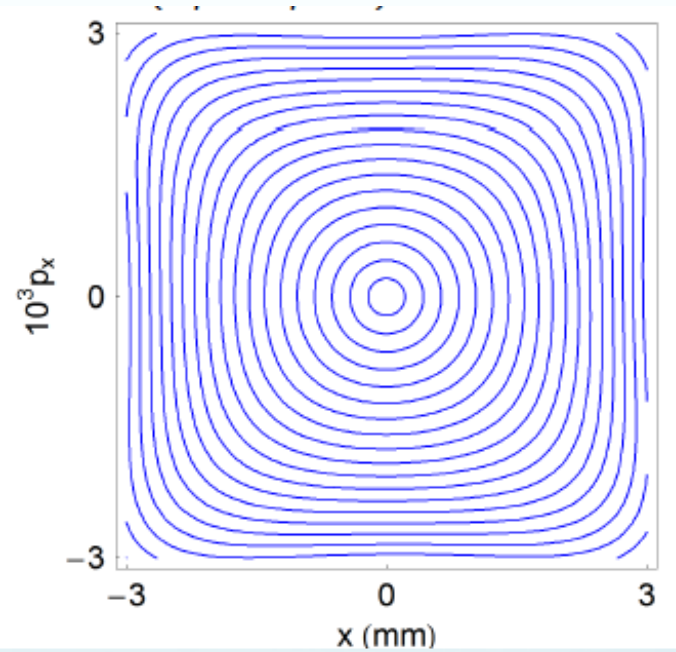
normalized map

Normal form treatment

Tracking for tunes near 4th order resonance is a bit tricky. Since the k_3l is positive, the tune shift with amplitude drives the tune up. Thus if the tune μ is $0.252\times 2\pi$, we barely see resonances. The two tracking results resemble



exact map

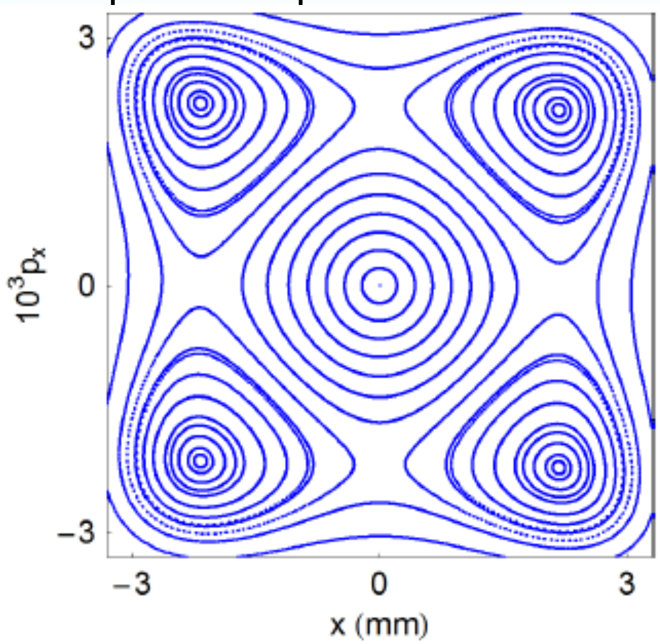


normalized map

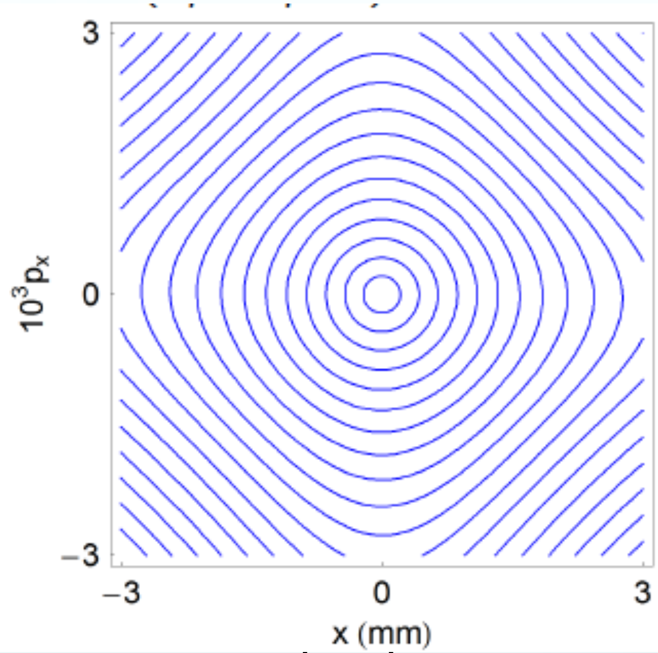
2500 turns

Normal form treatment

For a tune less than quarter integer, i.e.,μ is $0.248\times2\pi$, we see strong resonances from exact tracking while for the normalized map, we only see a rotation in phase space.



exact map



normalized map

2500 turns

Normal form of a one turn map preserves the information on tune amplitude dependence while loses the key phase information (when close to resonances). Need to retain higher order terms!

Resonance driving terms(RDTs)

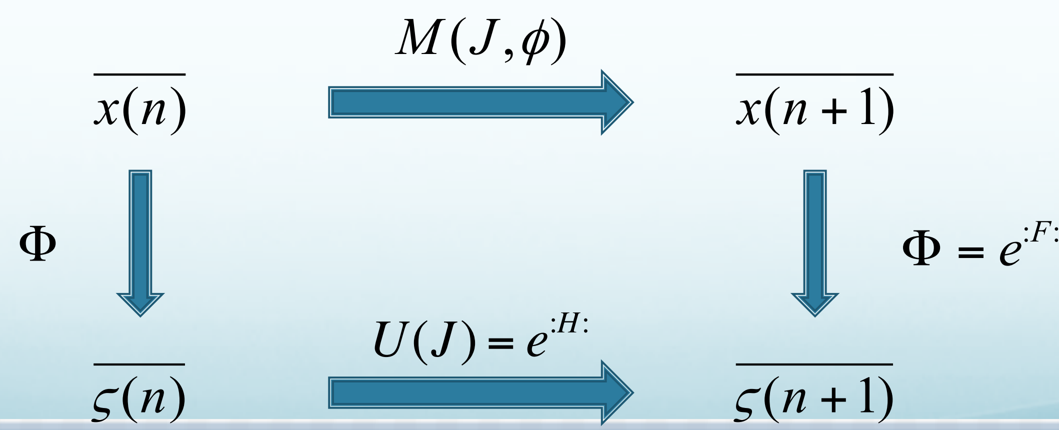
We can interpret the Fourier coefficients $\bar{f}_{3,m}(J)$ as resonance strengths. And the generating function diverges when resonance condition mμ=2π is satisfied, meaning such driving term has large effect. Put it into polynomial expression, the generating function can be written as

$$F = \sum_{jklm} f_{jklm} \varsigma_x^+ \varsigma_x^- \varsigma_y^+ \varsigma_y^- = F_3 + F_4 + \cdots$$

where

$$f_{jklm} = \frac{h_{jklm}}{1 - e^{i2\pi[(j-k)\nu_x + (l-m)\nu_y]}}$$

hjklm are called resonance driving terms in many accelerator tracking codes. The entire process of the normal form the one turn map can be visualized as



Resonance driving terms(RDTs)

Incorporating the optics of a lattice, the resonance driving term (RDT) coefficients h_{jklm} (1st order RDT) are usually calculated as

$$h_{jklm} = c\sum_{i=1}^{N} S_2 \beta_{xi}^{(j+k)/2} \beta_{yi}^{(l+m)/2} e^{i[(j-k)\mu_{xi}+(l-m)\mu_{yi}]}$$

It is very sensitive to linear lattice thus a carefully designed linear lattice with proper phase advance per periodic structure benefits greatly in reducing the RDTs (we will talk about a few tactics later).

Chromatic aberration

Sextupoles (and even higher order magnets) are necessary in an accelerator design (not only existing as the field error of strong linear magnets).

Sextupoles are used to correct the chromatic aberration, i.e., tune shift, that resides in linear lattice (in comparison to the aberration that exists in optics).

We can define chromaticities

$$\Delta\nu_x = \left[-\tfrac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \equiv C_x\delta, \quad C_x = d\nu_x / d\delta$$

$$\Delta\nu_y = \left[-\tfrac{1}{4\pi}\oint \beta_y(s)K_y(s)ds\right]\delta \equiv C_y\delta, \quad C_y = d\nu_y / d\delta$$

The chromaticity induced by quadrupole field is called natural chromaticity.

Chromaticity

Chromaticity can be very large. Taking FODO lattice as an example,

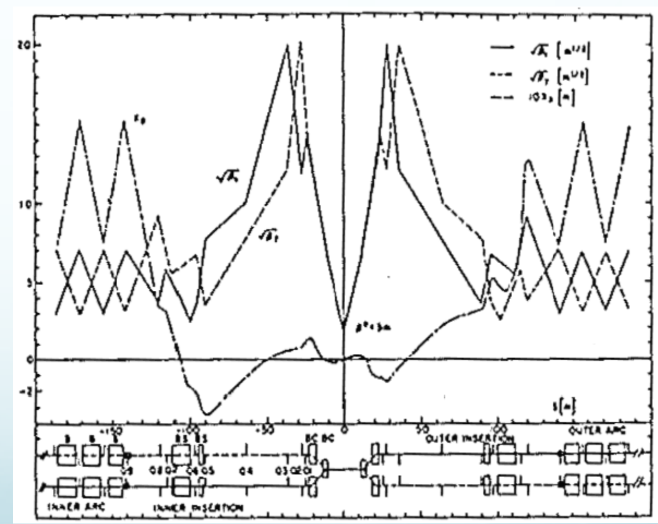
$$C_{X,\text{nat}}^{\text{FODO}} = -\frac{1}{4\pi}N\left(\frac{\beta_{\max}}{f} - \frac{\beta_{\min}}{f}\right) = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$

Natural chromaticity per cell is approaching cell tune (when phase advance per cell is not large).

Chromaticities from interaction region
(for colliders) can be huge due to the low beta.

$$C_{total} = N_{IR}C_{IR} + C_{ARCs}$$

$$C_{IR} = -\frac{2\Delta s}{4\pi\beta^*} \approx -\frac{1}{2\pi}\sqrt{\frac{\beta_{\max}}{\beta^*}}$$



For a ring with C~1000 m, the chromaticities can easily exceed negative few hundred units, which cause severe instablities.

Chromaticity correction

In existence of a sextupole element, the Hill's equation becomes

$$x''_\beta + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad y''_\beta + (K_y(s) - K_2 D\delta)y_\beta = 0$$

$$x = x_\beta + D\delta$$

$$\Delta K_x(s) = K_2(s)D(s)\delta, \quad \Delta K_y(s) = -K_2(s)D(s)\delta$$

thus

$$C_x = -\tfrac{1}{4\pi}\oint \beta_x(s)[K_x(s) - K_2(s)D(s)]ds$$

$$C_y = -\tfrac{1}{4\pi}\oint \beta_y(s)[K_y(s) + K_2(s)D(s)]ds$$

In order to minimize their strength, the chromatic sextupoles should be located near quadrupoles, where $\beta_x D_x$ and $\beta_y D_x$ are maximum.
A large ratio of β_x/β_y for the focusing sextupole and a large ratio of β_y/β_x for the defocussing sextupole are needed for optimal independent chromaticity control.

Chromaticity correction 2nd

To avoid head-tail instability, we need to satisfy:

$$C_x/\eta > 0, \quad \eta = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} > 0$$

The 2nd order chromaticity can be expressed as $C_x^{(2)} = -C_x^{(1)} - \frac{|J_{p,x}|^2}{4(\nu_x - p/2)\delta^2}$

By pairing adjacent sextupole families

$$S_{F1} \to S_{F1} + (\Delta S)_F, \qquad S_{D1} \to S_{D1} + (\Delta S)_D,$$
$$S_{F2} \to S_{F2} - (\Delta S)_F \qquad S_{D2} \to S_{D2} - (\Delta S)_D$$

$$\Delta J_{p,x} = \frac{\delta}{2\pi} N[\beta_F (\Delta S)_F D_F + \beta_D (\Delta S)_D D_D e^{i\pi/4}]$$

Under conditions $\quad p \approx 2\nu \quad$ $\Phi \approx \pi/2$

We design the linear lattice to have 90 deg phase advance per FODO cell to remove the potential cancellation between sextupoles and the change in stopband integral linearly depends on the change in sextupole strengths.

Dynamic aperture (DA)

Dynamic aperture determines the stable region in 2d real space (x-y) while particles travel along the accelerator. It is very important for particle dynamic study especially in effects that requires tracking over many revolutions (decided by system's damping time, could range from 1000 (light sources) to 1,000,000 (proton/heavy ion storage rings).

Dynamic aperture is a clear indication of nonlinear resonances that reside in an accelerator. Its size is limited by the utilize of nonlinear magnets to correct chromatic aberration. Thus designing the lattice with the nonlinear magnets' strengths reduced is crucial in improving DA.

Careful tuning of multipole nonlinear elements can also result in reducing the resonance driving terms thus improving the DA.

There are many ways of determining the DA of a specific lattice. Mostly commonly used techniques include line search mode (single-line, n-line,etc…) and frequency map analysis.

Line search analysis

Line search mode requires tracking particles with different initial positions (or gradually increasing the particle offset till it is lost) to determine the boundary of the stable region. Itself is machine expensive however can be easily parallelized.

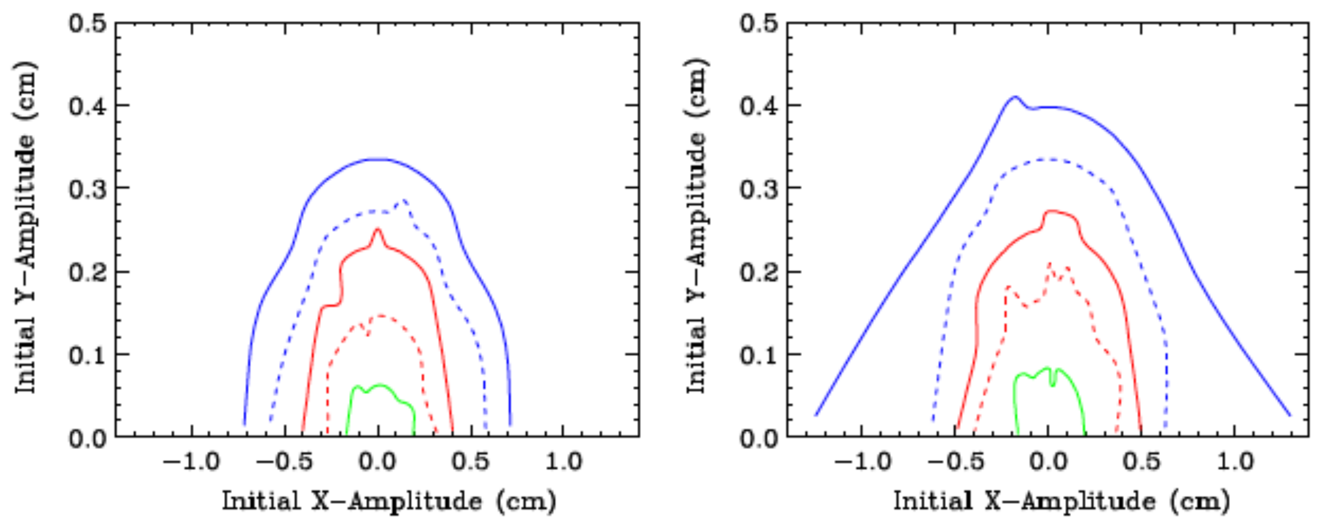


Figure 10: *Momentum dependent dynamic aperture without errors for OPA (left) and 4th-order geometric achromat (right) solutions with chromaticity set to zero, where:* $\delta = 0$ *(blue solid), 0.5% (blue dash), 1% (red solid), 1.5% (red dash), 2% (green).*

Frequency map analysis(FMA)

If we perform a discrete Fourier transform on the tracking data with initial position. We can obtain the betatron tunes (for N turn tracking, the precision is merely 1/N). If we repeat this process with different initial positions, we can obtain a tune map. To indicate the variation of the tunes over different turns of the ring, we can define a diffusion or regularity which describes the difference between the tunes over various periods (usually the first half of the tracking (Q_{x1}, Q_{y1}) and the second half(Q_{x2}, Q_{y2})). In other words, we define a diffusion constant D

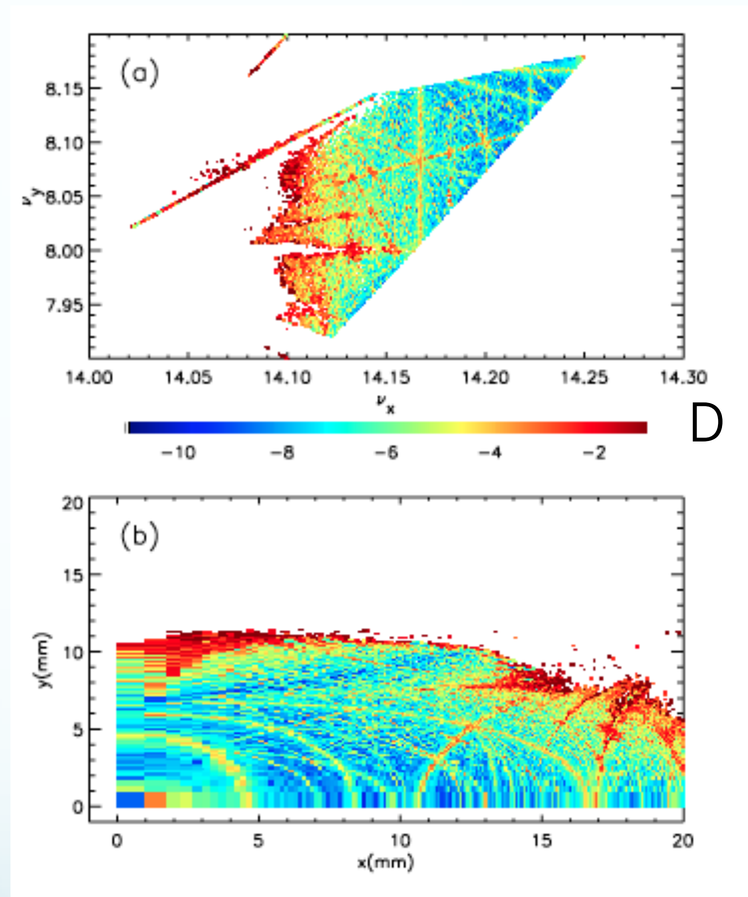
$$D = \log_{10}\sqrt{(Q_{y2} - Q_{y1})^2 + (Q_{x2} - Q_{x1})^2}$$

The rule of thumb is when D is small, the variation is low (or regular) and particle motion is stable. On the other hand, when D is large, the variation is high (or irregular) and particle motion is unstable (chaotic). The points in tune space with large variation (chaotic) usually lies on the crossing of different resonance lines.

Frequency map analysis(FMA)

The obtained resonance feature in frequency space (tune space) can then be easily related into 2 dimension x-y real space and used as an indicator of the size of stable region. It may discover some resonance islands that line search is not capable of finding as well as the important tune shifts and strong resonances that we need to avoid. FMA is often used in accelerator design to identify the dynamical behavior.

Experimental construction of FM requires very high precision measurements and some data mining techniques to further improve the precision, e.g., Hanning filter, data interpolation, NAFF, etc…



A plot showing the FM for an ideal lattice for ALS in tune space (a) and real space (b).