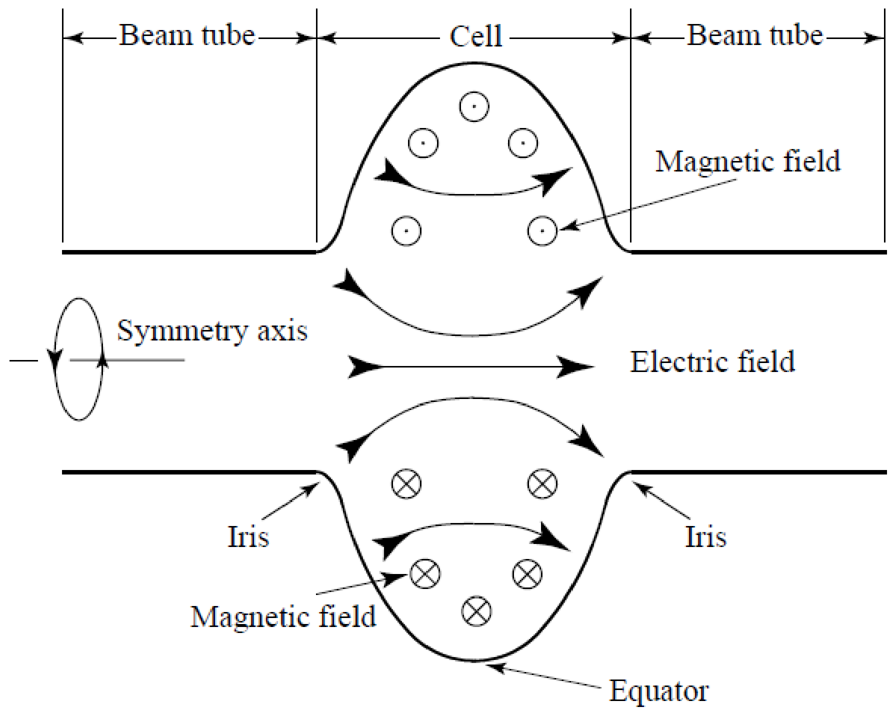
# PHY 554

# Fundamentals of Accelerator Physics

## Lecture 9: Introduction to RF accelerators

September 28, 2016

Vladimir N. Litvinenko




http://case.physics.stonybrook.edu/index.php/PHY554_fall_2016

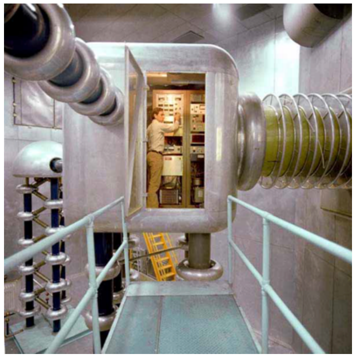
# Acknowledgement

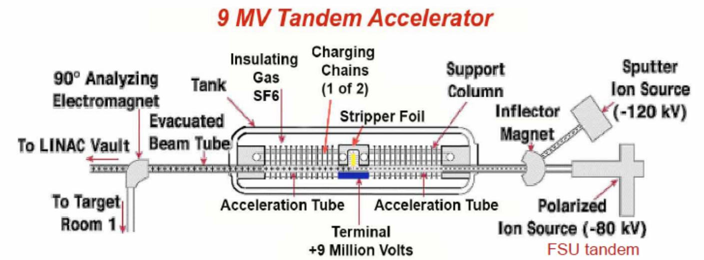
Next three lectures use some materials from courses on RF and Superconducting RF (SRF) accelerators taught by Prof. S. Belomestnykh at SBU/BNL and USPAS, which can be found on the following websites:

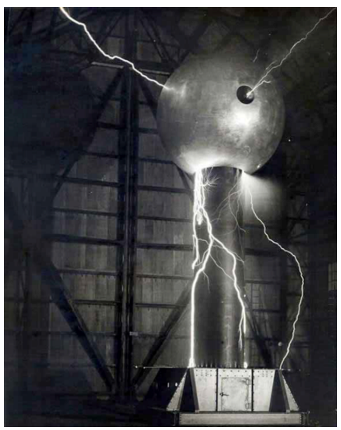
*http://case.physics.stonybrook.edu/index.php/Courses:_P554_Fundamentals_of_Accelerator_Physics,_Spring_2014*
*http://case.physics.stonybrook.edu/index.php/CASE:Courses*
*https://sites.google.com/site/srfsbu11/*
*http://uspas.fnal.gov/materials/materials-table.shtml*

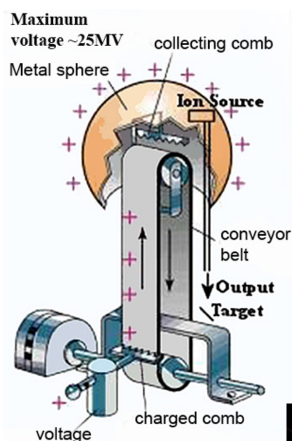
Plus some EM material of my own

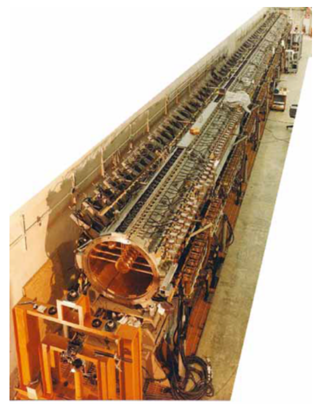
# Linear accelerators: from electrostatic to RF

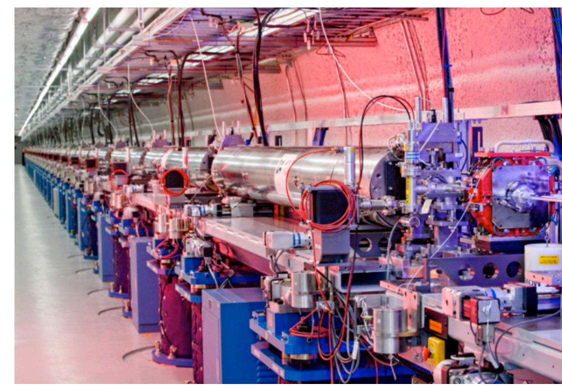


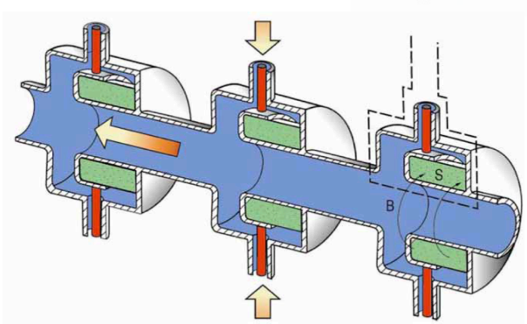


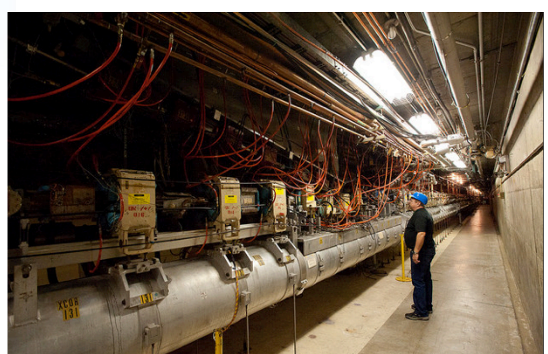









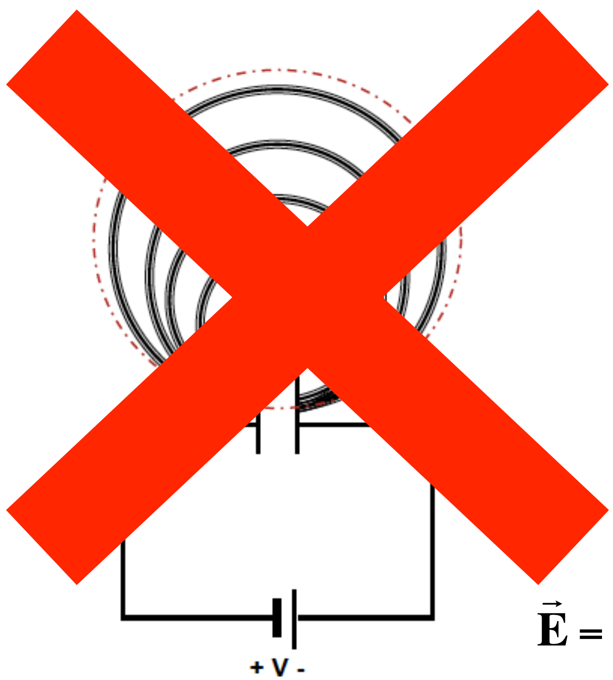


Can one gain the energy again and again by passing through a DC accelerating gap?

# Electrostatic: what is the limit ?

Maxwell equations and energy conservation law!

$$\Delta E = e\oint \vec{\mathbf{E}}\cdot d\vec{l} = -\frac{e}{c}\frac{\partial}{\partial t}\left(\int \cancel{\vec{\mathbf{H}}\cdot d\vec{s}}\right)$$



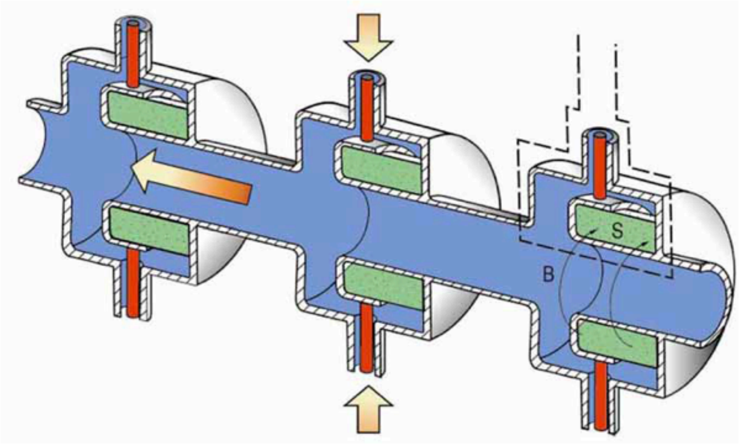

DC

$$\Delta E = e\oint \vec{\mathbf{E}}\cdot d\vec{l} = 0$$

$$\vec{\mathbf{E}} = -\vec{\nabla}\varphi \quad \rightarrow \quad E(\vec{r}) = E(0) - e\varphi(\vec{r})$$

Can not cheat the Maxwell equations

# Induction linacs: linear betatrons

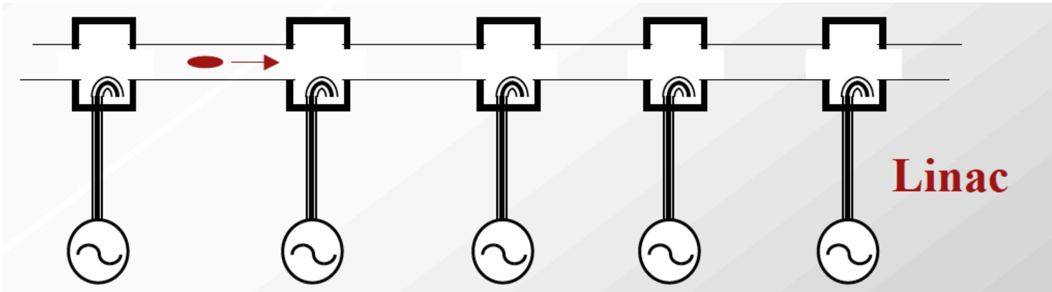



$$\oint \vec{\mathbf{E}} \cdot d\vec{l} = -\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}} \cdot d\vec{s}\right)$$

- Useful for high power and high current beams
- Have limited accelerating field
- By nature are pulsed, with relatedly low rep-rate (kHz)

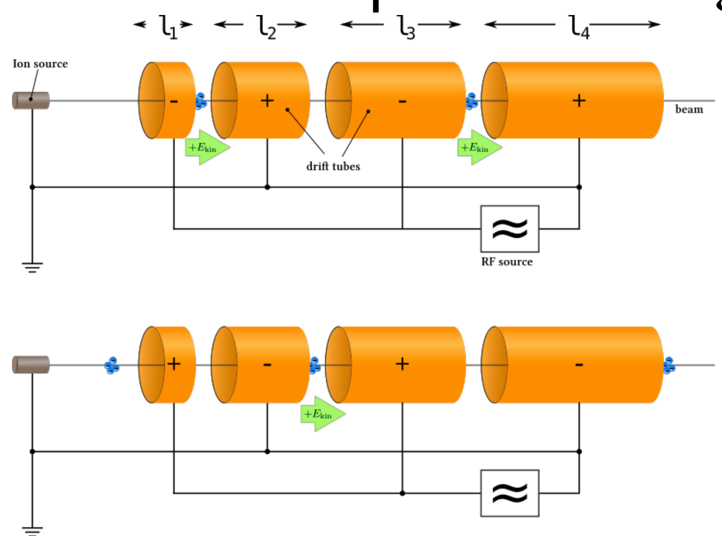
# How RF accelerator works

- It has oscillating (typically sinusoidal in time) longitudinal (along the particle's trajectory) eclectic field
- It also has longitudinal structure (cells) which alternates the direction of the filed
- When particle propagates through the RF accelerator, the field direction in each cell is synchronized with the particle arrival and the effect from all cells is added coherently

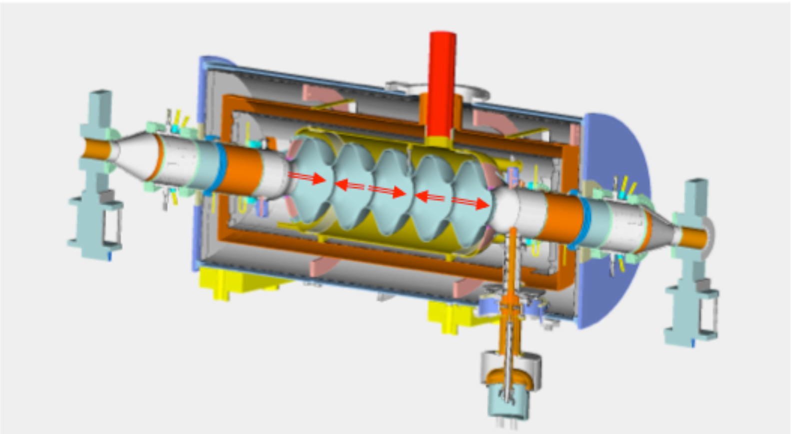



$$\frac{dE}{dt} = e\vec{\mathbf{E}} \cdot \vec{\mathrm{v}} \quad \rightarrow \mathrm{sign}(\vec{\mathbf{E}} \cdot \vec{\mathrm{v}}) = \mathrm{const}$$

Wideröe's linac: β=v/c is changing

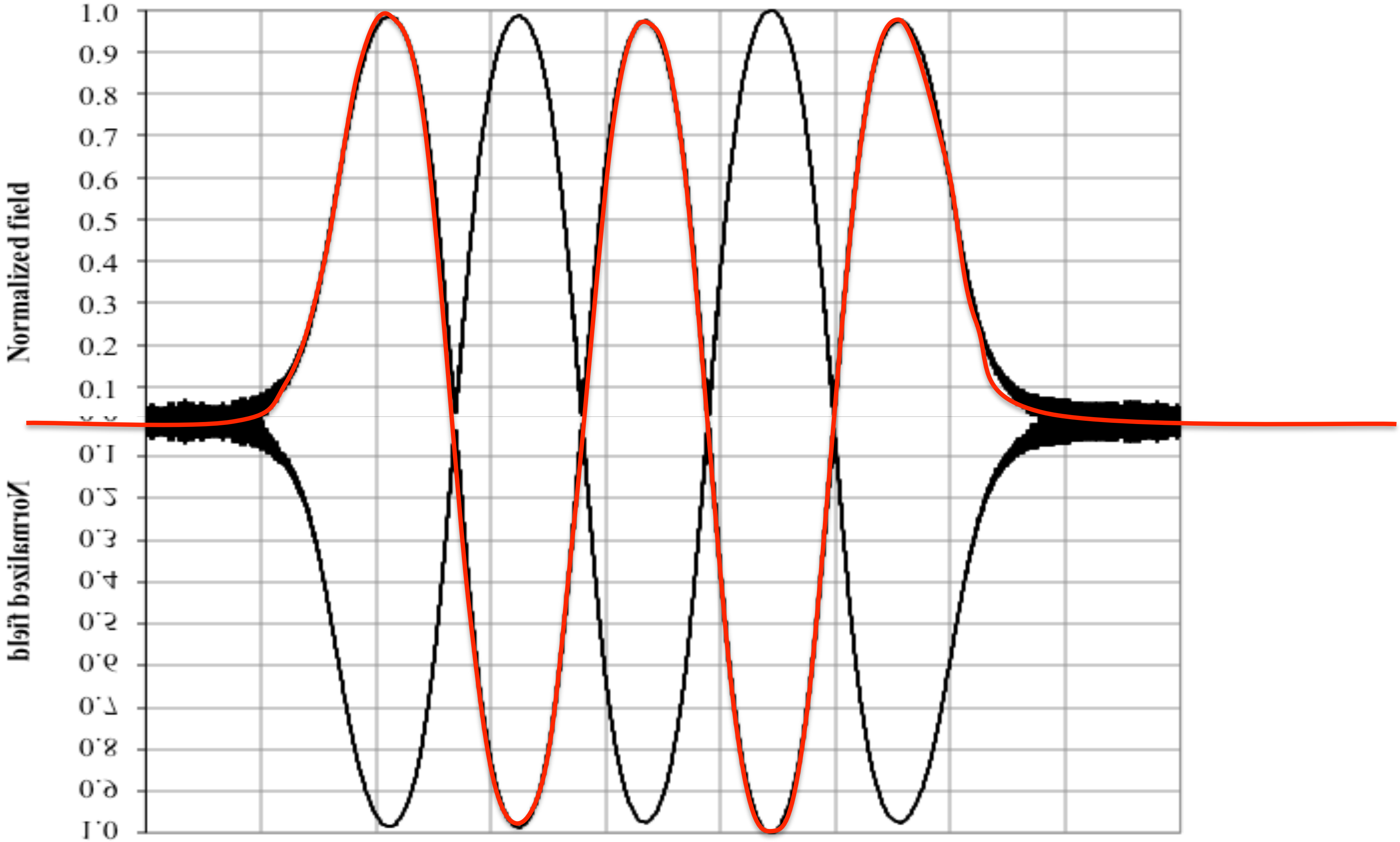



Electron linac

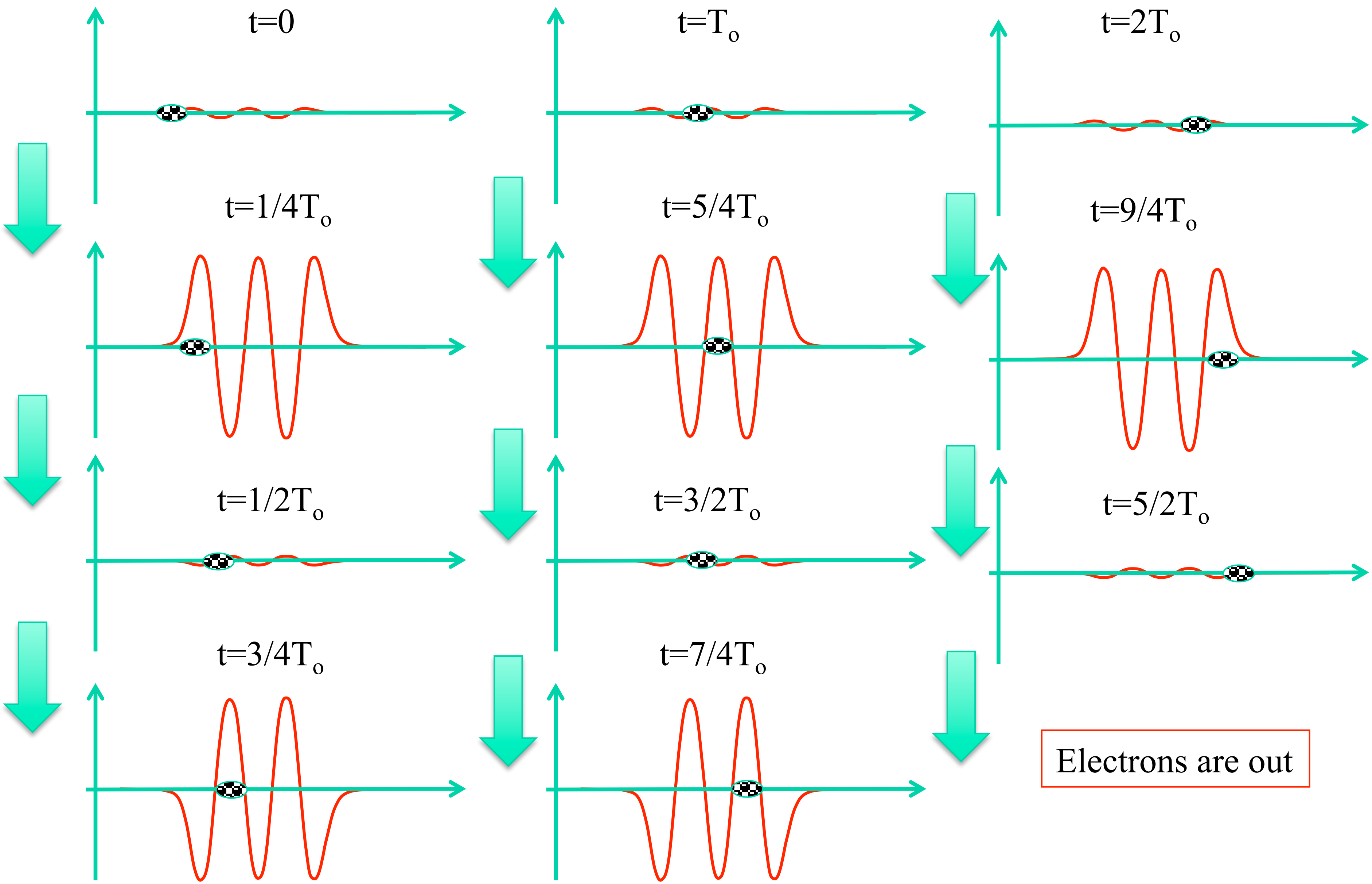


β=v/c ~1

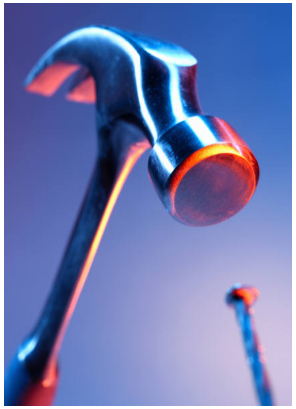
# Wave-form in 5-cell cavity



# How β=1 RF linac works?
# Example of 5-cell cavity



# Simple things to remember



- Acceleration in DC electrostatic is limited to the difference in terminal potential (e.g. voltage between the ground and the cathode)
- RF linear accelerators (RF linacs or simply linacs) are not limited in beam energy
- In RF linacs, the coherent addition/subtraction of the energy gain from cell to cell happens by design: period of the electric field oscillation is matched to the travel time of electron between the cells.
- Accurate synchronization of RF linac is important task for any linear accelerator

# A bit of EM and conducting media

$$\vec{j} = \sigma \vec{E};$$

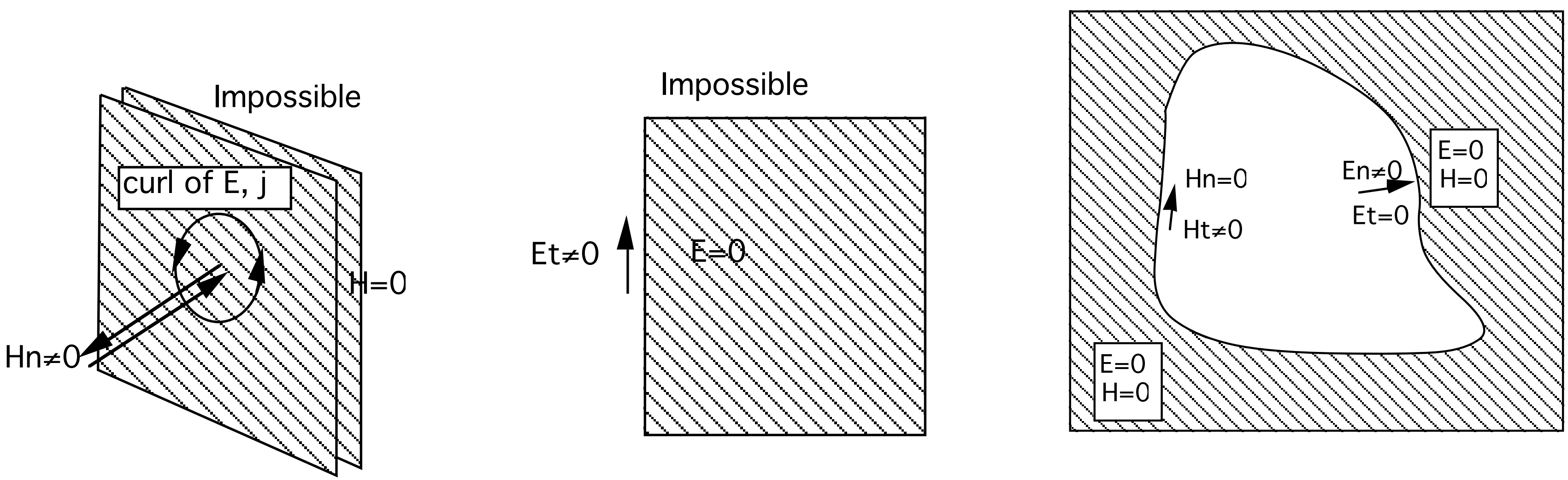
- Assuming oscillating field we can use Coulomb gauge for EM field

$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t)\right\}; \varphi = 0;$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = curl\vec{A}.$$

$$\left|\vec{H}\right| \propto \left|\frac{(\alpha + i\beta)}{k_o}\right|\left|\vec{E}\right| = \left|\sqrt{1 + \frac{4\pi i\sigma}{\omega}}\right|\left|\vec{E}\right|$$

$$\sigma \rightarrow \infty$$

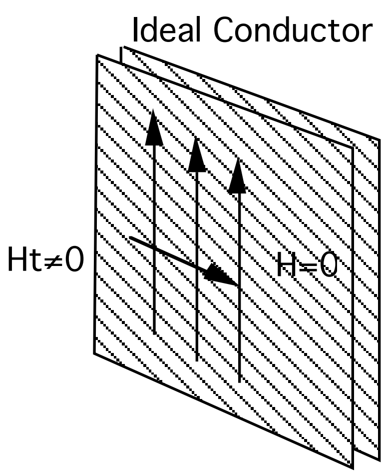


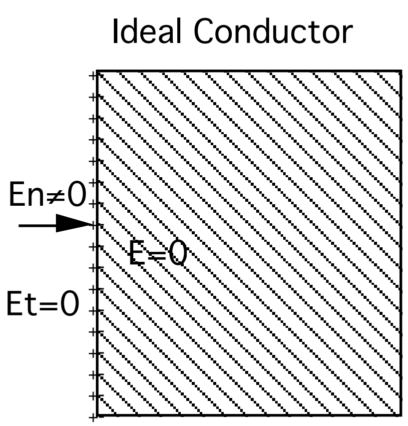
# Boundary conditions

- We are considering oscillating EM fields in RF structures
- RF structures are built from highly conducting material, both to contain EM filed inside and to provide low losses
- In first approximation we can consider an ideal boundary conditions and take finite conductivty as a perturbation later
- Q-factor: $Q_{\text{room temp}} \sim 10^4\text{-}10^5$, $Q_{SRF} \sim 10^9\text{-}10^{10}$

$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t - \alpha t)\right\};$$

$$\alpha = \frac{2\pi\omega}{Q}$$

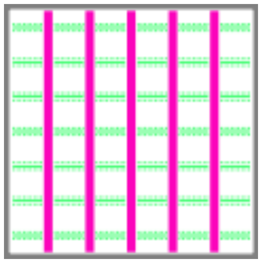


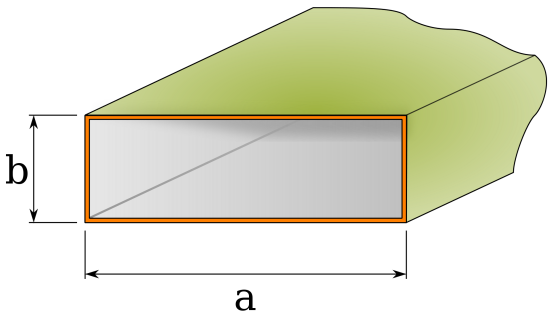



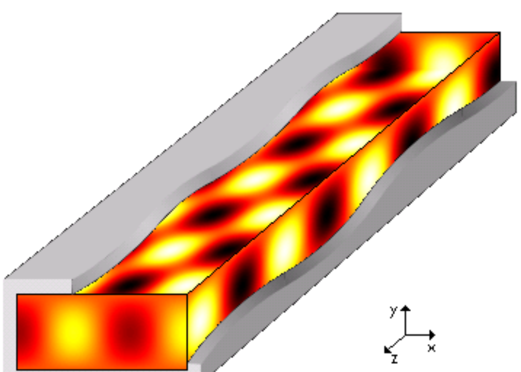

$$\vec{E} = \vec{n}\left(\vec{n}\vec{E}\right) + \vec{E}_{//}; \vec{B} = \vec{n}\left(\vec{n}\vec{B}\right) + \vec{B}_{//};$$

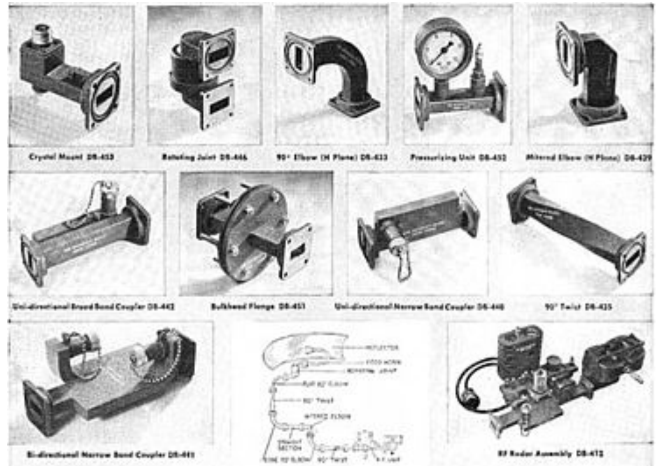
# Waveguides

## Rectangular







$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}=0;\quad \Delta\equiv\vec{\nabla}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2};$$

$$\vec{A}=\mathrm{Re}\left\{\vec{A}(\vec{r}_\perp)\exp\left(i\left(\omega t-k_z z\right)\right)\right\};$$

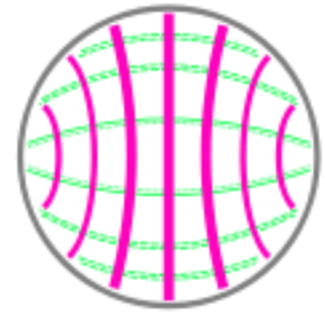
$$\vec{\nabla}_\perp^2\vec{A}+\left(k_o^2-k_z^2\right)\vec{A}=0;\quad k_o=\frac{\omega}{c}.$$

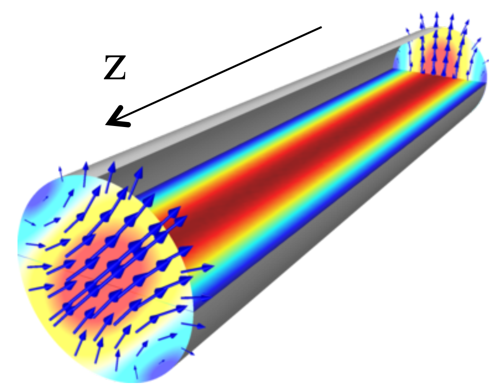
$$\vec{\nabla}_\perp^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

*At the surfaces*

$$\vec{n}\times\vec{E}\Big|_s=0;\ \vec{n}\cdot\vec{B}=0\rightarrow E_z\Big|_s=0;\ \frac{\partial B_z}{\partial n}\Big|_s=0$$

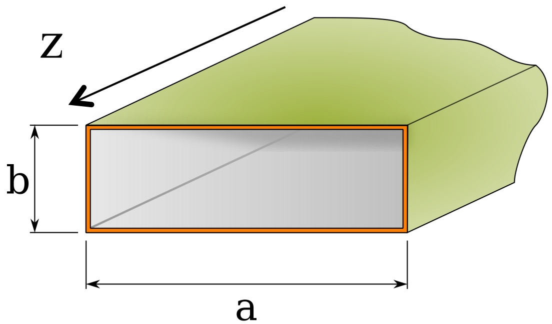
## Circular





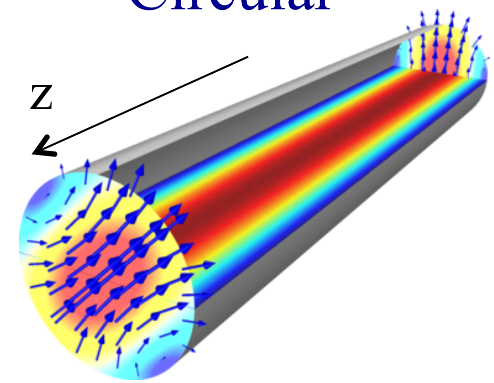
# TE and TM waves

Rectangular




- There is simplification
  - The modes are divided into two types: TE (transverse electric) and TM (transverse magnetic)

Circular




$$\vec{E}=\vec{E}_z+\vec{E}_\perp;\vec{B}=\vec{B}_z+\vec{B}_\perp;\ \vec{A}_z\equiv\hat{z}A_z;$$

$$\vec{\nabla}\times\vec{E}=ik_o\vec{B};\ \vec{\nabla}\times\vec{B}=-ik_o\vec{E};\quad\Rightarrow$$

$$ik_z\vec{E}_\perp+ik_o\left[\hat{z}\times\vec{B}_\perp\right]=\vec{\nabla}_\perp E_z;$$

$$ik_z\vec{B}_\perp-ik_o\left[\hat{z}\times\vec{E}_\perp\right]=\vec{\nabla}_\perp B_z;$$

*At the surfaces*

$$\vec{n}\times\vec{E}\Big|_s=0;\ \vec{n}\cdot\vec{B}=0\rightarrow E_z\Big|_s=0;\ \frac{\partial B_z}{\partial n}\Bigg|_s=0$$

$$TM: B_z\equiv 0;\ E_z\Big|_s=0;$$

$$TE: E_z\equiv 0;\ B_z\Big|_s=0;$$

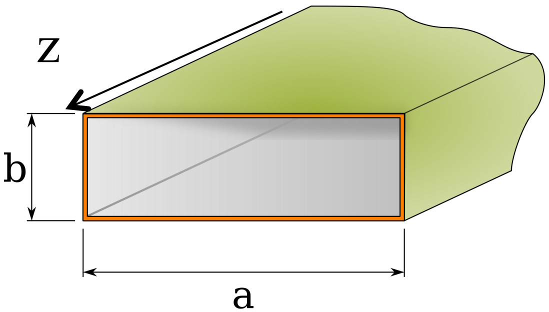
- Last two equations indicated that $E_z$ and $B_z$ fully determine transverse component of the EM field
- It means that we can always consider a linear combination of the fields with $E_z = 0$ *everywhere* (TE) and $B_z = 0$ *everywhere* (TM)
- Naturally, when we interested in accelerating particles, we will need TM mode with $E_z \neq 0$.

$\vec{B}_\perp=\pm\dfrac{k_z}{k_o}\left[\hat{z}\times\vec{E}_\perp\right]$ for both TE and TM modes

TM: $\vec{E}_\perp=\vec{\nabla}_\perp\psi_1(\vec{r}_\perp)$; TE: $\vec{B}_\perp=\vec{\nabla}_\perp\psi_2(\vec{r}_\perp)$;

# Cut-off frequency

## Rectangular

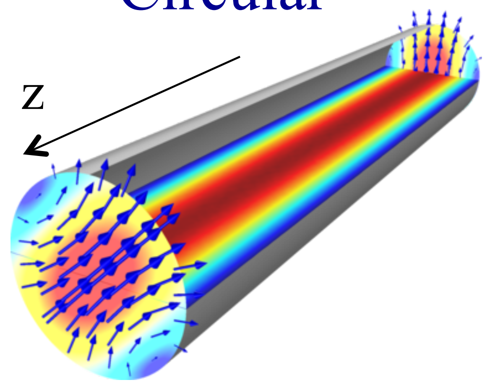



- EM field is a linear combination of modes with $E_z = 0$ *everywhere* (TE) and $B_z = 0$ *everywhere* (TM)

*At the surfaces*

$$\vec{n}\times\vec{E}\Big|_s = 0;\ \vec{n}\cdot\vec{B} = 0 \rightarrow E_z\Big|_s = 0;\ \frac{\partial B_z}{\partial n}\Big|_s = 0$$

## Circular




$\vec{B}_\perp = \pm\frac{k_z}{k_o}\left[\hat{z}\times\vec{E}_\perp\right]$ for both TE and TM modes

$TM: B_z \equiv 0;\ E_z\big|_s = 0;$

$TE: E_z \equiv 0;\ B_z\big|_s = 0;$

TM: $\vec{E}_\perp = \vec{\nabla}_\perp\psi_1(\vec{r}_\perp)$; TE: $\vec{B}_\perp = \vec{\nabla}_\perp\psi_2(\vec{r}_\perp)$;

$$TM: \psi\big|_s = 0;\ TE: \frac{\partial\psi}{\partial n}\Big|_s = 0.$$

$\vec{\nabla}_\perp^2\psi + \left(k_o^2 - k_z^2\right)\psi = 0$ + *boundary conditions*

*Different boundary conditions for TE and TM modes*
*In general case we need to find eigen function (modes)*

$$\boxed{\begin{gathered}\vec{\nabla}_\perp^2\psi_\lambda + \gamma_\lambda{}^2\psi_\lambda = 0;\\ \lambda = 1,2,3,\ldots\end{gathered}}$$

**Cut-off frequency**

$$k_{z,\lambda}^2 = k_o^2 - \gamma_\lambda^2 > 0$$

$$k_{o\,\min} = \gamma_\lambda \rightarrow \omega_{cut-off} = c\gamma_\lambda$$

Below cut-off $\omega < \omega_{cut-off}$

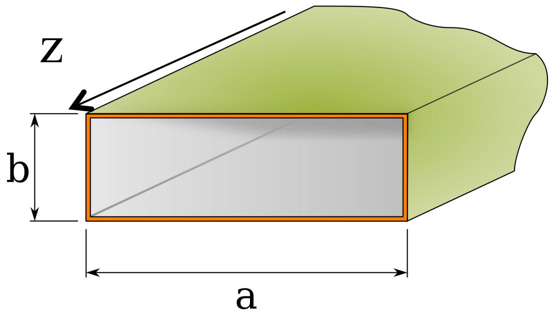
evanescent wave: $k_z = \pm i\sqrt{\omega^2{}_{cut-off} - \omega^2} = \pm i\kappa_z$

Exp decay

$$\psi = \psi_0 e^{\pm\kappa_z z}$$

# Cut-off frequency

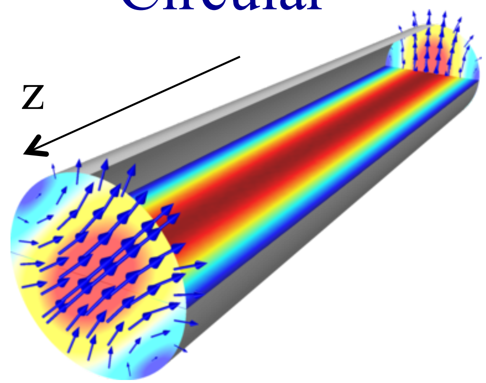
## Rectangular




*Different boundary conditions for TE and TM modes*

$$TM: \psi\big|_s = 0; \;\; TE: \frac{\partial \psi}{\partial n}\bigg|_s = 0.$$

## Circular




$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + \gamma_{mn}^2 \psi = 0$$

$$TE: \psi^{TE}{}_{mn} = \psi_o \cos k_m x \cos k_n x; \;\; m+n \geq 1;$$

$$TM: \psi^{TM}{}_{mn} = \psi_o \sin k_m x \sin k_n x; \;\; m \geq 1; n \geq 1;$$

$$k_m = \pi \frac{m}{a}; k_m = \pi \frac{n}{b}; \gamma_{mn} = k_m{}^2 + k_n{}^2.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \gamma_{mn}^2 \psi = 0$$

$$\psi_{mn} = \varphi_{mn}(r) e^{in\theta} \Rightarrow r \frac{\partial}{\partial r}\left(r \frac{\partial \psi_m}{\partial r}\right) + \left(r^2 \gamma_{mn}^2 - n\right) = 0$$

$$\varphi_{mn} = J_n(\gamma_{mn} r)$$

## Lowes cut-off frequency

### Rectangular

$$TE: a > b; m = 1; n = 0; \;\; \omega_{cut-off} = \frac{\pi c}{a};$$

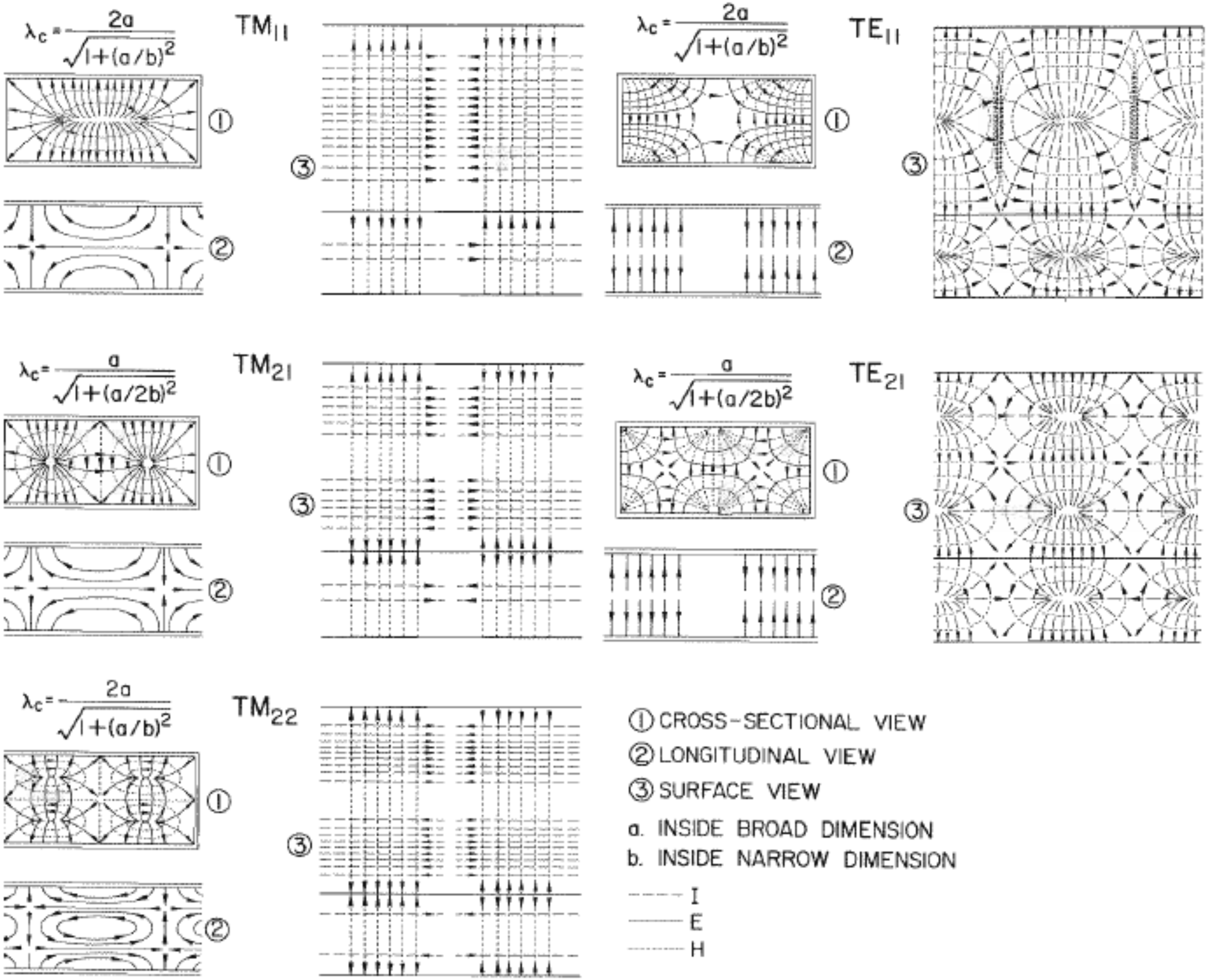
$$TM: \; m = 1; n = 1; \;\; \omega_{cut-off} = \frac{\pi c}{a}\sqrt{1 + \frac{a^2}{b^2}}.$$

### Circular

$$TM: J_o(\gamma_{01} R) = 0 \rightarrow \gamma_{01} \cong \frac{2.40483..}{R}; \;\; \omega_{cut-off} \cong \frac{2.40\, c}{R};$$

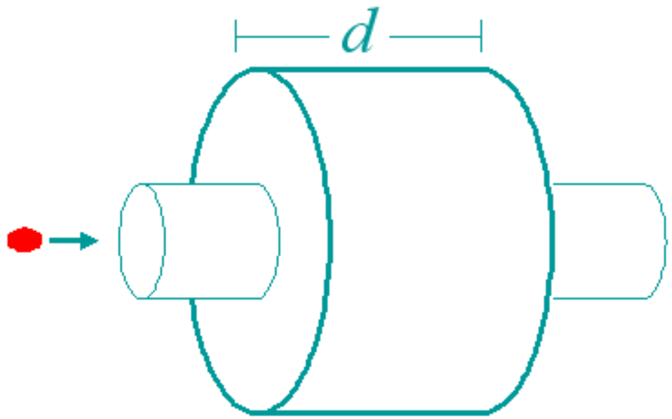
$$TE: J_1(\gamma_{11} R) = 0 \rightarrow \gamma_{11} \cong \frac{1.84118....}{R}; \;\; \omega_{cut-off} \cong \frac{1.84\, c}{R}.$$

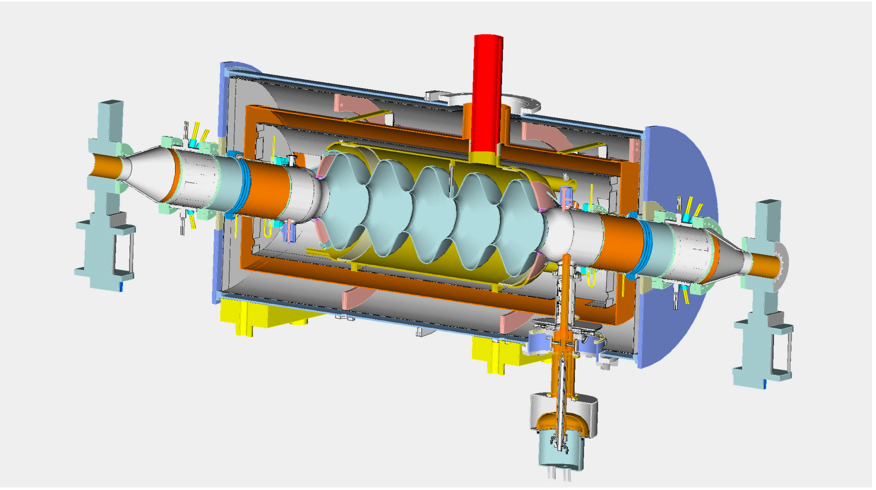
# Modes in rectangular waveguide

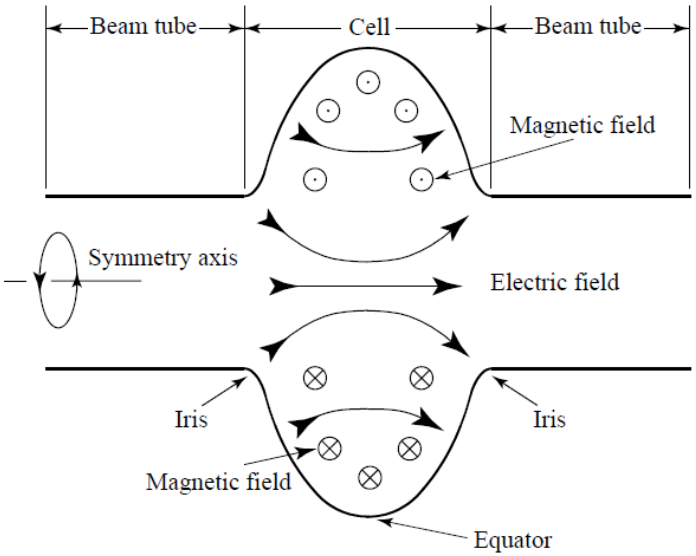


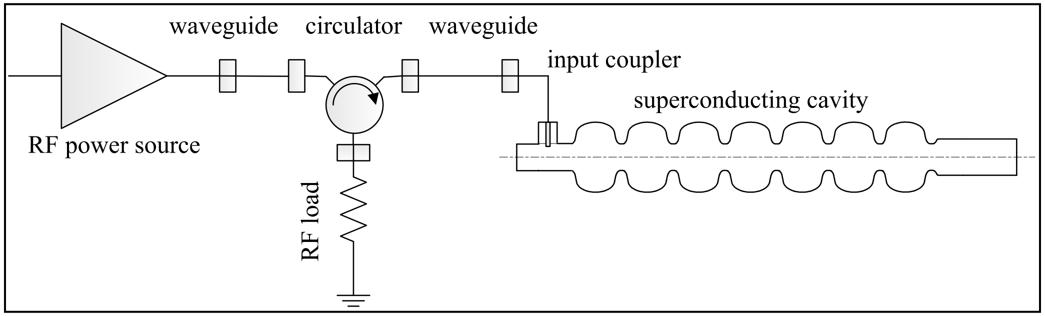
# RF cavities

are designed to confine the EM field inside: It means that they operate at frequency below cut-off of the beam-pipes attached to them





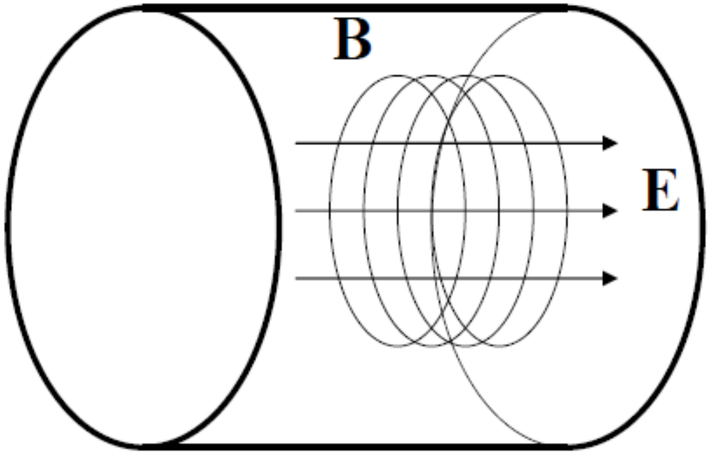
# RF Cavity Modes:
## the lowest accelerating is $TM_{010}$ mode

- Fields in the cavity are solutions of the equation $$\left(\nabla^2 - \frac{1}{c}\frac{\partial^2}{\partial t}\right)\begin{Bmatrix}\mathbf{E}\\\mathbf{H}\end{Bmatrix} = 0$$
- Subject to the boundary conditions $\hat{n}\times\mathbf{E}=0,\ \hat{n}\cdot\mathbf{H}=0$
- Two extra surfaces (z=0 and z=d): but this is no problem for TM mode
- An infinite number of solutions (eigen modes) belong to two families of modes with different field structure and eigen frequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency $TM_{010}$ mode is used.
- For the pillbox cavity w/o beam tubes
- Note that frequency does not depend of the cavity length! But only its radius.

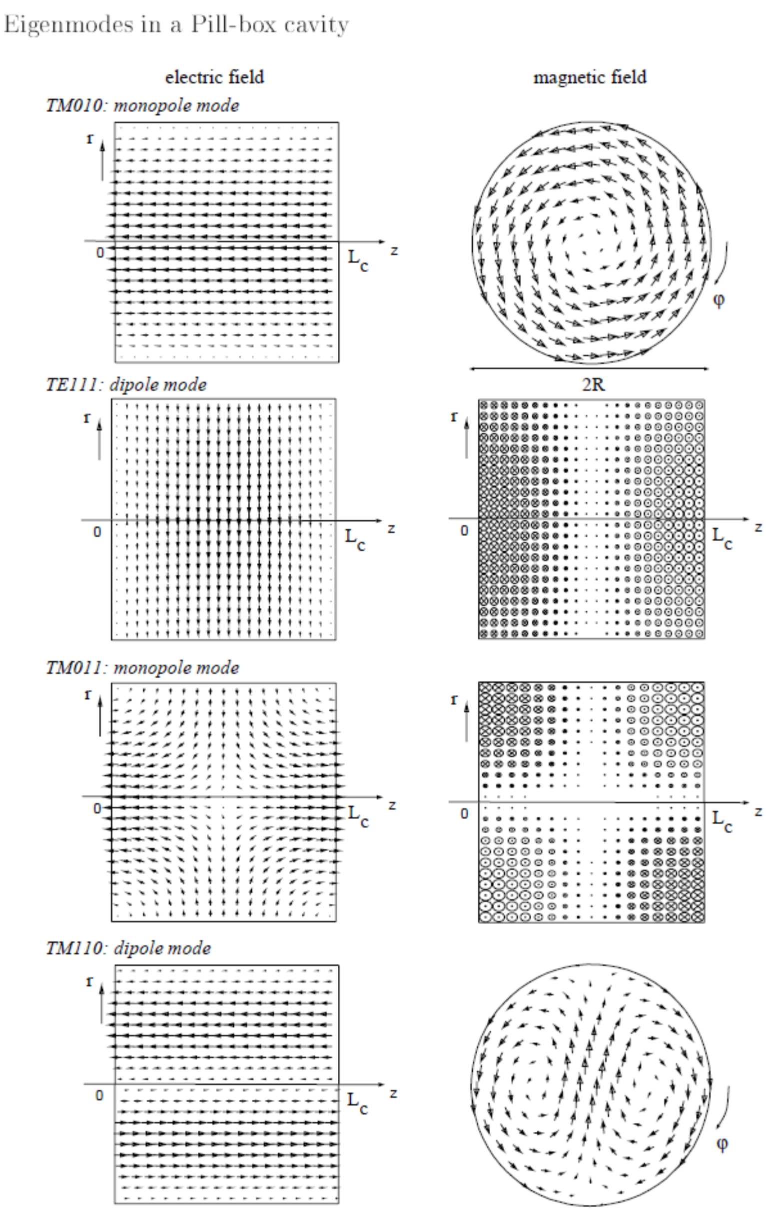
$$E_z = E_0 J_0\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

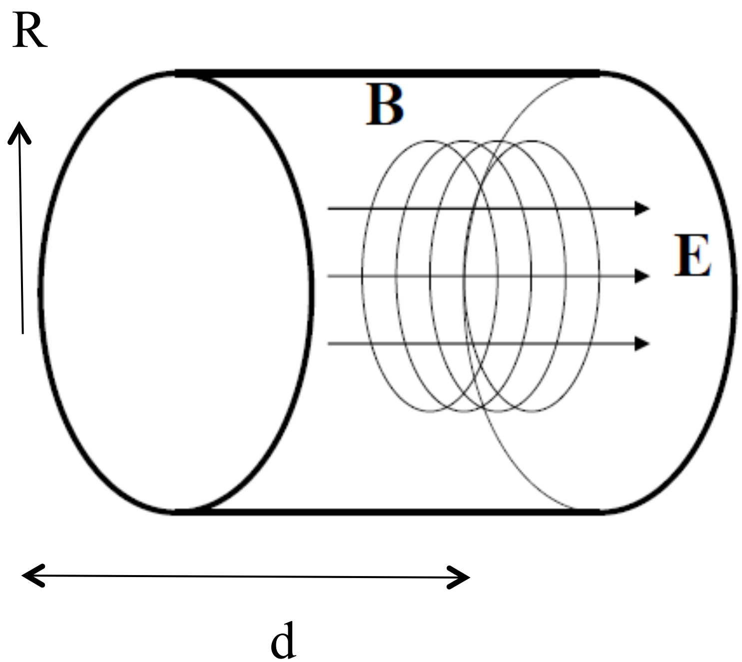
$$H_\phi = -iE_0 J_1\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

$$\omega_{010} = \frac{2.405c}{R},\quad \lambda_{010} = 2.61R$$



# Fundamental and high order modes (HOMs)



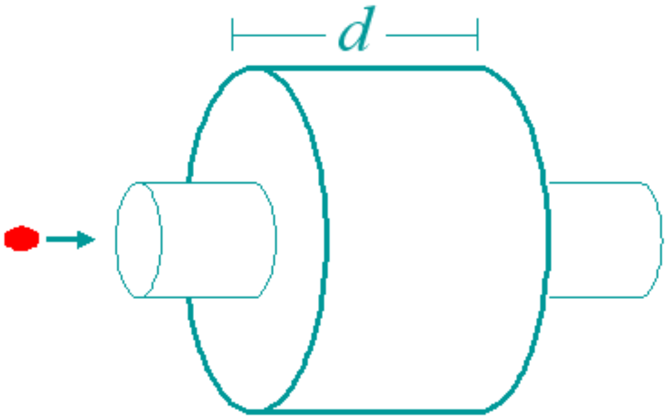

$$TM:\ \varphi_{mnl} = J_n(\gamma_{mn} r)\cos k_{z,l} z;\ \ k_{z,l} = l\frac{\pi}{d};\ \ J_n(\gamma_{mn} R) = 0;$$

$$\omega_{res} = c\sqrt{\gamma_{mn}^2 + l^2\frac{\pi^2}{d^2}};\ l = 0,1,2....$$

$$TE:\ \varphi_{mnl} = J_n(\kappa_{mn} r)\sin k_{z,l} z;\ \ k_{z,l} = l\frac{\pi}{d};\ \ J_n'(\kappa_{mn} R) = 0;$$

$$\omega_{res} = c\sqrt{\kappa_{mn}^2 + l^2\frac{\pi^2}{d^2}};\ l = 1,2....$$

# Accelerating voltage & transit time




- Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

$$V_c = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i\omega_0 z/\beta c} dz \right|$$

For the pillbox cavity one can integrate this analytically:

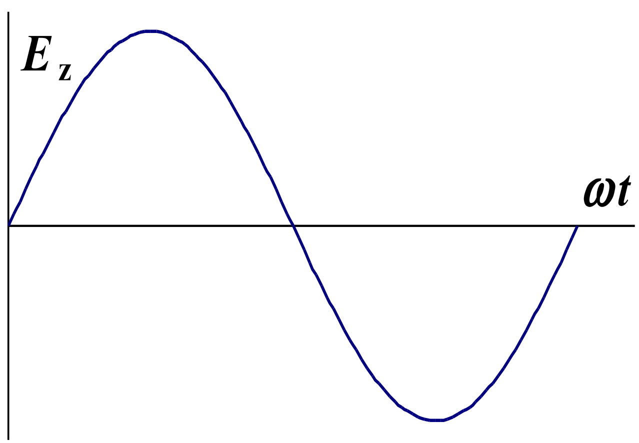
$$V_c = E_0 \left| \int_0^d e^{i\omega_0 z/\beta c} dz \right| = E_0 d \, \frac{\sin\left(\frac{\omega_0 d}{2\beta c}\right)}{\frac{\omega_0 d}{2\beta c}} = E_0 d \cdot T$$

where $T$ is the transit time factor.

- To get maximum acceleration:

$$T_{transit} = t_{exit} - t_{enter} = \frac{T_0}{2} \Rightarrow d = \beta\lambda/2 \Rightarrow \quad V_c = \frac{2}{\pi} E_0 d$$

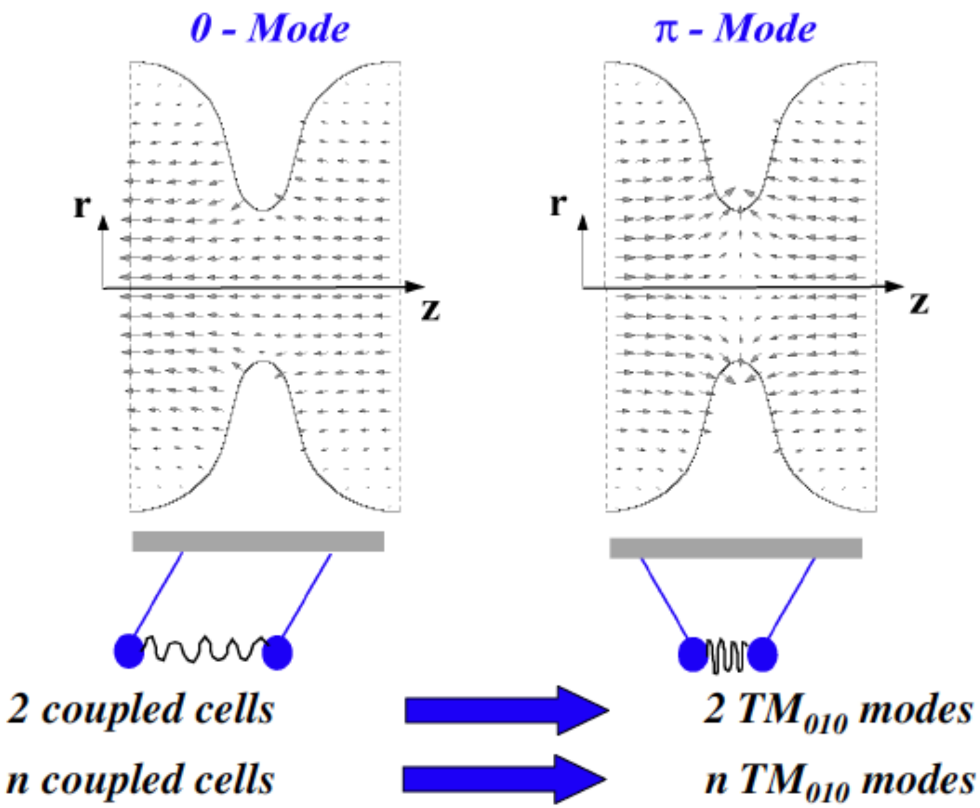
Thus for the pillbox cavity $T = 2/p$ .




- The accelerating field $E_{acc}$ is defined as $E_{acc} = V_c/d$ . Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be $d = bl/2$ . This works OK for multi-cell cavities, but poorly for single-cell ones.

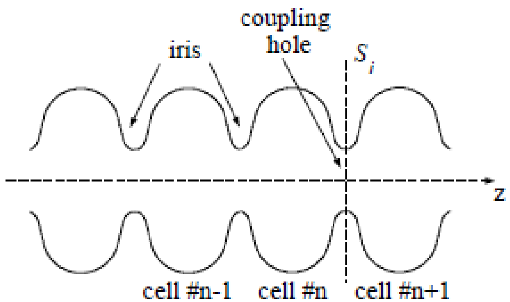
# Multicell cavities: coupled oscillators

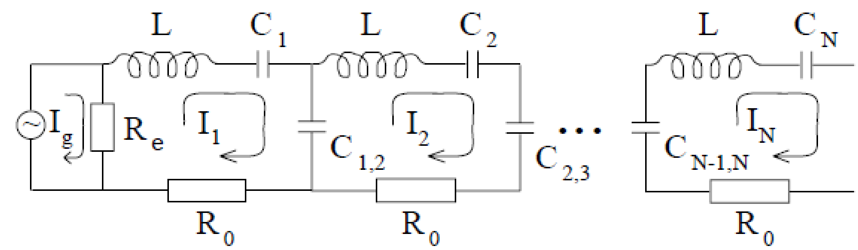
- Several cells can be connected together to form a multicell cavity.
- Coupling of $TM_{010}$ modes of the individual cells via the iris (primarily electric field) causes them to split:



# Multicell cavities (2)

- The split mode forms a passband of closely spaced modes equal in number to the number of cells.

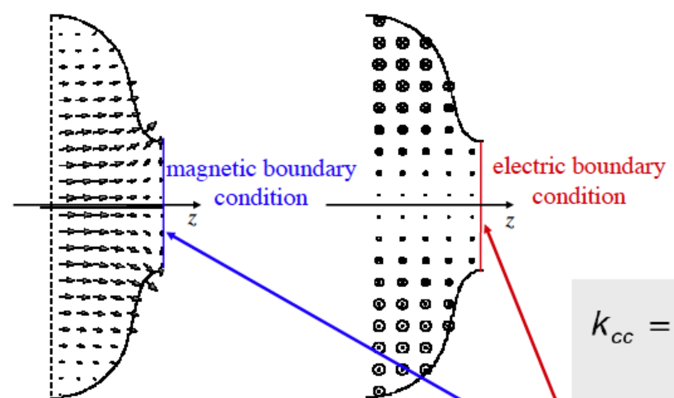



- The width of the passband is determined by the strength of the cell-to-cell coupling $k$ and the frequency of the *n*-th mode can be calculated from the dispersion formula

$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

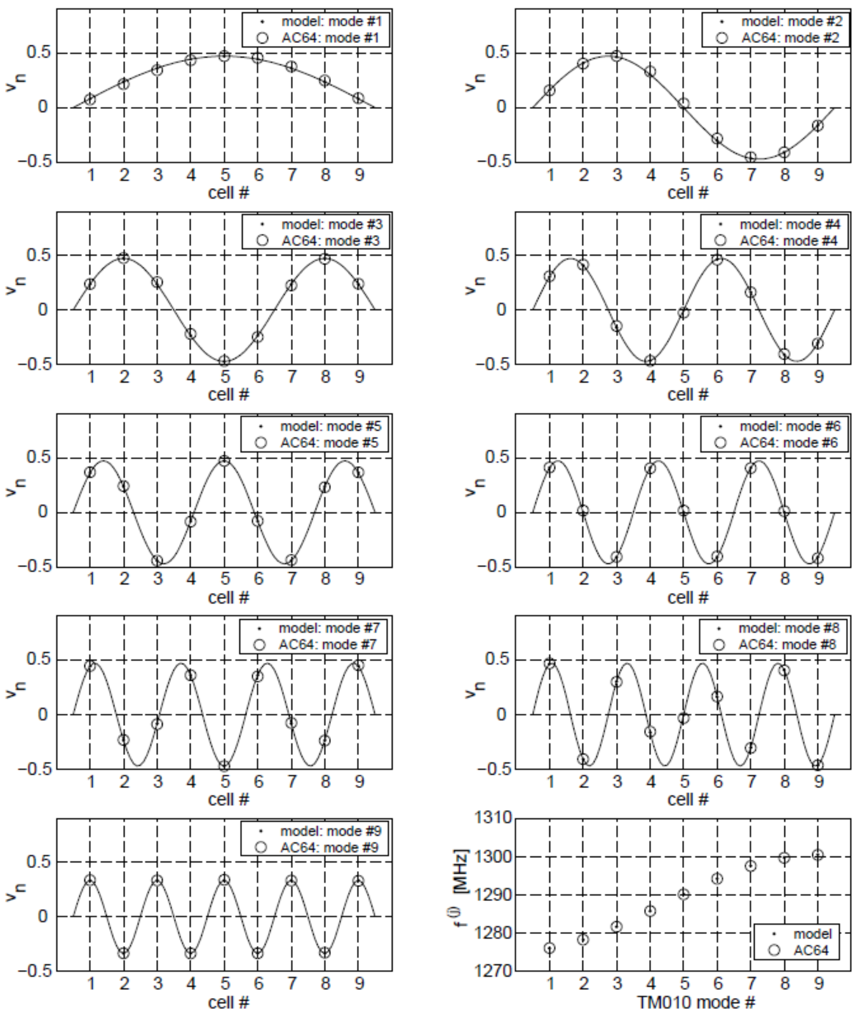
where $N$ is the number of cells,
$n = 1 \ldots N$ is the mode number.



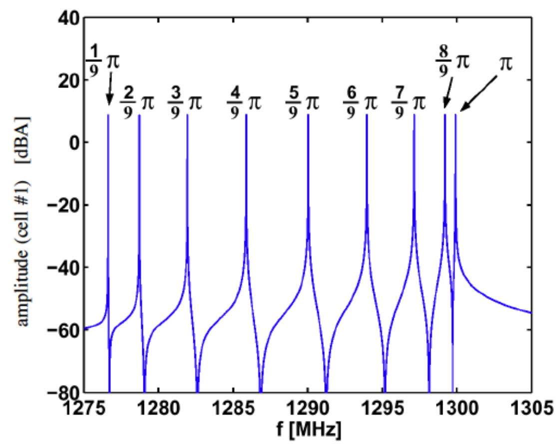

$$k_{cc} = \frac{\omega_\pi - \omega_0}{\frac{\omega_\pi + \omega_0}{2}}$$

*Coupling factor:* $k_{cc} = \frac{c}{\omega_0} \int_{S_i} \left(\vec{e} \times \vec{h}\right) \cdot \vec{u}_z ds$

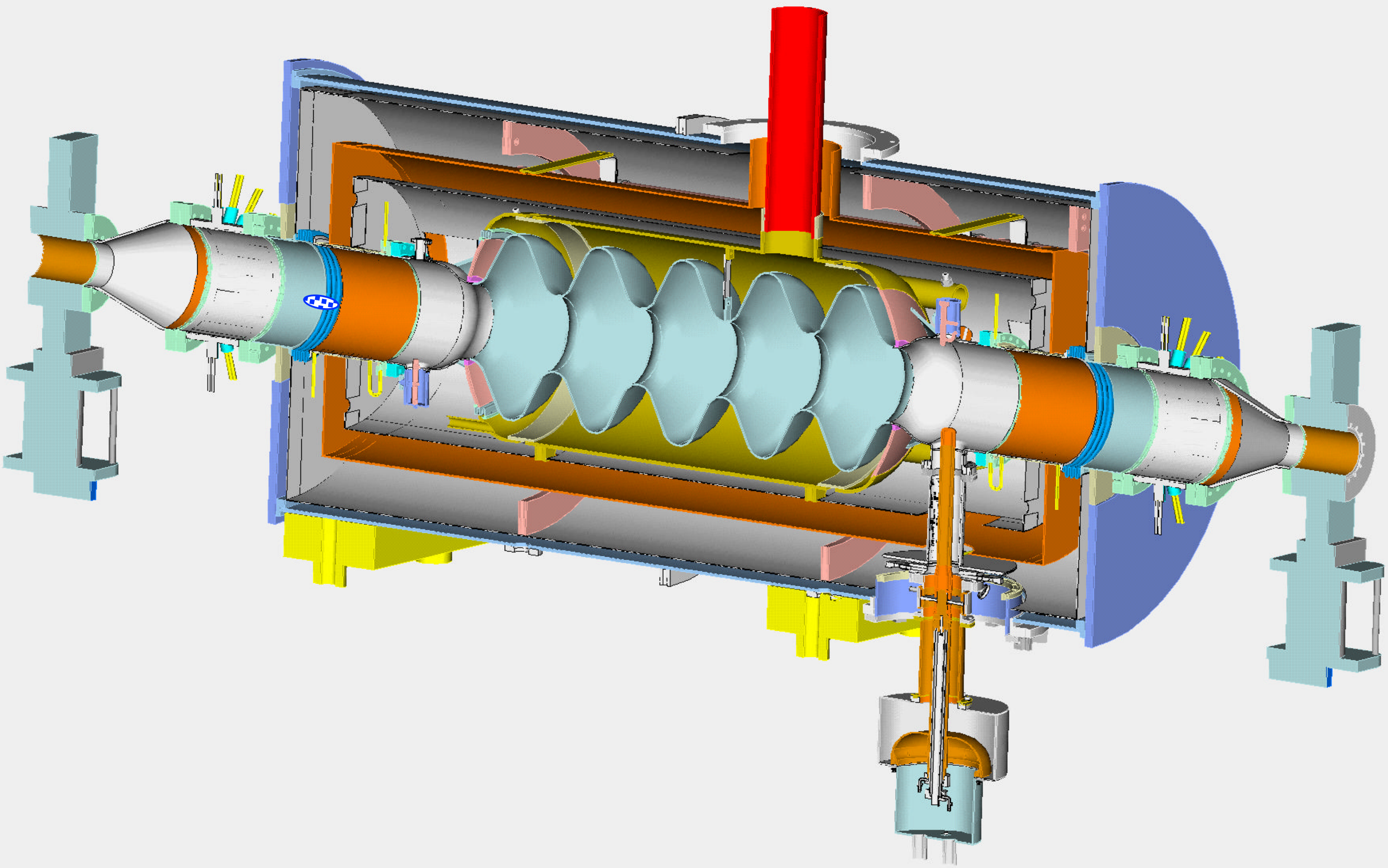
# Multicell cavities (2)




- Figure shows an example of calculated eigenmodes amplitudes in a 9-cell TESLA cavity compared to the measured amplitude profiles. Also shown are the calculated and measured eigenfrequencies.
- A longer cavity with more cells has more modes in the same frequency range, hence the reduction in frequency difference between adjacent modes. The number of cells is usually a result of the accelerating structure optimization.
- The accelerating mode for SC cavities is usually the **p**-mode, which has the highest frequency for electrically coupled structures.
- The same considerations are true for HOMs.



# How β=1 RF accelerator works?
# In pictures



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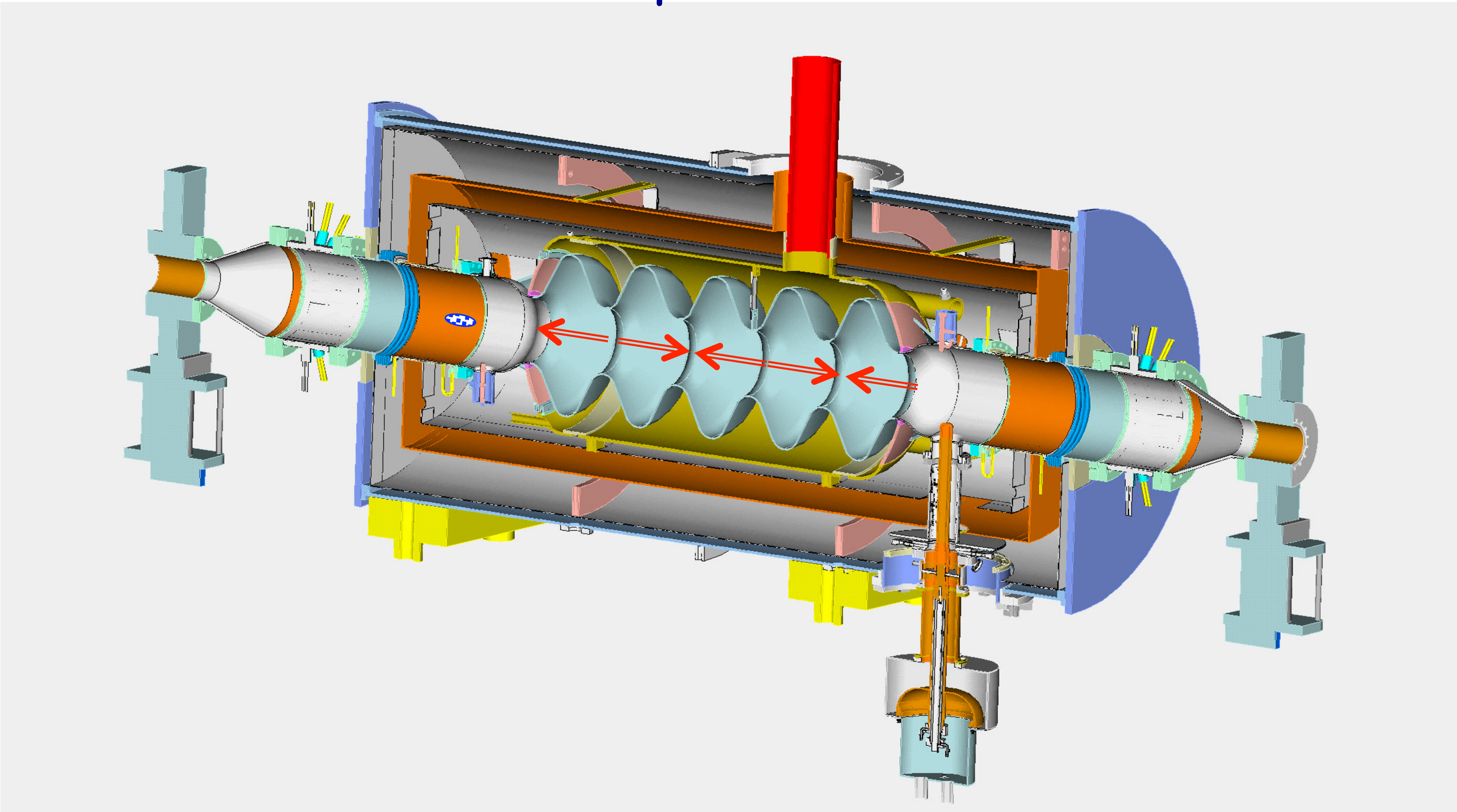

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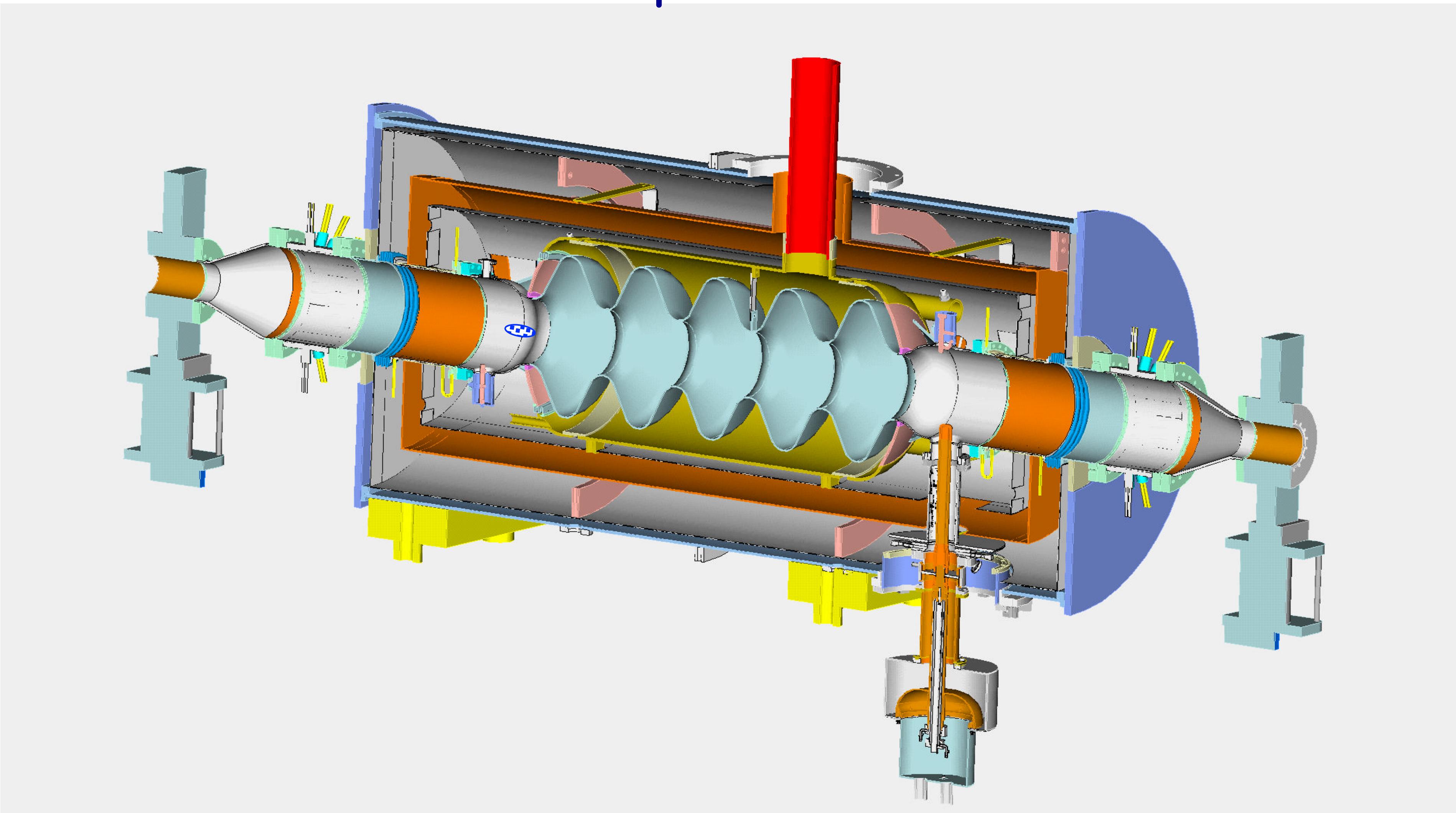

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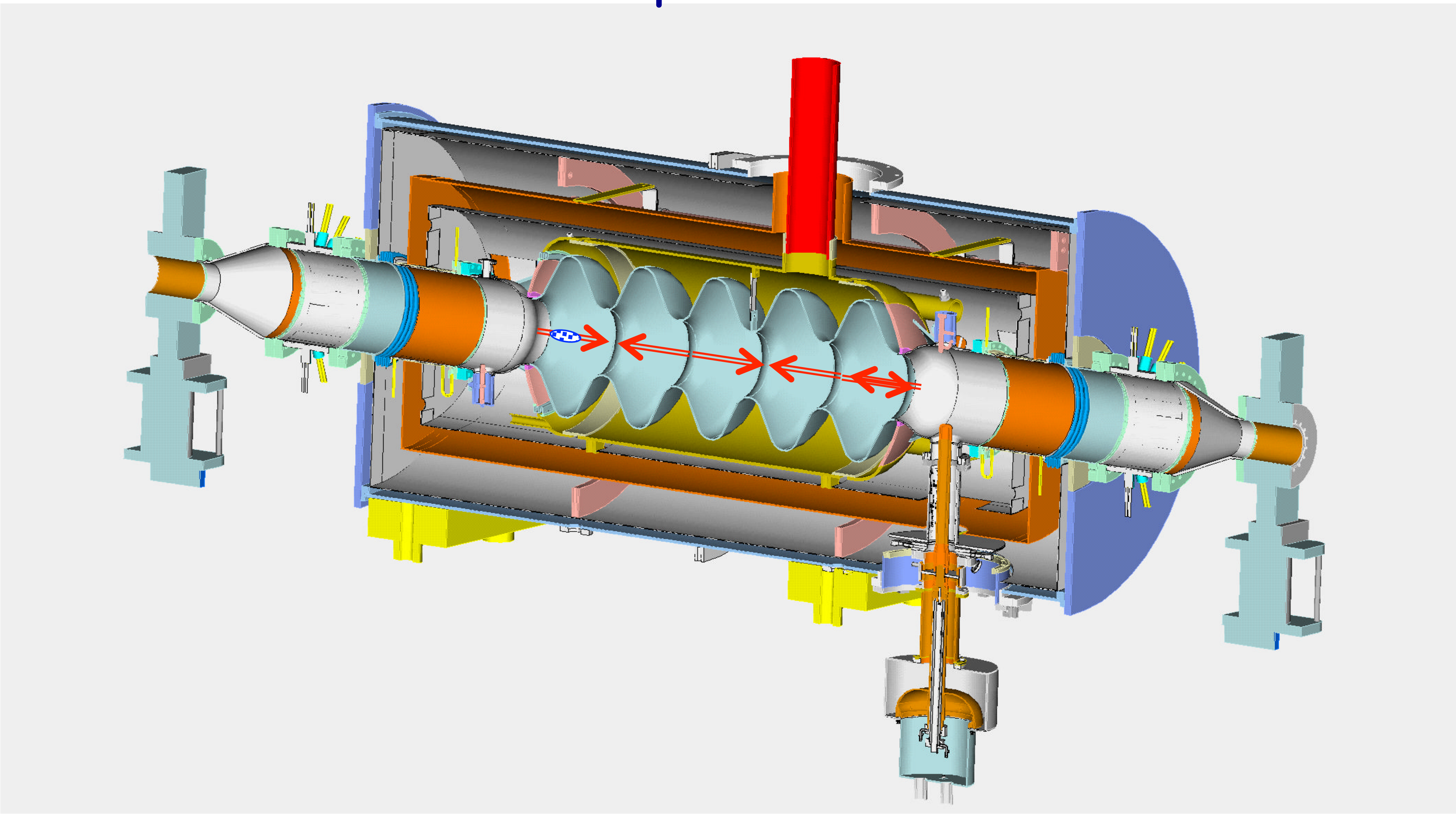

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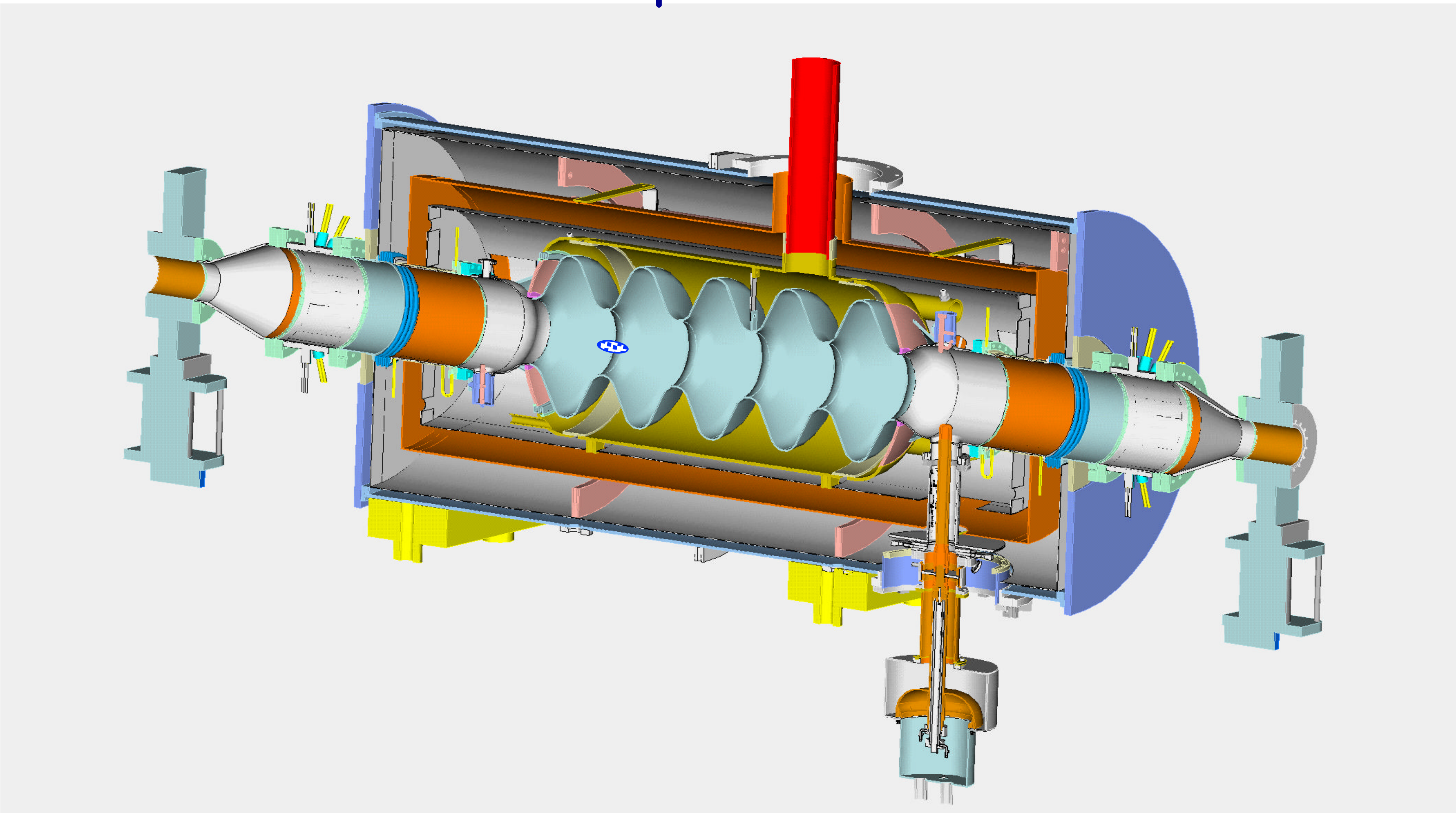

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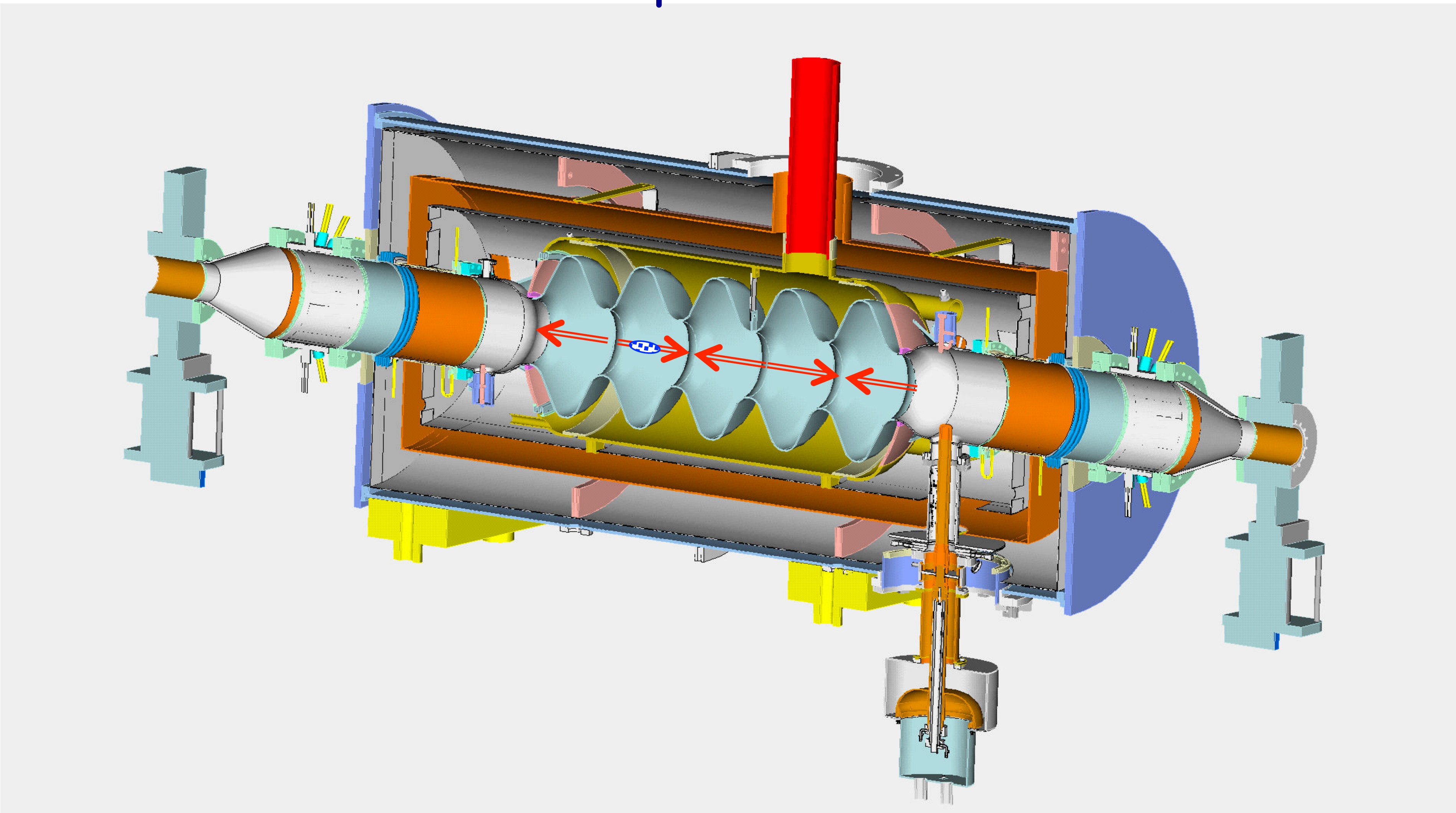

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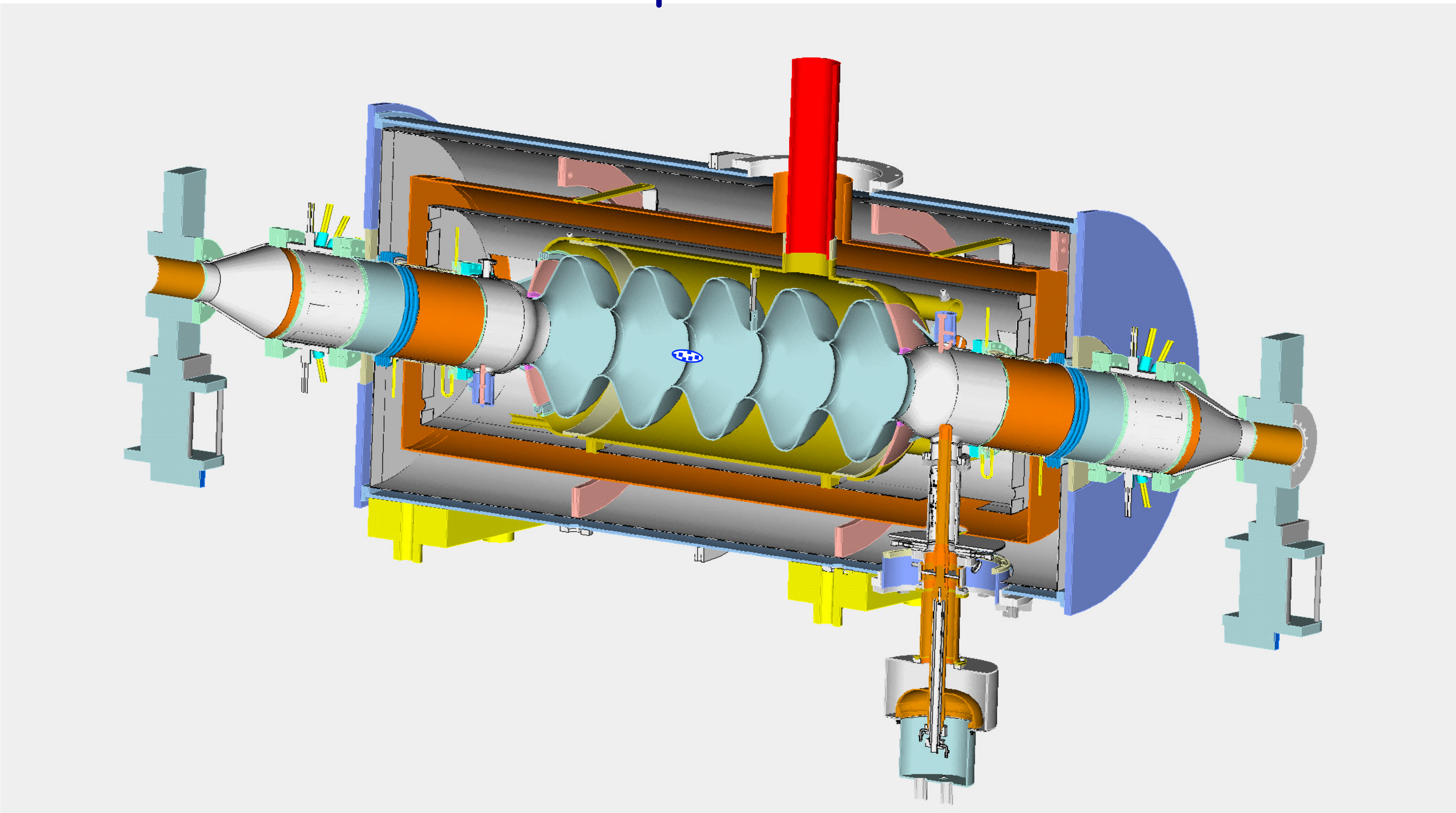

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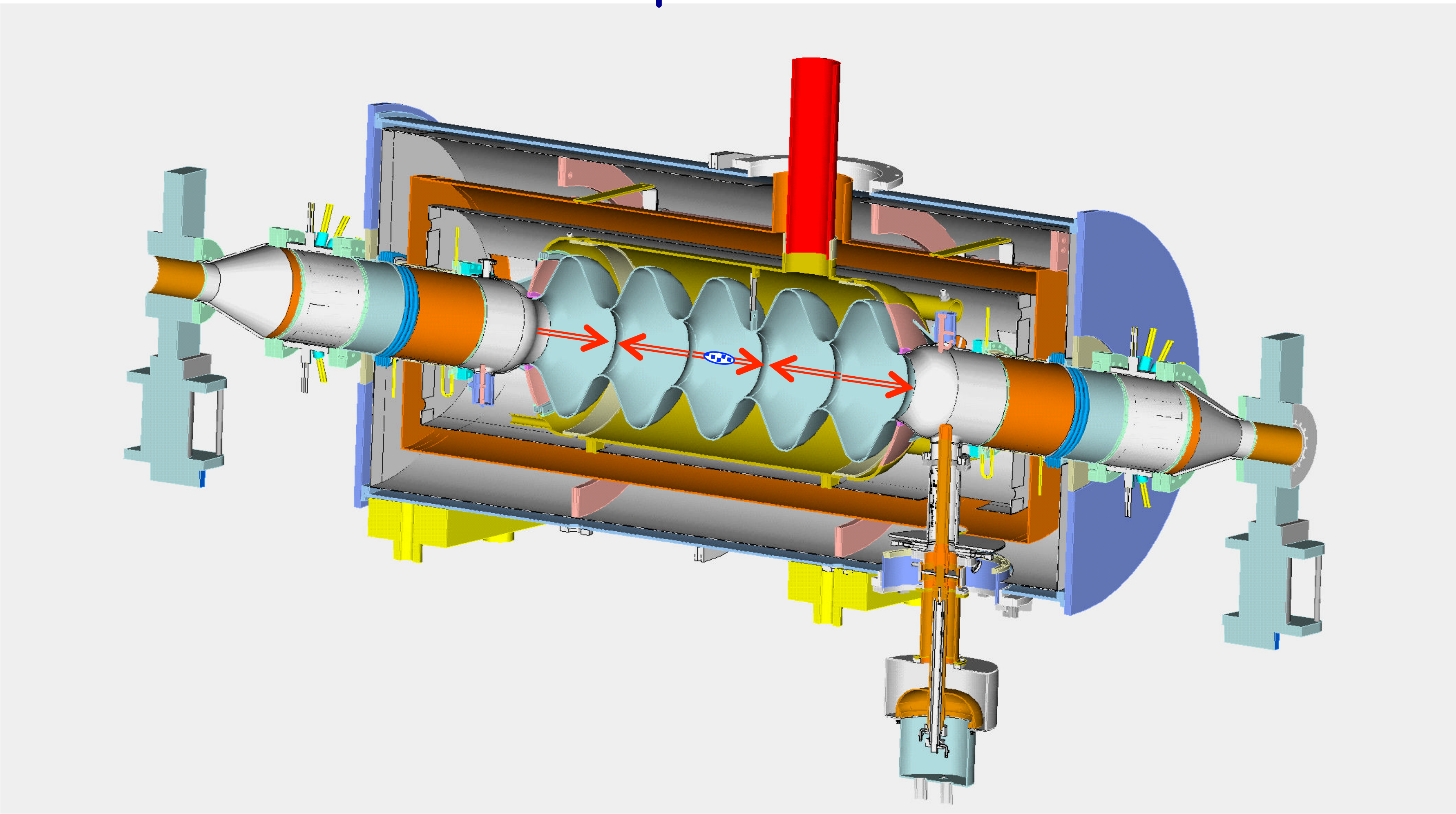

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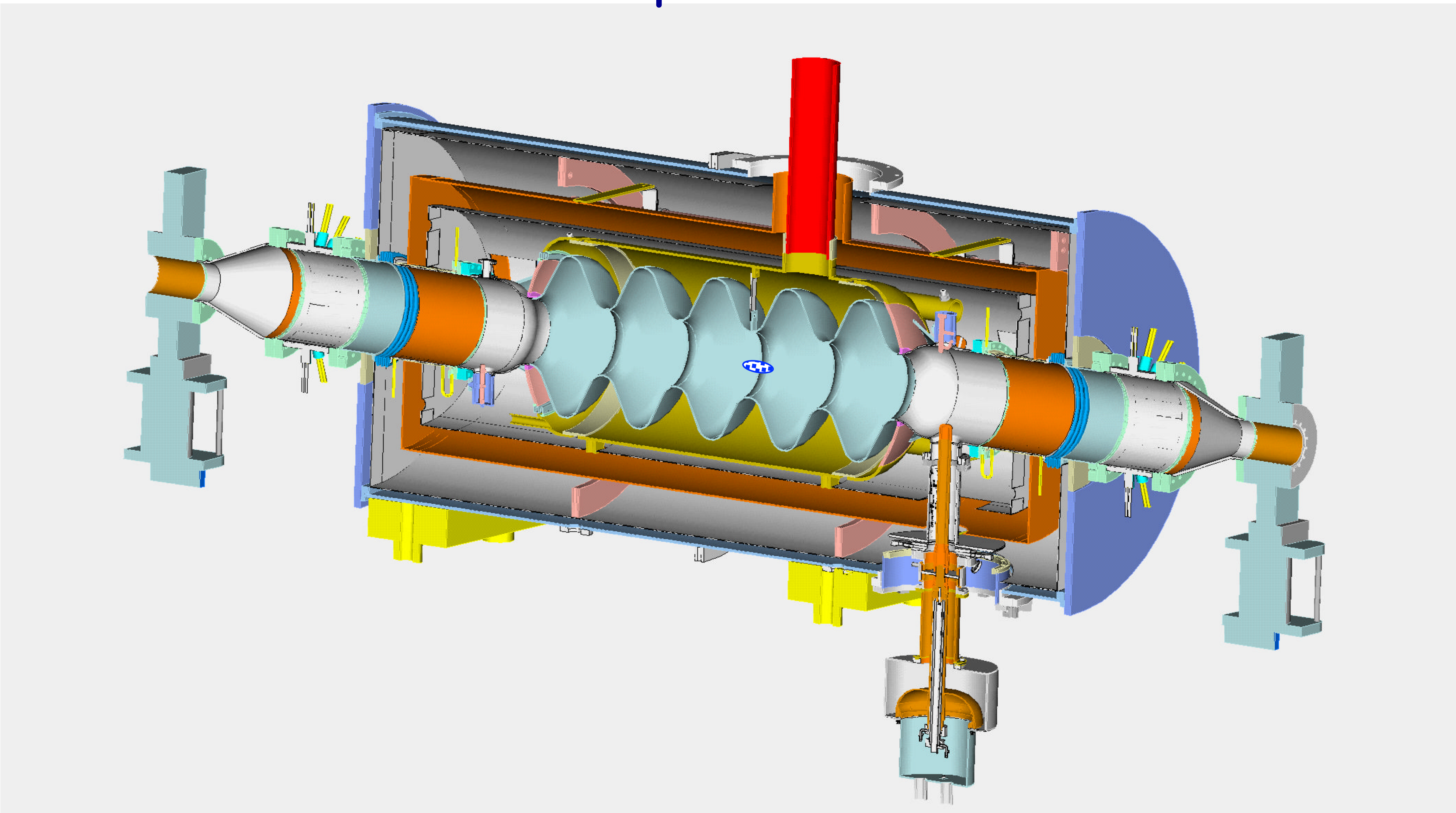

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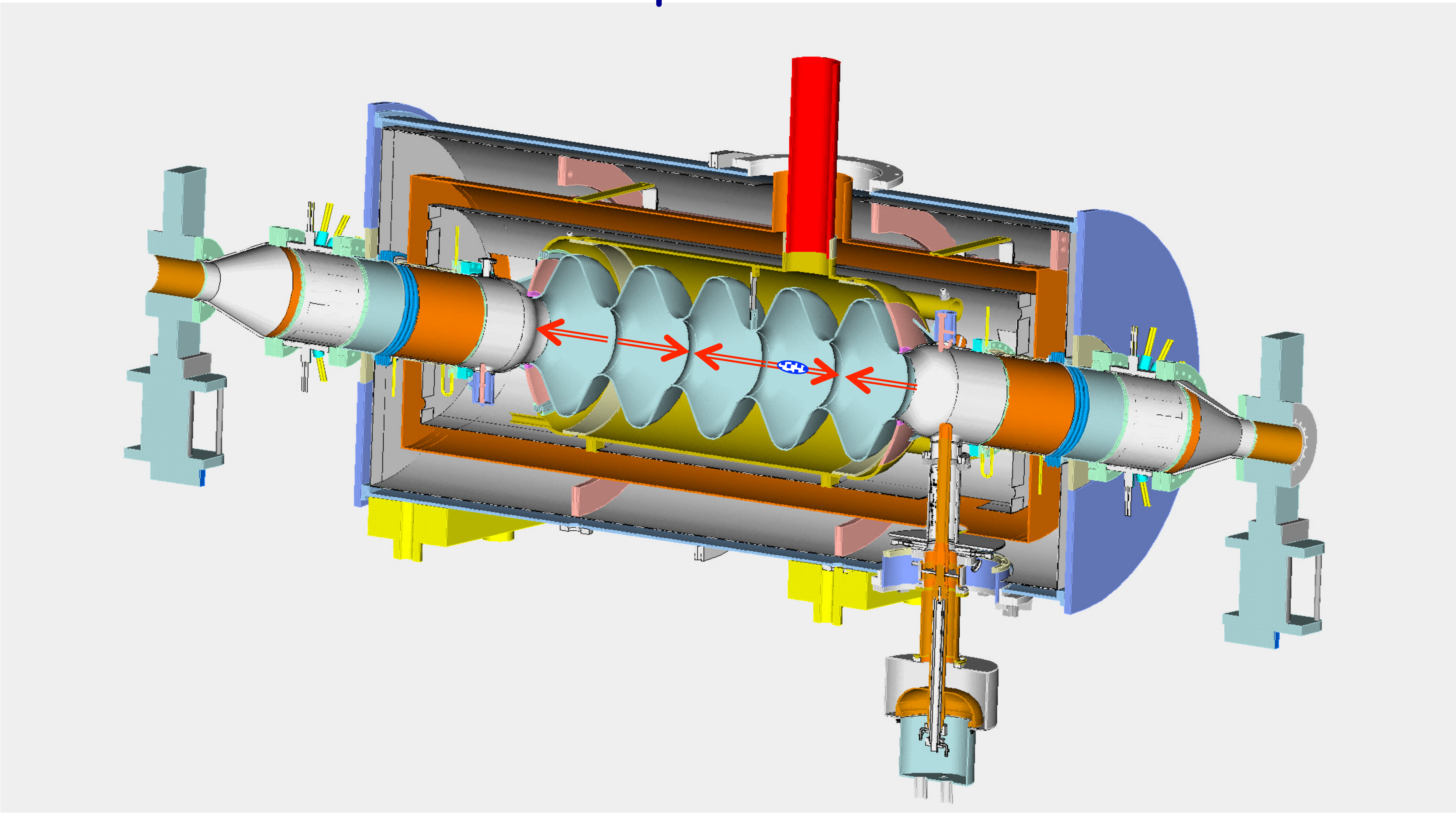

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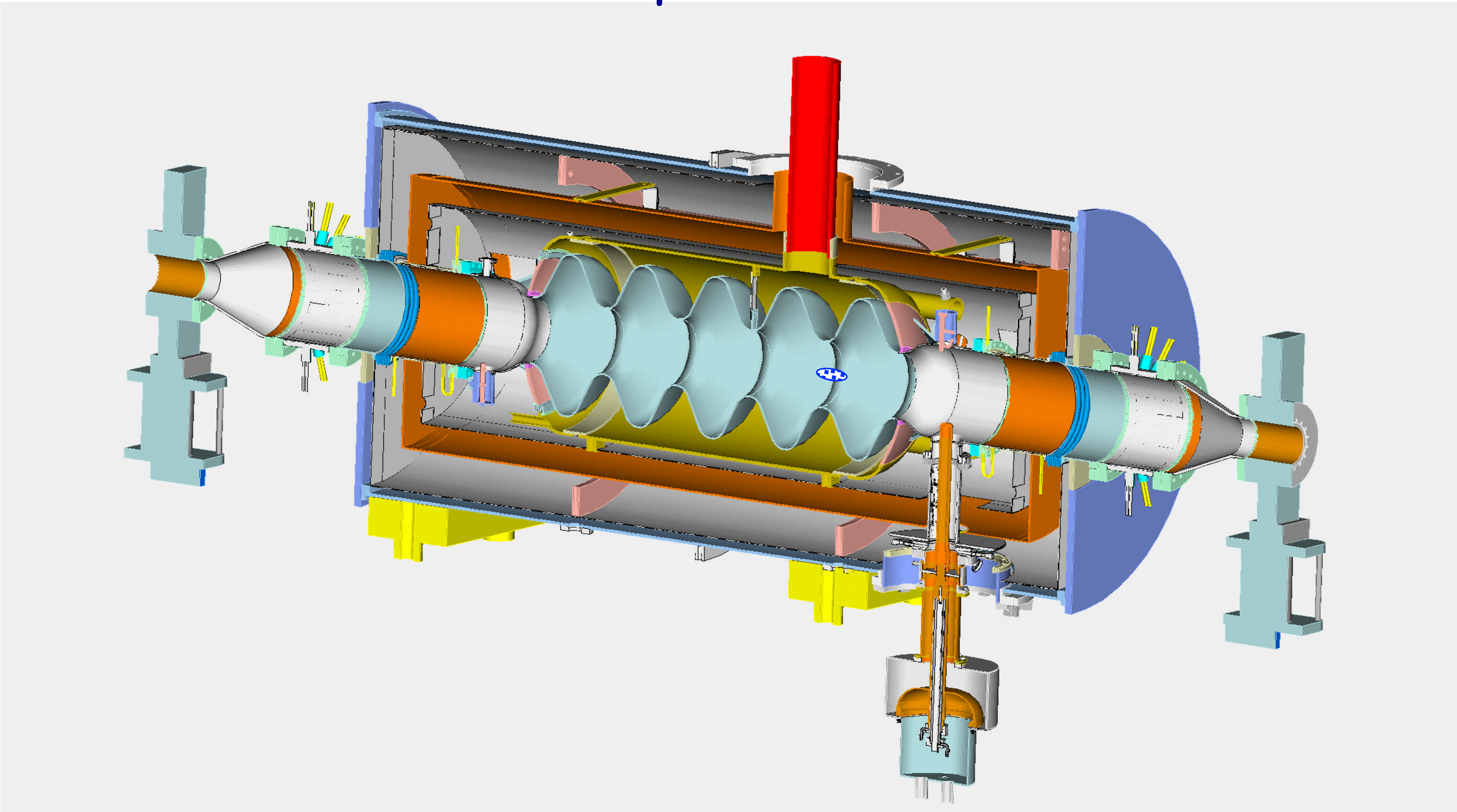

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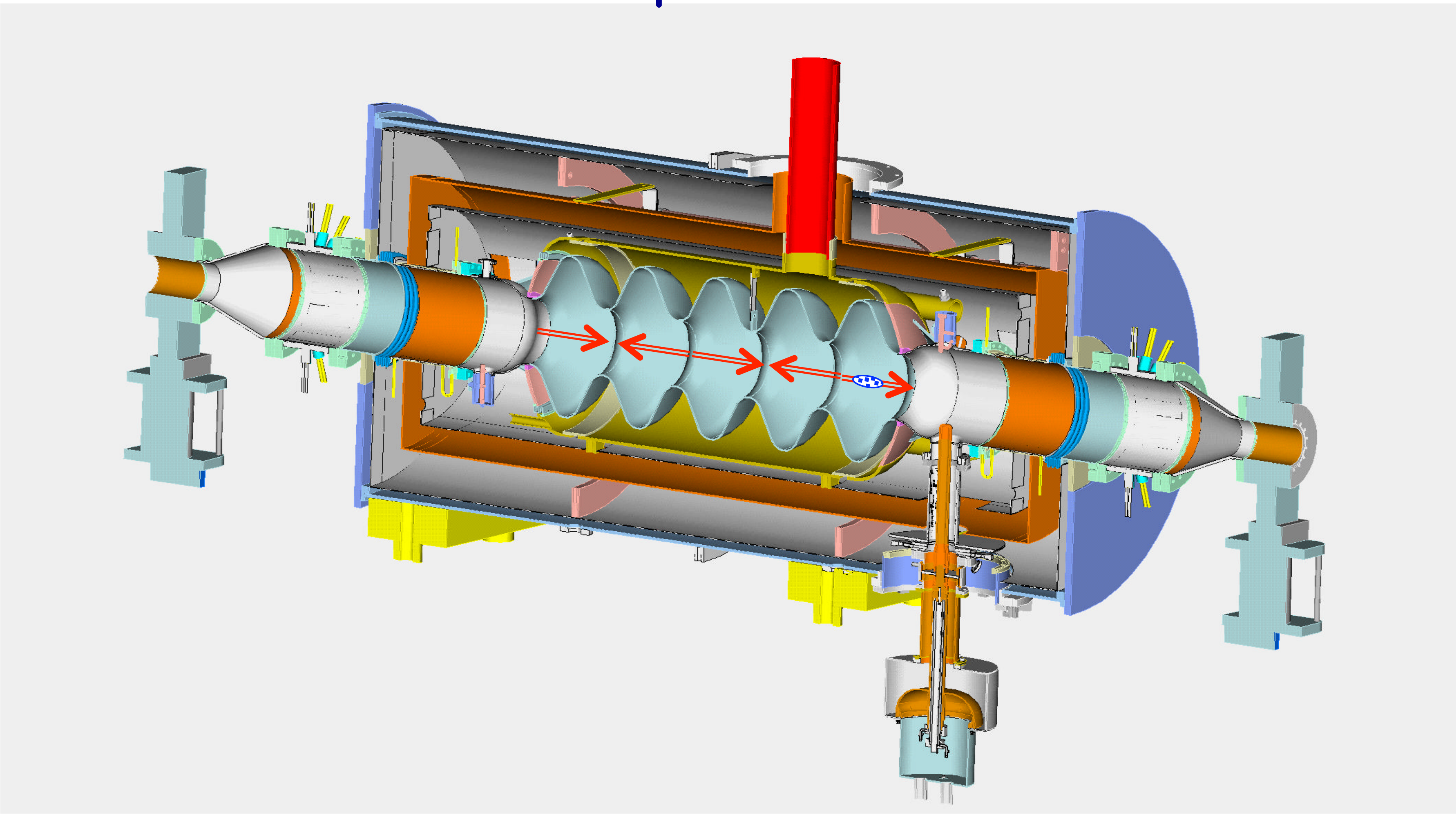

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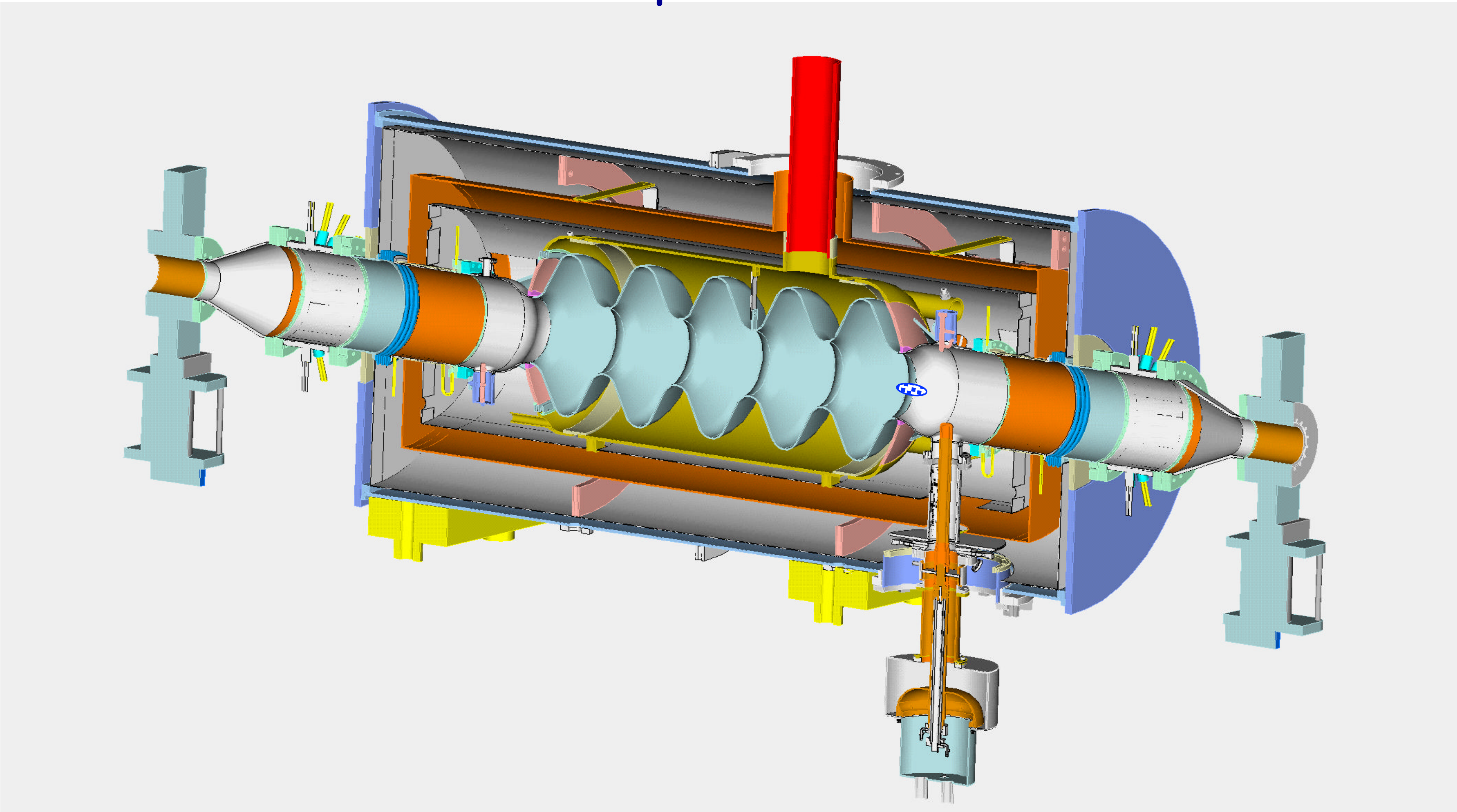

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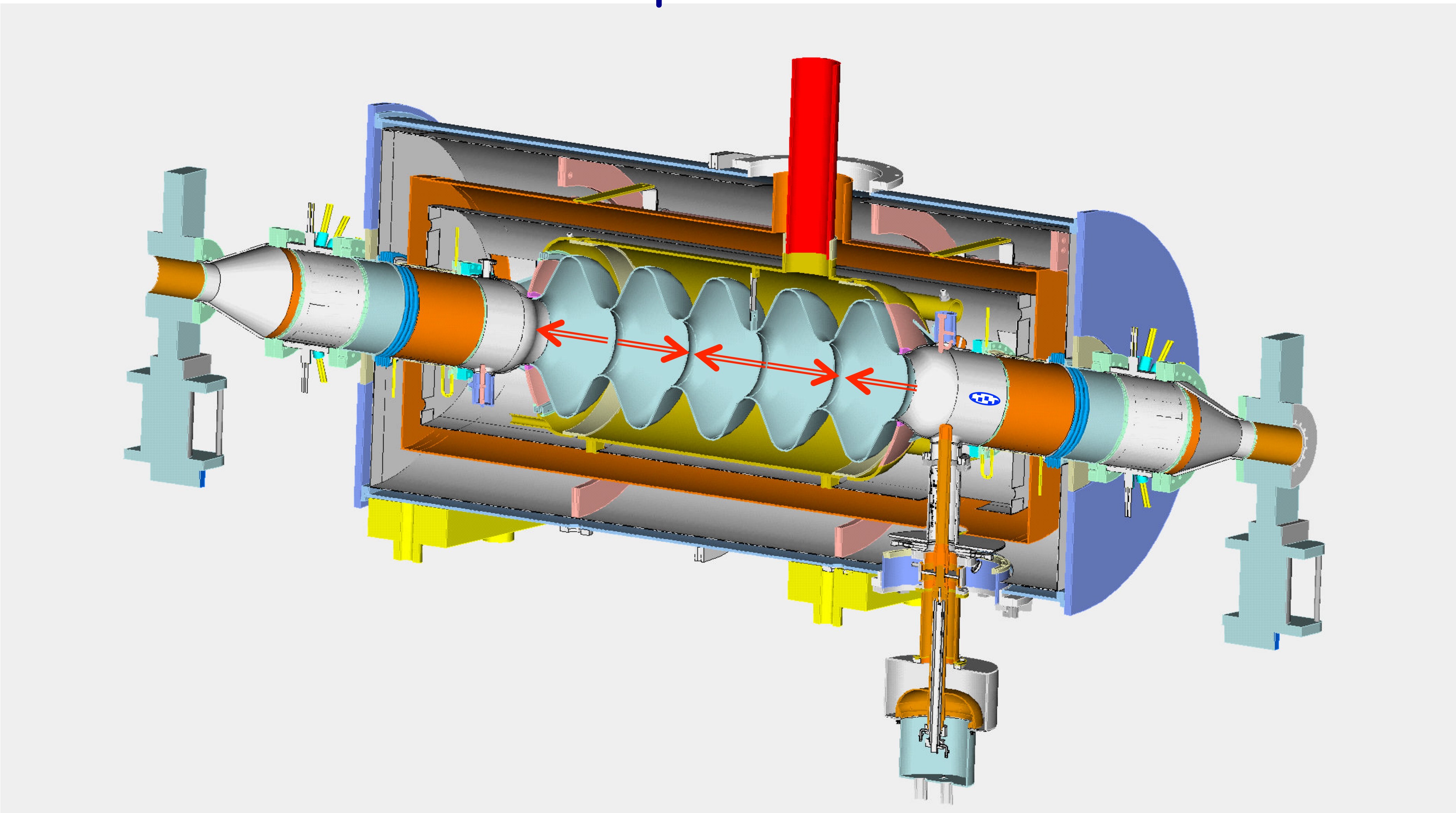

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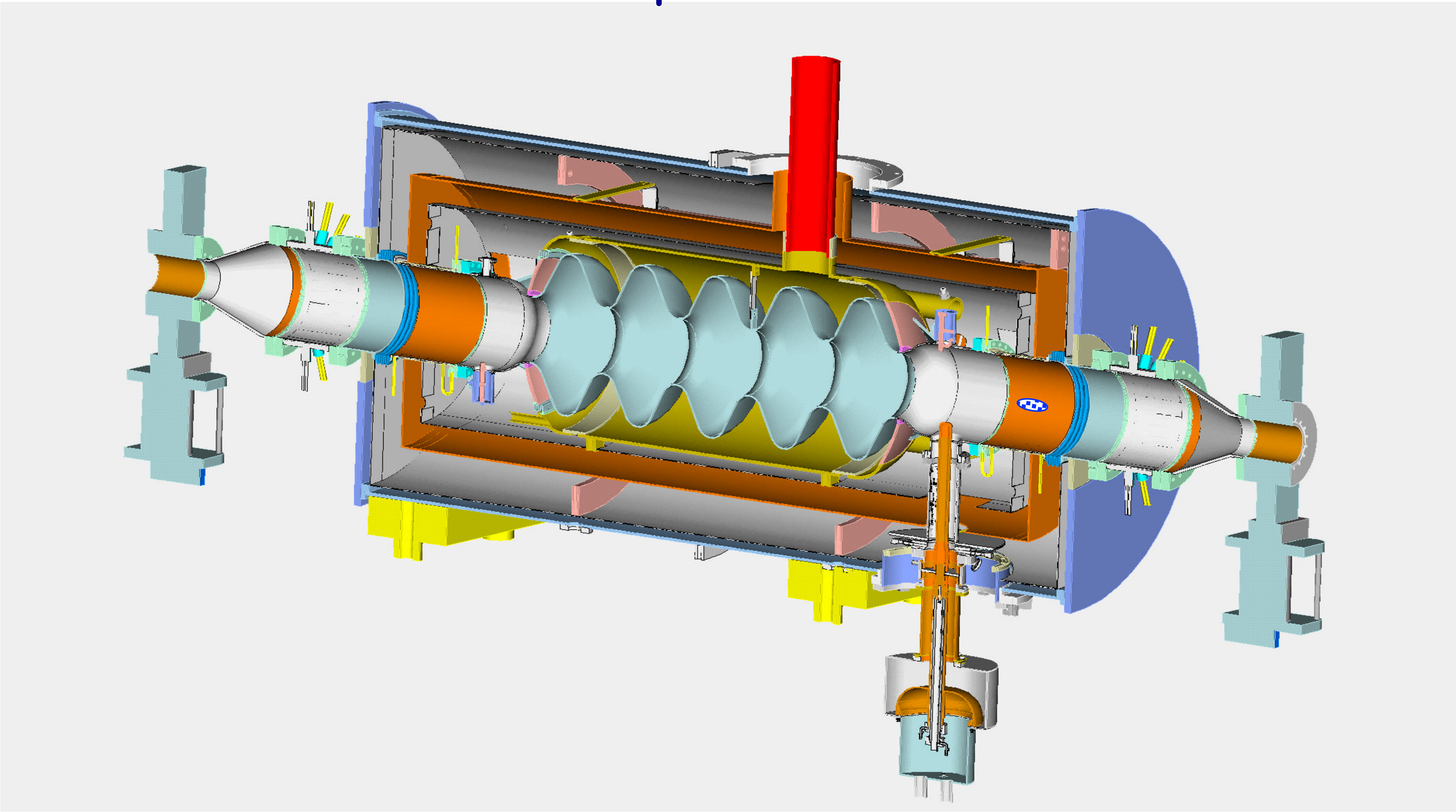

# What we learned

- Resonant modes in a cavity resonator belong to two families: TE and TM.
- There is an infinite number of resonant modes.
- The lowest frequency TM mode is usually used for acceleration.
- All other modes (HOMs) are considered parasitic as they can harm the beam.
- Several figures of merits are used to characterize accelerating cavities: main are accelerating voltage, transit time and Q-factor.
- In a multi-cell cavity every mode splits into a pass-band.
- The number of modes in each pass-band is equal to the number of cavity cells.
- The width of the pass-band is determined by the cell-to-cell coupling.
- Accelerating cavities operate at frequency below the cut-off frequency of vacuum pipes connected to them. The RF field decay exponentially along the pipes and reduces to a negligible level at length ~ few beam-pipe radii (assuming $R \ll \lambda_{RF}$)
- Coaxial lines and rectangular waveguides are commonly used in RF systems for power delivery to cavities

- Homework will be posted on the website this evening: due on October 12, 2016