PHY 564
Advanced Accelerator Physics
Lecture 10

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Let's start from HW1/2 solutions

Problem 1. Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with $v = \dfrac{v_1 + v_2}{1 + (v_1 v_2 / c^2)}$.

Solution: Each Lorentz transformations along x-axis corresponds to the block-diagonal matrix with parameterization of :

$$L_i = \begin{bmatrix} L_i & O \\ O & I \end{bmatrix}; L_i = \gamma_i \begin{bmatrix} 1 & \beta_i \\ \beta_i & 1 \end{bmatrix}; O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \ \det L_i = \gamma_i^2 (1 - \beta_i^2) = 1$$

and we should find parameters of L by brining it to the same form

$$L = \begin{bmatrix} L & O \\ O & I \end{bmatrix} = L_2 L_1 = \begin{bmatrix} L_2 & O \\ O & I \end{bmatrix} \begin{bmatrix} L_1 & O \\ O & I \end{bmatrix} = \begin{bmatrix} L_2 L_1 & O \\ O & I \end{bmatrix}; \ L = L_2 L_1 .$$

The fact that $\det L = \gamma^2 (1 - \beta^2) = 1$ for any L is taking care of the rest:

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$$L = L_2 L_1 = \gamma_1 \gamma_2 \begin{bmatrix} 1 & \beta_2 \\ \beta_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta_1 \\ \beta_1 & 1 \end{bmatrix} = \gamma_1 \gamma_2 \begin{bmatrix} 1 + \beta_1 \beta_2 & \beta_1 + \beta_2 \\ \beta_1 + \beta_2 & 1 + \beta_1 \beta_2 \end{bmatrix} = \gamma_1 \gamma_2 (1 + \beta_1 \beta_2) \begin{bmatrix} 1 & \dfrac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \\ \dfrac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} & 1 \end{bmatrix}$$

Problem 2. Show that trace of a tensor is 4-invariant, i.e. $F^i{}_i \equiv \sum_i F^i{}_i = inv$.

Solution: $$Trace(F') = F'^i{}_i = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^j}{\partial x'^i}F^k{}_j = \frac{\partial x^j}{\partial x^k}F^k{}_j = \delta^j_k F^k{}_j = F^k{}_k = Trace(F) \ \#$$

Problem 3. Lorentz group

a) **5 points.** For the Lorentz boost and rotation matrices **K** and **S** show that

$$\left(\vec{\varepsilon}\vec{\mathbf{S}}\right)^3 = -\vec{\varepsilon}\vec{\mathbf{S}}; \left(\vec{\varepsilon}\vec{K}\right)^3 = \vec{\varepsilon}\vec{K}; \forall\, \vec{\varepsilon} = \vec{\varepsilon}^{\,*}; |\vec{\varepsilon}| = 1;$$

$$\text{or } \left(\vec{a}\vec{\mathbf{S}}\right)^3 = -\vec{a}\vec{\mathbf{S}}\cdot\vec{a}^{\,2}; \left(\vec{a}\vec{\mathbf{K}}\right)^3 = \vec{a}\vec{\mathbf{K}}\cdot\vec{a}^{\,2}; \forall\, \vec{a} = \vec{a}.$$

b) **5 points.** use this results to show that

$$e^{\vec{\omega}\vec{\mathbf{S}}} = I + \frac{\vec{\omega}\vec{\mathbf{S}}}{|\vec{\omega}|}\sin|\vec{\omega}| + \frac{\left(\vec{\omega}\vec{\mathbf{S}}\right)^2}{\vec{\omega}^2}(\cos|\vec{\omega}| - 1);$$

$$e^{\vec{\beta}\vec{K}} = I + \frac{\vec{\beta}\vec{\mathbf{K}}}{|\vec{\beta}|}\sinh|\vec{\beta}| + \frac{\left(\vec{\beta}\vec{\mathbf{K}}\right)^2}{\vec{\beta}^2}(\cosh|\vec{\beta}| - 1);$$

Draw connection to Lorentz transformations (e.g. boosts and rotations).

$$L = -\vec{\omega}\vec{S} - \vec{\varsigma}\vec{K}; A = e^{-\vec{\omega}\vec{S} - \vec{\varsigma}\vec{K}}; \tag{B-44}$$

with

$$\vec{S} = \hat{e}_x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \hat{e}_y \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + \hat{e}_z \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \tag{B-45}$$

$$\vec{K} = \hat{e}_x \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \hat{e}_y \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \hat{e}_z \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \tag{B-46}$$

Solution: it is possible to do it by multiplying three matrices and getting confirmation. Otherwise, we can test that:

$$\left(\vec{a}\vec{K}\right)^3 = \left(\vec{a}\vec{K}\right)^2 \cdot \vec{a}\vec{K};$$

$$K_\alpha K_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \delta_{\alpha\beta} + \begin{bmatrix} 0 & 0 \\ 0 & u_{\chi\varepsilon} \end{bmatrix} \delta_{\chi\alpha} \delta_{\varepsilon\beta}; u = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix};$$

and use it to calculate square of the matrix:

$$\left(\vec{a}\vec{K}\right)^2 \equiv \sum_{\alpha,\beta=1,2,3} a_\alpha a_\beta K_\alpha K_\beta = \begin{bmatrix} \vec{a}^2 & 0 \\ 0 & 0 \end{bmatrix} + \sum_{\alpha,\beta=1,2,3} \begin{bmatrix} 0 & 0 \\ 0 & a_\alpha a_\beta u_{\chi\varepsilon} \end{bmatrix} \delta_{\chi\alpha} \delta_{\varepsilon\beta} = \vec{a}^2 I + X;$$

$$X = \left(\sum_{\alpha,\beta=1,2,3} \begin{bmatrix} 0 & 0 \\ 0 & a_\alpha a_\beta u_{\chi\varepsilon} \end{bmatrix} \delta_{\chi\alpha} \delta_{\varepsilon\beta} - \vec{a}^2 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right).$$

First term gives us desirable answer if product of matrix X and $\vec{a}\vec{K}$ is zero. It is easy to show:

$$\vec{a}\vec{K} = \begin{bmatrix} 0 & \vec{a} \\ \tilde{\vec{a}} & 0_{3\times3} \end{bmatrix}; -\vec{a}^2 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} 0 & \vec{a} \\ \tilde{\vec{a}} & 0_{3\times3} \end{bmatrix} = -\vec{a}^2 \begin{bmatrix} 0 & 0 \\ \tilde{\vec{a}} & 0_{3\times3} \end{bmatrix};$$

$$\sum_{\alpha,\beta=1,2,3} a_\alpha a_\beta \delta_{\chi\alpha} \delta_{\varepsilon\beta} \begin{bmatrix} 0 & 0 \\ 0 & u_{\chi\varepsilon} \end{bmatrix} \cdot \begin{bmatrix} 0 & \vec{a} \\ \tilde{\vec{a}} & 0_{3\times3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \tilde{\vec{b}} & 0_{3\times3} \end{bmatrix}$$

$$b_\chi = \sum_{\alpha,\beta,\varepsilon=1,2,3} a_\alpha a_\beta \delta_{\chi\alpha} \delta_{\varepsilon\beta} a_\varepsilon = a_\chi \sum_{\alpha,\beta,\varepsilon=1,2,3} a_\beta \delta_{\varepsilon\beta} a_\varepsilon = a_\chi \cdot \vec{a}^2 \Rightarrow \vec{b} = \vec{a} \cdot \vec{a}^2$$

. #K

For **S** it is even easier, noting that it is already block-diagonal matrix:

$$S_\alpha = e_{\alpha\beta\gamma}\begin{bmatrix} 0 & 0 \\ 0 & u_{\beta\gamma} \end{bmatrix}$$

and further we can drop all time components operating with 3x3 matrix:

$$\left[S_\alpha\right]_{\beta\gamma} = e_{\alpha\beta\gamma}; \left[\vec{a}\vec{S}\right]_{\beta\gamma} = a_\alpha e_{\alpha\beta\gamma};$$

$$\left(\vec{a}\vec{S}\right)^2{}_{\beta\eta} = \left[a_\alpha a_\varepsilon S_\alpha S_\varepsilon\right] = a_\alpha a_\varepsilon e_{\alpha\beta\gamma} e_{\varepsilon\gamma\eta}; \; e_{\alpha\beta\gamma} e_{\varepsilon\gamma\eta} = -e_{\alpha\beta\gamma} e_{\varepsilon\eta\gamma} = -\delta_{\alpha\varepsilon}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\varepsilon};$$

$$\delta_{\alpha\varepsilon} a_\alpha a_\varepsilon = \vec{a}^2; a_\alpha a_\varepsilon \delta_{\alpha\eta}\delta_{\beta\varepsilon} = a_\eta a_\beta; \left(\vec{a}\vec{S}\right)^2{}_{\beta\eta} = I\vec{a}^2 + a_\beta a_\eta; \; a_\beta a_\eta a_\mu e_{\mu\eta\theta} \equiv 0!$$

which is equivalent to

$$\left(\vec{a}\vec{S}\right)^2 \left(\vec{a}\vec{S}\right) = -\vec{a}^2 \left(\vec{a}\vec{S}\right) \quad \#\mathrm{S}$$

b) is trivial for any matrix

$$M^3 = (-1)^n x^2 M;\ n = 0,1$$

which also means that

$$M^4 = (-1)^n x^2 M^2;$$

Separating series into zero order, odd and even terms:

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + \sum_{k=0}^{\infty} \frac{M^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty} \frac{M^{2k}}{(2k)!}$$

and then use induction principle to remove all powers higher then two:

$$\sum_{k=0}^{\infty} \frac{M^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(M^2)^k}{(2k+1)!} M = M \sum_{k=0}^{\infty} \frac{\left\{(-1)^n x^2\right\}^k}{(2k+1)!};$$

$$\sum_{k=1}^{\infty} \frac{M^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{(M^2)^k}{(2k)!} = M^2 \sum_{k=1}^{\infty} \frac{\left\{(-1)^n x^2\right\}^k}{(2k)!}$$

$$\sum_{k=0}^{\infty}\frac{M^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}\frac{(M^2)^k}{(2k+1)!}M=M\sum_{k=0}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k+1)!};$$

$$\sum_{k=1}^{\infty}\frac{M^{2k}}{(2k)!}=\sum_{k=1}^{\infty}\frac{(M^2)^k}{(2k)!}=M^2\sum_{k=1}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k)!}$$

brining the rest of the problem to known exponents:

$$i^n x\sum_{k=0}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k+1)!}=\frac{1}{2}\left\{e^{i^n x}-e^{-i^n x}\right\};$$

$$1+(-1)^n x^2\sum_{k=1}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k)!}=\frac{1}{2}\left\{e^{i^n x}+e^{-i^n x}\right\};$$

Therefore, both cases are identical with exception of the split between regular sin/cos and their hyperbolic twins.

In addition:

$$M=\vec{a}\vec{S}\Rightarrow x=|\vec{a}|;\Rightarrow\frac{M}{x}=\frac{\vec{a}\vec{S}}{|\vec{a}|}=\hat{e}\vec{S};$$

$$M=\vec{a}\vec{K}\Rightarrow x=|\vec{a}|;\Rightarrow\frac{M}{x}=\frac{\vec{a}\vec{K}}{|\vec{a}|}=\hat{e}\vec{K};\#\#$$

$$\hat{e}\rightarrow\vec{\beta};\ |\vec{a}|\rightarrow\zeta$$

Since had typo ("-" sign) – everybody who got correct sign, had extra 10 points!

Problem 1. 10 points Motion of non-radiating charged particle in constant uniform magnetic field is a well known spiral:

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\left[\vec{v}\times\vec{H}\right] = \frac{e}{c}H\left[\hat{e}_x v_y - \hat{e}_y v_x\right]; \vec{H} = \hat{e}_z H$$

$$\mathrm{E} = c\sqrt{m^2c^2 + \vec{p}^2 =} conts; \gamma = const; v = const;$$

$$p_z = const;\ z = v_{oz}t + z_o;$$

$$p_x^2 + p_y^2 = const;\ p_x + ip_y = p_\perp e^{i\varphi(t)} = m\gamma v_\perp e^{i\varphi(t)}$$

simple substitution gives:

$$m\gamma v_\perp \frac{de^{i\varphi(t)}}{dt} = \frac{e}{c}\left[\vec{v}\times\vec{H}\right] = -i\frac{e}{c}Hv_\perp e^{i\varphi(t)}$$

$$r_\perp = x + iy = i\omega m\gamma v_\perp \frac{de^{i\varphi(t)}}{dt}$$

$$\varphi(t) = \omega t + \varphi_o;\ \omega = -\frac{eH}{m\gamma c}$$

and trajectory: $z = v_{oz}t + z_o; x + iy = v_\perp / \omega \cdot e^{i\omega t}$. Do not forget to apply Re or Im to all necessary formulae. Use analytical extension of the Lorentz transformation to complex values by going into a reference frame with x-velocity going approaching infinity $\beta \Rightarrow \infty; \chi \to 0; \chi\beta \to 1$. Show that transverse electric field becomes a magnetic field (with an imaginary value) and visa versa. Follow this path and transfer 4-coordinates to that frame. Use analytical extension of *exp, sin, cos* to complex values and transform the solution above in that for motion in constant magnetic field. Compare it with known solution is your favorite EM book .

Solution: Let's, formally, apply Lorentz transformation as it is not limited to velocities lower than the speed of the light. In this sense, we are analytically extending Lorentz transformation into all range if the velocities: both real and imaginary.

Lorentz transformation with speed exceeding the speed of the light are analytic extensions of real transformations for $\beta > 1$ and two choices of sign (±):

$$L = \pm \begin{bmatrix} i\chi & i\chi\beta & & \\ i\chi\beta & i\chi & & \\ & & 1 & \\ & & & 1 \end{bmatrix}; \chi = 1/\sqrt{\beta^2 - 1}.$$

Applying this transformation to a pure electric field E_y, we got

$$E'_y = \pm i\chi E'_y; H'_z = \pm i\chi\beta E'_y;$$

and by applying

$$\beta \Rightarrow \infty; \chi \to 0; \chi\beta \to 1$$

we got desirable transformation:

$$E'_y = 0; H'_z = \pm iE_y$$

with imagine magnetic field instead of real electric field.

Our solutions for pure magnetic case should extended (I use specific initial conditions):

$$z' = v_{oz} t'; \omega = -\frac{eH'_z}{p_o c};$$
$$x' = v_\perp / \omega \cdot \sin(\omega t);$$
$$y' = v_\perp / \omega \cdot \cos(\omega t)$$

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using coordinate and fields transformation have two branches

$$t' = \pm ix; x' = \pm it, z' = z; y' = y \qquad and \qquad H'_z = \pm iE_y$$
$$E' = \pm ip_x; p'_x = \pm iE; p'_y = p_y; p'_z = p_z;$$

into (I use $+i$ branch): $t' = +ix; x' = -it, z' = z; y' = y \qquad and \qquad H'_z = iE_y$:

$$z = iv_{oz} x; \omega = -i\frac{eE_y}{p_o c};$$
$$-it = v_\perp / \omega \cdot \cos(i\omega x + \varphi);$$
$$y = v_\perp / \omega \cdot \sin(i\omega x + \varphi)$$

And choosing constants easily transformable to well know result for an uniform electric field (we have to rename some of the constants of motion):

$$\frac{z}{x} = \frac{v_{oz}}{v_{ox}}; \quad \Omega = \frac{eE_y}{p_o c};$$

$$t = \frac{E_o}{ecE_y} \cdot \sinh\left(\frac{eE_y}{p_o c} x\right);$$

$$y = \frac{E_o}{eE_y} \cdot \cosh\left(\frac{eE_y}{p_o c} x\right).$$

Naturally we used that $\sin(i\varphi) = -i \sinh\varphi; \cos(i\varphi) = \cosh\varphi$.

The goal of this problem was to demonstrate close connection of Lorentz transformations, special relativity and E&M fields. Not only that fields are transform into each other but also that solutions for particle's trajectory are analytical extensions of each other.

Problem 2. 4 points

Find maximum energy of a charged particle (with unit charge *e*!) which can be circulating in Earth's larges possible storage ring: the one going around Earth equator with radius of 6,384 km.

First, find it for storage ring using average bending magnetic field of a super-conducting magnet with strength of 10 T (100 kGs).

Second, find it for a very strong DC electric dipole fields of 10 MV/m.

Compare these energies with current largest (27 km in circumference) circular collider, LHC, circulating 6.5 TeV (1 TeV = 10^{12} eV).

Hint: assume that particles move with speed of the light. Check the final result for protons having rest mass of 938.27 MeV/c^2

Solution:

Magnetic field:

$$\frac{d\vec{p}}{dt} = -\left[\vec{\Omega}\times\vec{p}\right]; \Omega = \frac{|\vec{v}|}{R}; \vec{v}\perp\vec{H}; \ \left|\frac{d\vec{p}}{dt}\right| = \left|\frac{d\vec{p}}{ds}\right||\vec{v}| = \frac{pv}{R} = \frac{e}{c}v\left|\vec{B}\right|$$

$$pc = eBR$$

Using ratio from the class notes: 1 Tm is 0.3 GeV (0.3*10^9 eV). It means that 1 T km is 0.3 TeV. Then we get

$$E \cong pc = eBR = 19{,}150 \ \ TeV \sim 1.9\cdot 10^{16} \ \ eV$$

which is ~2,700 higher than LHC energy. One should note that there will be another problem, which we will study when we look into synchrotron radiation. Still, a long way to go! Relativistic factor for proton is >2 10^7.

Electric field:

$$\frac{d\vec{p}}{dt} = -\left[\vec{\Omega}\times\vec{p}\right]; \Omega = \frac{|\vec{v}|}{R}; \vec{v}\perp\vec{H}; \ \left|\frac{d\vec{p}}{dt}\right| = \left|\frac{d\vec{p}}{ds}\right||\vec{v}| = e\left|\vec{E}\right|$$

$$pc \cong eER = 63.84 \ TeV \sim 6\cdot 10^{13} eV$$

Just short of 10-fold higher than LHC, but 300-fold lower that possible with magnets.

Matrices and matrix functions

- When somebody wants to build an accelerator, she or he should use some approximations
- One of VERY popular design approximation is called "an element (usually a magnet)" with nearly constant parameters
- Then our Hamiltonian is step-wise *s*-independent and the linear motion is easy:

$$\mathbf{H} = \mathbf{H}_i(s); \;\; \mathbf{H}_i(s) = const; \{s_i < s < s_{i+1}\}$$
$$\mathbf{M}(s_o, s) = \prod_{i=1} \mathbf{M}_i; \quad \mathbf{M}_i(s_i, s) = \exp(\mathbf{SH}_i(s - s_i)) \tag{187}$$

- E.g. we just need to learn how to calculate $\exp(\mathbf{SH}_i(s - s_i))$
- Finally, she or he than should try to build such elements. They never ideal but can be relatively close to the ideal boxes…

Accelerator Hamiltonian

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau; \quad (143)$$

with

$$\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_oc}B_y - \frac{e}{p_oc}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_oc}\right)^2\right] - \frac{e}{p_o\mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o\mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2;$$

$$\frac{G}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_oc}\right)^2\right] - \frac{e}{p_o\mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2; \quad (144)$$

$$\frac{2N}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial x} - \frac{e}{p_oc}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_oc}B_x - \frac{e}{p_o\mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o\mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right)$$

$$L = \kappa + \frac{e}{2p_oc}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3};$$

$$F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

If momentum p_o is constant, we can use (134) and rewrite Hamiltonian of the linearized motion (143) as

$$\tilde{h}_n = \frac{\pi_1{}^2 + \pi_3{}^2}{2} + f\frac{x^2}{2} + n\cdot xy + g\frac{y^2}{2} + L(x\pi_3 - y\pi_1) + \frac{\pi_o{}^2}{2}\cdot\frac{m^2c^2}{p_o{}^2} + u\frac{\tau^2}{2} + g_x x\pi_o + g_y y\pi_o + f_x x\tau + f_y y\tau; \quad (143\text{-n})$$

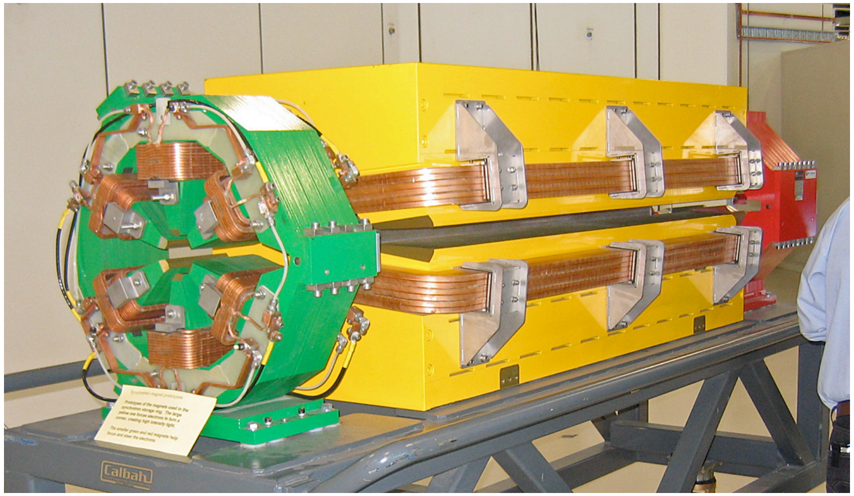
with

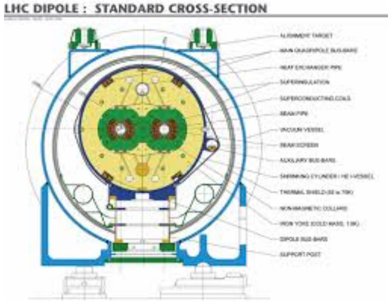
$$f = \frac{F}{p_o};\ n = \frac{N}{p_o};\ g = \frac{G}{p_o};\ u = \frac{U}{p_o};\ f_x = \frac{F_x}{p_o};\ f_y = \frac{F_y}{p_o}; \quad (144\text{-n})$$

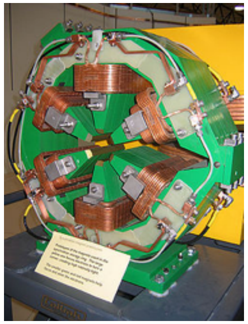
Note that

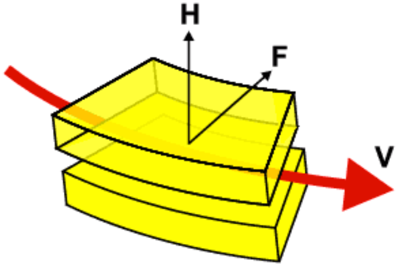
$$x' = \frac{\partial h_n}{\partial \pi_1} = \pi_1 - Ly; \quad y' = \frac{\partial h_n}{\partial \pi_3} = \pi_3 + Lx; \quad ; \quad (145)$$

Types of magnetic elements

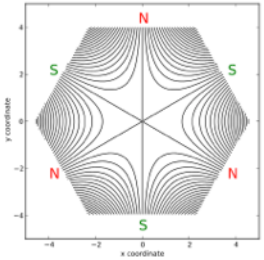




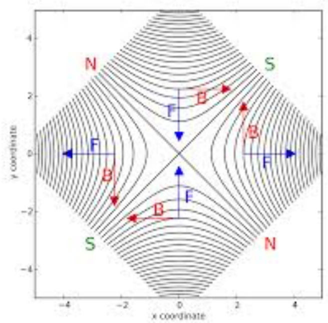


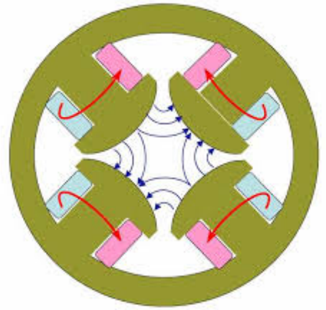


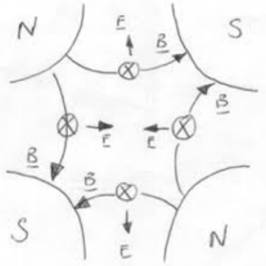
dipole



Typical elements of accelerators are dipoles and quadrupoles (or their combination), sextupoles and octupoles (they a nonlinear), solenoids, wigglers…. Let’s start from a linearized Hamiltonian (143) magnetic DC elements – this is typical accelerator beam-line.







$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L\left(xP_3 - yP_1\right) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + g_x x\delta \;; \qquad (188)$$

with

$$\frac{F}{p_o} = \left[\left(\frac{e}{p_oc}B_y\right)^2 - \frac{e}{p_oc}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_oc}\right)^2\right]; \frac{G}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_oc}\right)^2\right]$$

$$\frac{N}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial x}\right]; \;\; L = \kappa + \frac{e}{2p_oc}B_s; \qquad g_x = -K\frac{c}{\mathrm{v}_o}; \qquad (189)$$

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}; \frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y};$$

If momentum p_o is constant, we can use (134) and rewrite Hamiltonian of the linearized motion as

$$\tilde{h}_n = \frac{\pi_1^2 + \pi_3^2}{2} + f\frac{x^2}{2} + n\cdot xy + g\frac{y^2}{2} + L\left(x\pi_3 - y\pi_1\right) + \frac{\pi_o^2}{2}\cdot\frac{m^2c^2}{p_o^2} + g_x x\pi_o \;; \qquad (188\text{-n})$$

with

$$f = \frac{F}{p_o}; n = \frac{N}{p_o}; g = \frac{G}{p_o}; \;; \qquad (189\text{-n})$$

Focusing/defocusing in transverse direction can come from

(a) a dipole field B_y or in other words, form the curvature of trajectory. Note that it is always focusing.

(b) from quadrupole field $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$. Note that quadrupole is focusing in one direction and defocusing in the other.

(c) from solenoidal field, B_s. Note that it is always focusing.

The other terms, are responsible for coupling

(a) the transverse motion (x & y): solenoidal field, B_s and torsion κ as well as SQ-quadrupole $\frac{\partial B_x}{\partial x}$.

(b) or transverse and longitudinal motion: $g_x x \delta$ - it is responsible of dependence of the time of flight on transverse coordinate.

Finally, there is $\frac{\delta^2}{2p_o} \cdot \frac{m^2c^2}{p_o^2}$ term which is corresponds to the velocity dependence on the particle energy. It is frequently neglected at very high energies when $m^2c^2 / p_o^2 \approx \gamma^{-2} <<< 1$. But it should be kept for many accelerators, including RHIC.

We should not forget one of the most common element in any accelerator lattice – an empty space, call drift.

In standard accelerator physics book you will find solution (matrices) for various elements of the lattice: drift, bending magnet (with or with field gradient), quadrupole. Then, piecewise, you can see introduction of solenoids, SQ-quadrupoles…. Instead of solving dozen of second, fourth and sixth order differential equations… we will use matrix function approach to find all solutions at once.

Calculating matrices

Next, we focus on the question of how matrices are calculated. We already discussed general idea than they can be integrates piece-wise wherein the coefficients in the Hamiltonian expansion do not change significantly. In practice, accelerators are build from elements, which, to a certain extent, offers such conditions. The typical elements in high-energy accelerators contributing to the linear part of the equations are the drift (free space, vacuum), the dipoles with and without transverse gradient, the quadrupoles (both normal and SQ), and the RF cavities for acceleration and bunching. Except for the last, the typical elements are magnetic and DC (or varying very slowly compared with the passing or turn-around time for particles). Electric elements are rare, so simplifying the Hamiltonian to some degree:

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta; \quad (190)$$

$$\frac{F}{p_o} = \left[K^2 - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \frac{1}{2}\left(\frac{eB_s}{p_o c}\right)^2\right]\frac{G}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial y} + \frac{1}{2}\left(\frac{eB_s}{p_o c}\right)^2\right]; \quad \frac{2N}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right]; (191)$$

$$\frac{L}{p_o} = \kappa + \frac{e}{2p_o c}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = -K\frac{c}{\mathrm{v}_o};$$

In general, many elements of accelerators are designed to keep constant the coefficients of Hamiltonian expansion, with exception of edges. Here we will consider simple rigid-edge elements, where the magnetic field ends abruptly (compared with the wavelength of betatron oscillations).

Hence, initially we will explore a general way of calculating matrices, and then consider few examples. When the matrices **D** are piece-wise constant and the **D** from different elements do not commute, we can write

$$\mathbf{M}(s_o|s) = \prod_i \mathbf{M}(s_{i-1}|s_i); \mathbf{M}(s_{i-1}|s) = \prod_{elements} \exp[\mathbf{D_i}(s - s_{i-1})] \tag{193}$$

The definition of the matrix exponent is very simple

$$\exp[\mathbf{A}] = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{A}^k}{k!}; \quad \exp[\mathbf{D} \cdot s] = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{D}^k s^k}{k!} \tag{194}$$

According to the general theorem of Hamilton-Kelly, the matrix is a root of its characteristic equation:

$$d(\lambda) = \det[\mathbf{D} - \lambda I]; \quad d(\lambda_k) = 0 \tag{195}$$

$$d(\mathbf{D}) \equiv 0 \tag{196}$$

i.e., a root of a polynomial of order ≤ 2n. There is a theorem in theory of polynomials (rather easy to prove) that any polynomial $p_1(x)$ of power n can be expressed via any polynomial $p_2(x)$ of power m<n as

$$p_1(x) = p_2(x) \cdot d(x) + r(x)$$

where r(x) is a polynomial of power less than m. Accordingly, series (194) can be always truncated to

$$\exp[\mathbf{D}] = I + \sum_{k=1}^{2n-1} c_k \mathbf{D}^k, \tag{197}$$

with the remaining daunting task of finding coefficients c_k! There are two ways of doing this; one is a general, and the other is case specific, but an easy one.

Starting from a specific case when the matrix **D** is nilpotent (m<2n+1), i.e.,

$$\mathbf{D}^m = 0.$$

In this case, $D^{m+j} = 0$ the truncation is trivial:

$$\exp[D] = I + \sum_{k=1}^{m-1} \frac{D^k}{k!}. \tag{198}$$

We lucky to have such a beautiful case in hand – a drift, where all fields are zero and K=0 and κ=0:

$$\tilde{h} = \frac{\pi_1{}^2 + \pi_3{}^2}{2} + \frac{\pi_\delta{}^2}{2} \cdot \frac{m^2c^2}{p_o{}^2}; \quad \mathbf{D} = \begin{vmatrix} D_1 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{vmatrix}; D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; D_2 = \begin{bmatrix} 0 & \frac{m^2c^2}{p_o{}^2} \\ 0 & 0 \end{bmatrix}; \tag{199}$$

where it is easy to check: $D^2 = 0$. Hence, the 6x6 matrix of drift with length l will be

$$\mathbf{M}_{drift} = \exp[\mathbf{D} \cdot l] = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{D}^k l^k}{k!} = \mathbf{I} + \mathbf{D} \cdot l = \begin{vmatrix} M_t & 0 & 0 \\ 0 & M_t & 0 \\ 0 & 0 & M_\tau \end{vmatrix}; M_t = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}; M_\tau = \begin{bmatrix} 1 & l/(\beta_o\gamma_o)^2 \\ 0 & 1 \end{bmatrix}; \text{(200}$$

The general evaluation of the matrix exponent in (193) is straightforward using the eigen values of the D-matrix:

$$\det[\mathbf{D} - \lambda \cdot \mathbf{I}] = \det[\mathbf{SH} - \lambda \cdot \mathbf{I}] = 0 \qquad (201)$$

When the eigen values are all different (2n numerically different eigen values, $\lambda_i = \lambda_i \Rightarrow i = j$, no degeneration, i.e., D can be diagonalized),

$$\mathbf{D} = \mathbf{U}\Lambda\mathbf{U}^{-1};\ \Lambda = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \dots & 0 \\ 0 & 0 & 0 & \lambda_{2n} \end{pmatrix}; \qquad (202)$$

we can use Sylvester's formula that is correct for any analytical f(D), http://en.wikipedia.org/wiki/Sylvester's_formula for evaluating (193):

$$\exp[\mathbf{D}s] = \sum_{k=1}^{2n} e^{\lambda_k s} \prod_{j \neq k} \frac{\mathbf{D} - \lambda_j \mathbf{I}}{\lambda_k - \lambda_j} \qquad (203)$$

Let's prove this very useful formula. First, let consider a polynomial function

$$f_N(x) = \sum_{k=0}^{N} a_k x^k \tag{204}$$

and apply it to (202)

$$f_N(D) = \sum_{k=0}^{N} a_k D^k = \sum_{k=0}^{N} a_k \left(\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}\right)^k = \mathbf{U}\left\{\sum_{k=0}^{N} a_k \boldsymbol{\Lambda}^k\right\}\mathbf{U}^{-1} = \mathbf{U}\cdot f_N(\boldsymbol{\Lambda})\cdot\mathbf{U}^{-1}$$
$$f_N(\boldsymbol{\Lambda}) \equiv \begin{bmatrix} \ldots & 0 & 0 \\ 0 & f_N(\lambda_i) & 0 \\ 0 & 0 & \ldots \end{bmatrix} \tag{205}$$

e.g. function of diagonalizable matrix is a similarity transformation of the diagonal matrix with function of it eigen values. Goin to infinite series, we get

$$\exp(D) = \sum_{k=0}^{\infty} \frac{D^n}{k!} = \mathbf{U}\sum_{k=0}^{\infty} a_k (\Lambda)^k \mathbf{U}^{-1} = \mathbf{U}\exp(\Lambda)\mathbf{U}^{-1}$$
$$\exp(\Lambda) \equiv \begin{bmatrix} \ldots & 0 & 0 \\ 0 & e^{\lambda_i} & 0 \\ 0 & 0 & \ldots \end{bmatrix} \tag{206}$$

Now we start using our refresher on linear algebra. Each eigen value of diagonalizable matrix corresponds to an eigen vector

$$D \cdot Y_i = \lambda_i Y_i \,. \tag{207}$$

(existence comes from statement that $\left(D-\lambda_i I\right)Y_i = 0$ has non-trivial solution if $\det\left(D-\lambda_i I\right)=0$). The set of eigen vectors is a full set of vectors, e.g. any arbitrary vector can be expanded as

$$X = \sum_i \alpha_i Y_i \,. \tag{208}$$

This eigen vectors are columns of the matrix used for similarity transform to its diagonal form:

$$\mathbf{U} = \left[Y_1, Y_2, ..., Y_{2n}\right] \tag{209}$$

which is trivial to prove using (208(and (209) and comparing it with (202)

$$\begin{aligned} &\mathbf{DU} = \mathbf{U}\Lambda; \quad \rightarrow \mathbf{D} = \mathbf{U}\Lambda\mathbf{U}^{-1} \\ &\mathbf{U}\Lambda \equiv \left[\lambda_1 Y_1, \lambda_2 Y_2 ..., \lambda_{2n} Y_{2n}\right] \end{aligned} \tag{210}$$

Now, let build a unit projection operator on Y_k:

$$P_k = \prod_{i \neq k} \frac{M - \lambda_i I}{\lambda_k - \lambda_i} \tag{211}$$

It is easy to show that

$$P_k Y_k = Y_k; \quad P_k Y_{i \neq k} = 0; \tag{212}$$

First, each of the elements of the product (211) is unit on Y_k

$$\frac{M - \lambda_i I}{\lambda_k - \lambda_i} \cdot Y_k = \frac{\lambda_k - \lambda_i}{\lambda_k - \lambda_i} Y_k = 1; i \neq k \tag{213}$$

while there is a zero-operator for all other eigen vectors:

$$\frac{M - \lambda_i I}{\lambda_k - \lambda_i} \cdot Y_i = \frac{\lambda_i - \lambda_i}{\lambda_k - \lambda_i} \cdot Y_i = 0 \tag{214}$$

Now we write

$$P_k\mathbf{U} = [\ldots 0, Y_k, 0\ldots] \tag{215}$$

and

$$\begin{aligned} f(D) &= \mathbf{U}\cdot f(\Lambda)\cdot\mathbf{U}^{-1} \\ \mathbf{U}\cdot f(\Lambda) &= \sum_{k=1}^{2n} f(\lambda_k)[\ldots 0, Y_k, 0\ldots] = \sum_{k=1}^{2n} f(\lambda_k)\cdot P_k\cdot\mathbf{U} \end{aligned} \tag{216}$$

and finally

$$\begin{aligned} &= \mathbf{U}\cdot f(\Lambda)\cdot\mathbf{U}^{-1} \\ f(D) = \mathbf{U}\cdot f(\Lambda)\cdot\mathbf{U}^{-1} &== \sum_{k=1}^{2n} f(\lambda_k)\cdot P_k\cdot\mathbf{U}\cdot\mathbf{U}^{-1} = \sum_{k=1}^{2n} f(\lambda_k)\cdot P_k \end{aligned} \tag{217}$$

e.g.

$$f[\mathbf{D}] = \sum_{k=1}^{2n} f(\lambda_k)\prod_{j\neq k}\frac{\mathbf{D}-\lambda_j\mathbf{I}}{\lambda_k-\lambda_j} \tag{218}$$

equivalent to

$$f[\mathbf{D}s] = \sum_{k=1}^{2n} f(\lambda_k s)\prod_{j\neq k}\frac{\mathbf{D}-\lambda_j\mathbf{I}}{\lambda_k-\lambda_j} \tag{219}$$

we got famous Sylvester formula.

We will us mostly $f:\exp$ and Sylvester formula in form of (203). Naturally, (219) is comprised of power of matrix **D** up to 2n-1 – perfectly with agreement that **D** is a root its characteristic equation (196).

Since **D** is real matrix, any of its complex eigen values paired with their complex conjugates:

$$\mathbf{D}Y_k = \lambda_k Y_k \iff \mathbf{D}Y_k^* = \lambda_k^* Y_k^* \quad (220)$$

meanwhile real eigen values not always related. One more important ratio for accelerators: trace of **D** is equal zero, e.g. sum of it eigen values is also equal zere:

$$Trace[\mathbf{D}] = Trace\left[\mathbf{U}\Lambda\mathbf{U}^{-1}\right] = Trace\left[\mathbf{U}^{-1}\mathbf{U}\Lambda\right] = Trace[\Lambda] = \sum_{k=1}^{2n}\lambda_k \quad (221)$$

It is especially useful for n=1 – you will see it in your home work.

$$\exp[\mathbf{D}s] = \sum_{k=1}^{2n} e^{\lambda_k s} \prod_{j\neq k} \frac{\mathbf{D} - \lambda_j \mathbf{I}}{\lambda_k - \lambda_j}$$

$n = 1$

$$\exp[\mathbf{D}s] = \sum_{k=1, j\neq k}^{2} e^{\lambda_k s} \frac{\mathbf{D} - \lambda_j \mathbf{I}}{\lambda_k - \lambda_j} = e^{\lambda_1 s} \frac{\mathbf{D} - \lambda_2 \mathbf{I}}{\lambda_1 - \lambda_2} + e^{\lambda_2 s} \frac{\mathbf{D} - \lambda_1 \mathbf{I}}{\lambda_2 - \lambda_1}$$