PHY 554. Homework 1.

Handed: August 27 Return by: September 10
Bring solution to class or -better- email solutions to vladimir.litvinenko@stonybrook.edu

HW 1.1 (3 points): Find available energy (so called C.M. energy) for a head-on collision in these scenarios:

(a) In CEBAF, polarized 12 GeV electrons collide with protons at rest;
(b) In one of scenarios for future collides, CERN consider collide 50 TeV protons with 180 GeV electrons;

The rest energy of proton is 938.257 MeV, and rest energy of electron is 0.511 MeV.

Solution.

First, the remined of the center of mass energy available for creating new particle:

$$p^\mu = \left(p_o = \frac{E}{c}, \vec{p}\right);\ p^\mu p_\mu = p_o^2 - \vec{p}^{\,2} = p_{cm}^2 = \left(M_{cm}c\right)^2;$$

$$E = E_1 + E_2 = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2;\ \vec{p} = \vec{p}_1 + \vec{p}_2 = \gamma_1 m_1 \vec{v}_1 + \gamma_2 m_2 \vec{v}_2$$

$$\left(M_{cm}c^2\right)^2 = \left(E_1 + E_2\right)^2 - \left(\vec{p}_1 + \vec{p}_2\right)^2 c^2 =$$

$$\left(E_1^2 - \vec{p}_1^{\,2} c^2\right) + \left(E_2^2 - \vec{p}_2^{\,2} c^2\right) + 2\left(E_1 E_2 - 2\vec{p}_1 \vec{p} c^2\right)$$

and using $E_{1,2}^2 - \vec{p}_{1,2}^{\,2} c^2 = \left(m_{1,2}c^2\right)^2$

we get

$$\left(M_{cm}c^2\right)^2 = \left(m_1 c^2\right)^2 + \left(m_2 c^2\right)^2 + 2\left(\gamma_1 m_1 c^2\right)\left(\gamma_2 m_2 c^2\right)\left(1 - \vec{\beta}_1 \vec{\beta}_2\right);\ \vec{\beta}_{1,2} = \frac{\vec{v}_{1,2}}{c}$$

$$M_{cm}c^2 = \sqrt{\left(m_1 c^2\right)^2 + \left(m_2 c^2\right)^2 + 2E_1 E_2\left(1 - \vec{\beta}_1 \vec{\beta}_2\right)}$$

The available c.m. energy depends not only on particles energy but dimensionless term $\vec{\beta}_1 \vec{\beta}_2$. The product is equal to zero is one of the particles is at rest or particles collide at 90 degrees and available energy is maximized for head-on collisions when

$$\vec{\beta}_1 \vec{\beta}_2 = -\left|\vec{\beta}_1\right|\left|\vec{\beta}_2\right| = -\beta_1 \beta_2 \Rightarrow M_{cm}c^2 = \sqrt{\left(m_1 c^2\right)^2 + \left(m_2 c^2\right)^2 + 2E_1 E_2\left(1 + \beta_1 \beta_2\right)};$$

(a) Electrons are moving with velocity close to speed of light, but protons are at rest (β_2=0) and $E_2 = m_2 c^2$ and we get

$$M_{cm}c^2 = \sqrt{\left(m_1 c^2\right)^2 + \left(m_2 c^2\right)^2 + 2E_1 m_2 c^2} =$$

$$\sqrt{(0.000511)^2 + (2 \cdot 12 + 0.938257) \cdot 0.938257} = 4.837\ GeV$$

If we use an approximate formular neglecting the c.m. of the proton, we would miss ~ 90 keV:

$$M_{cm}c^2 \cong \sqrt{2E_1 m_2 c^2} = 4.745\ GeV$$

(b) Beams are colliding head-on, i.e. we need to use

$$M_{cm}c^2 = \sqrt{\left(m_1c^2\right)^2 + \left(m_2c^2\right)^2 + 2E_1E_2\left(1+\beta_1\beta_2\right)}$$

Both beams are ultra-relativistic with γ_e=3.52x10^5 and γ_p=5.33x10^4. Off cause, we can calculate β for both beams and find that β_e=1-4.03 x10^{-12} and β_e=1-1.76 x10^{-10} to get

$$M_{cm}c^2 = \sqrt{\left(m_1c^2\right)^2 + \left(m_2c^2\right)^2 + 2E_1E_2\left(1+\beta_1\beta_2\right)} = 6.0000000731\ TeV$$

But in this case we really can neglect first two terms and replace βs by 1s:

$$M_{cm}c^2 \cong 2\sqrt{E_1E_2} = 6\ TeV$$

with very tiny 1.2x10^{-8} relative difference from exact answer of 73 keV.

HW 1.2 (2 points): Let's consider a futuristic case that humanity decided – and figured out how - to build a storage ring surrounding the equator, i.e. with circumference of 40,075 km.

(a) 1 point: If bending dipole magnets fill 75% of the ring circumference, and we used dipole magnetic field of 8.3 T, demonstrated in LHC, what would be energy of the circulating protons?

(b) 1 point: What magnetic field would be needed if storage ring with the same energy of the ring should fit at Earth largest continent of Asia. Asia could fit a ring with radius of 2,500 km. Assume 75% filling factor.

P.S. USA can fit ring with radius of 1.465 km.

Solution:

(a) Circumference of 40.075 km means that average radius is R=6.378 km and ff=75% of the trajectory has radius of ρ=4,783.6 km. With B_o=8.3 T it would correspond to $B\rho$=39.704 T km and pc=11.90 PeV (1 PeV = 10^{15} eV)– not to shabby at all. Beam would ultra-relativistic and E= pc=11.90 PeV with very high accuracy/

(b) To have the same energy, i.e. the same momentum and beam rigidity $B\rho$=39.704 T km. With the same filling factor ρ scale proportionally to the average radius of the storage ring:

$$B_{req} = \frac{B\rho}{\rho_{asia}} = \frac{B_o \cdot \rho_{earth}}{\rho_{asia}} = \frac{B_o \cdot ff \cdot R_{earth}}{ff \cdot R_{asia}} = B_o \frac{R_{earth}}{R_{asia}} = 21.18\ T$$

It would require field exceeding 36 T to fit it in the USA, or operate it at 2.7 PeV suing 8.3T magnetic field.

HW 1.3 (2 points): For a classical microtron with orbit factor k=1 and energy gain per pass of 1.022 MeV and operational RF frequency 3 GHz (3 x 10^9 Hz) find required magnetic field. What will be radius of first orbit in this microtron?

Hint: Note that rest energy of electron with γ=1 is 0.511 MeV. This is energy gain per pass will define available n numbers in eq. (2.6)

Solution: The main equations for the microtrons are derived from following equations for beam going on nth pass through the cavity:

$$E_n = mc^2 + n \cdot e \cdot V_{rf} \Rightarrow \gamma_n = \frac{E_n}{mc^2} = 1 + n \cdot \frac{e \cdot V_{rf}}{mc^2};\ \beta_n = \frac{v_n}{c} = \sqrt{1 - \gamma_n^{-2}};$$

$$p_n c = \beta_n \cdot E_n = e \cdot B \cdot \rho_n;\ \rho_n = \frac{\beta_n \cdot E_n}{e \cdot B}\ ;\ T_n = \frac{2\pi \cdot \rho_n}{v_n} = \frac{2\pi \cdot E_n}{e \cdot B} \cdot \frac{\beta_n}{v_n} = \frac{2\pi \cdot E_n}{e \cdot B \cdot c} = \frac{2\pi \cdot mc}{e \cdot B} \cdot \left(1 + n \cdot \frac{e \cdot V_{rf}}{mc^2}\right).$$

where T_n is the time required for particle turn around n^{th} path. The resonant condition for microtron is that each pass takes integer number of RF cycles:

$$T_n = \frac{2\pi \cdot mc}{e \cdot B}\left(1 + n \cdot \frac{e \cdot V_{rf}}{mc^2}\right) = N_n \cdot T_{RF} = N_n \cdot \frac{1}{f_{RF}};$$

$$N_n = \frac{2\pi \cdot mc \cdot f_{RF}}{e \cdot B}\left(1 + n \cdot \frac{e \cdot V_{rf}}{mc^2}\right) = \frac{2\pi \cdot mc^2}{e \cdot B \cdot \lambda_{RF}} \cdot \left(1 + n \cdot \frac{e \cdot V_{rf}}{mc^2}\right);\ \lambda_{RF} = \frac{c}{f_{RF}};$$

$$N_n = A \cdot C_n;\ A = \frac{2\pi \cdot mc^2}{e \cdot B \cdot \lambda_{RF}};\ C_n = 1 + n \cdot \frac{e \cdot V_{rf}}{mc^2}.$$

For product of two numbers A and C_n to be integers for all n=1,2,…. it is required that A is an integer and that

$$A \cdot \frac{e \cdot V_{rf}}{mc^2}$$

is also integer. In our case, $eV_{rf}/mc^2=2$ and second condition is satisfied if A is an integer, which determining

$$B = \frac{2\pi}{A} \cdot \frac{mc^2}{e \cdot \lambda_{RF}}$$

We are selecting A=1 to get

$$B = \frac{2\pi \cdot mc^2}{e \cdot \lambda_{RF}};\ 2\pi \cdot \frac{mc^2}{e} = 2\pi \cdot \frac{0.511[MeV]}{0.29979} = 10.71[kGs\ cm];$$

$$\lambda_{RF} = \frac{c}{f_{RF}} = \frac{2.9979 \cdot 10^{10}\left[\frac{cm}{sec}\right]}{3 \cdot 10^9\left[\frac{1}{sec}\right]} = 9.993\ cm;\ B = 1.071\ kGs\ (0.1071\ T)$$

At first orbit (n=1), the beam energy is E=3 mc^2 =1.533 MeV and particle rigidity

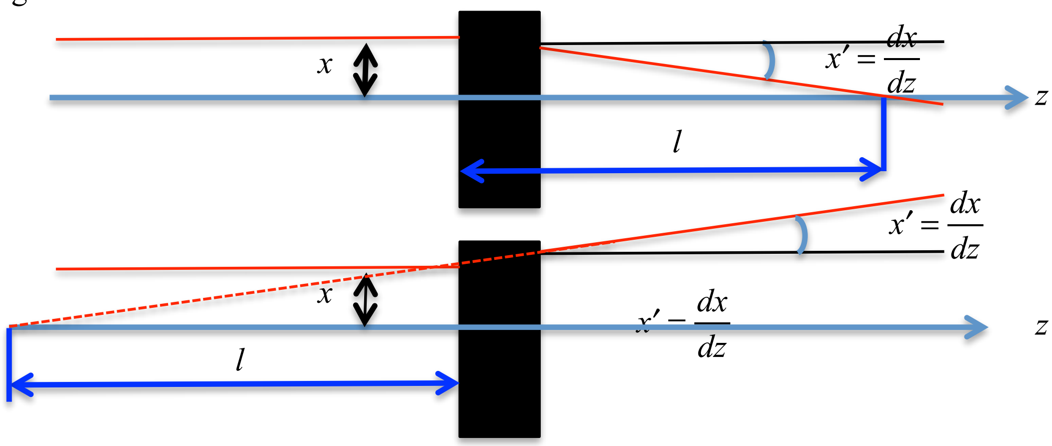
$$pc = \sqrt{E^2 - (mc^2)^2} = 1.445\ MeV; B\rho[kGs\ cm] = \frac{pc[MeV]}{0.29979} = 4.821\ kGs\ cm$$

$$B = 1.071\ kGs;\quad \rho = \frac{B\rho}{B} = 4.498\ cm$$

HW 1.4 (8 point): Let's first determine an effective focal length, F, of a paraxial (e.g. small angles!) focusing object (a black-box) as ratio between a parallel displacement of trajectory at its entrance to corresponding change of the angle at its exit (see figure below):

$$F = -\frac{x}{x'}; x' \equiv \frac{dx}{dz}$$

see figure below for



Let consider a triplet of thin lenses: horizontal focusing (F) and defocusing (D) lenses with centers are separated by distance L as in Fig. 1. Since quadrupole focusing is opposite for horizontal and vertical directions, the lattice - in accelerator lingo it is magnetic structure – looks like (a) in horizontal direction and (b) in vertical direction. Let's assume that that in Fig 1(a) Q_1 are focusing lenses in horizontal plane with focal length of $\mathbf{F=+1/q_1}$ and, naturally, they are defocusing in vertical direction, Fig 1 (b), with focal length of $\mathbf{F=-1/q_1}$. Similarly, Q_2 is defocusing in horizontal plane (Fig 1(a)) with focal length of $\mathbf{F=-1/q_2}$ and is focusing in vertical plane with focal length of $\mathbf{F=+1/q_2}$.

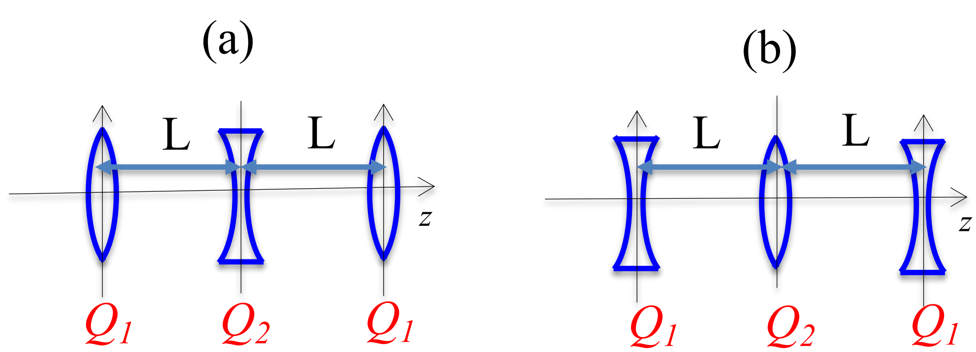
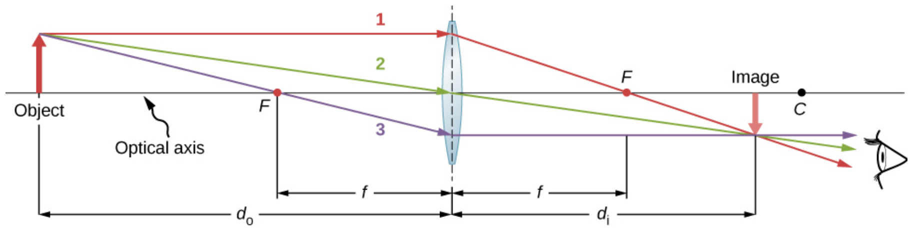


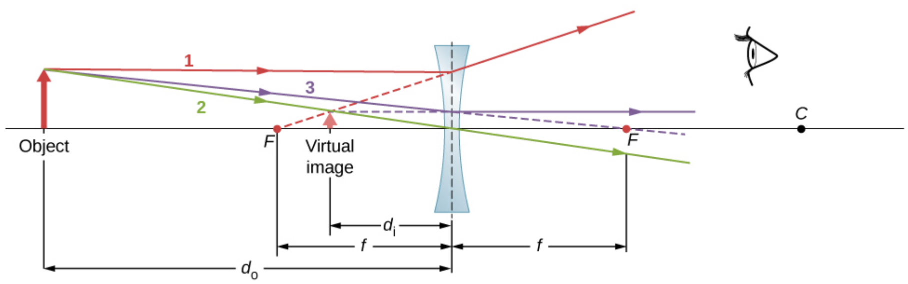
Fig.1. Triplet of short quadrupole lenses: (a) in horizontal direction FDF and (b) in vertical direction DFD.

1. (5 points) Find of horizonal (*FDF)* and vertical (*DFD)* focal lengths of the triplet.
2. (3 points) Find ratio between $\mathbf{q_1}$ and $\mathbf{q_2}$ when horizontal and vertical focal lengths are equal, specifically in a form of $\mathbf{q_2}$=f($\mathbf{q_1}$,$\mathbf{L}$).

P.S. Definition (picture) of thin lens:



(a)



(b)

Solution. Instead of doing repetitive calculations of effect on x and x' by lenses and drifts (space between lenses) we will write it in compact, matrix form.
The thin lens with focal length F=1/q does not change position but changes angle if trajectory:

$$x_{out} = x_{in},\ x'_{out} = x'_{in} - \frac{x_{in}}{f} = x'_{in} - q \cdot x_{in} \Rightarrow \begin{bmatrix} x_{out} \\ x'_{out} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{in} \\ x'_{in} \end{bmatrix}$$

Sign of q determines is this focusing (q>0) or defocusing (q<0) length.
The drift of length L does not changes angle of trajectory but changes position:

$$x_{out} = x_{in} + L \cdot x'_{in},\ x'_{out} = x'_{in} \Rightarrow \begin{bmatrix} x_{out} \\ x'_{out} \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{in} \\ x'_{in} \end{bmatrix}$$

Hence the triplet will result in multiplying 5 matrices – 3 of thin lenses and two drift between them. For horizontal case (a) we have

$$\begin{bmatrix} x_{out} \\ x'_{out} \end{bmatrix} = M_a \cdot \begin{bmatrix} x_{in} \\ x'_{in} \end{bmatrix} \quad M_a = \begin{bmatrix} 1 & 0 \\ -q_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ q_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -q_1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 - L\left(2q_1 - q_2\right) - L^2 q_1 q_2 & L\left(2 + Lq_2\right) \\ -\left(1 - Lq_1\right)\left(q_1\left(2 + Lq_2\right) - q_2\right) & 1 - L\left(2q_1 - q_2\right) - L^2 q_1 q_2 \end{bmatrix}$$

For vertical case (b) we just need to reverse signs of q's:

$$\begin{bmatrix} x_{out} \\ x'_{out} \end{bmatrix} = M_b \cdot \begin{bmatrix} x_{in} \\ x'_{in} \end{bmatrix} \quad M_b = \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -q_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 + L\left(2q_1 - q_2\right) - L^2 q_1 q_2 & L\left(2 - Lq_2\right) \\ \left(1 + Lq_1\right)\left(q_1\left(2 - Lq_2\right) - q_2\right) & 1 + L\left(2q_1 - q_2\right) - L^2 q_1 q_2 \end{bmatrix}$$

We define the focal length if the "black box" as distance from it exit of the system till it crossing the axis for initial trajectory without angle: $x'_{in}=0$:

$$\begin{bmatrix} x_{out} \\ x'_{out} \end{bmatrix}_a = M_a \cdot \begin{bmatrix} x_o \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - L(2q_1 - q_2) - L^2 q_1 q_2 \\ -(1 - Lq_1)(q_1(2 + Lq_2) - q_2) \end{bmatrix} \cdot x_o;$$

$$\begin{bmatrix} x_{out} \\ x'_{out} \end{bmatrix}_b = M_b \cdot \begin{bmatrix} x_o \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + L(2q_1 - q_2) - L^2 q_1 q_2 \\ (1 + Lq_1)(q_1(2 - Lq_2) - q_2) \end{bmatrix} \cdot x_o;$$

$$F_a = -\frac{(x_{out})_a}{(x'_{out})_a} = \frac{1 - L(2q_1 - q_2) - L^2 q_1 q_2}{(1 - Lq_1)(q_1(2 + Lq_2) - q_2)};$$

$$F_b = -\frac{(x_{out})_b}{(x'_{out})_b} = \frac{1 + L(2q_1 - q_2) - L^2 q_1 q_2}{-(1 + Lq_1)(q_1(2 - Lq_2) - q_2)}$$

It will be sufficient to write $F_a=F_b$ to get the full points, but if you go and solved it:

$$Solve\left[\frac{1 - L(2q_1 - q_2) - L^2 q_1 q_2}{(1 - Lq_1)(q_1(2 + Lq_2) - q_2)} = -\frac{1 + L(2q_1 - q_2) - L^2 q_1 q_2}{(1 + Lq_1)(q_1(2 - Lq_2) - q_2)}\right];$$

$$q_2 = \frac{1}{2L}\frac{1 - 3u \pm \sqrt{1 + 2u^2 - 15u^4 + 16u^6}}{u^3 - u}; \; u = Lq_1$$

I will double your points! It turns out that solution with minus sign in front ot the square root is focusing when $0<u<0.708$ and defocusing at $u>0.708$. At $u=0.708$, the focal spot is located at the exit of the triplet, i.e. parallel beam merges in a point at its exit. Solution with positive sign is defocusing at $0<u<1$ and focusing at $u>0$.