PHY 554
Fundamentals of Accelerator Physics
Lecture 9: Introduction to RF accelerators
September 28, 2016
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Acknowledgement

Next three lectures use some materials from courses on RF and Superconducting RF (SRF) accelerators taught by Prof. S. Belomestnykh at SBU/BNL and USPAS, which can be found on the following websites:

http://case.physics.stonybrook.edu/index.php/CASE:Courses
https://sites.google.com/site/srfsbu11/

Plus some EM material of my own
Linear accelerators: from electrostatic to RF
Can one gain the energy again and again by passing through a DC accelerating gap?

**Electrostatic: what is the limit?**

Maxwell equations and energy conservation law!

\[ \Delta E = e \int \vec{E} \cdot d\vec{l} = -\frac{e}{c} \frac{\partial}{\partial t} \left( \int \vec{H} \cdot ds \right) \]

\[ \Delta E = e \int \vec{E} \cdot d\vec{l} = 0 \]

\[ \vec{E} = -\nabla \varphi \quad \rightarrow \quad E(\vec{r}) = E(0) - e\varphi(\vec{r}) \]

Can not cheat the Maxwell equations
**Induction linacs: linear betatrons**

- Useful for high power and high current beams
- Have limited accelerating field
- By nature are pulsed, with relatively low rep-rate (kHz)

\[
\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \left( \int \vec{H} \cdot d\vec{s} \right)
\]
How RF accelerator works

- It has oscillating (typically sinusoidal in time) longitudinal (along the particle’s trajectory) eclectic field
- It also has longitudinal structure (cells) which alternates the direction of the field
- When particle propagates through the RF accelerator, the field direction in each cell is synchronized with the particle arrival and the effect from all cells is added coherently

\[ \frac{dE}{dt} = e \vec{E} \cdot \vec{v} \rightarrow \text{sign}(\vec{E} \cdot \vec{v}) = \text{const} \]

Wideröe’s linac: \( \beta = \frac{v}{c} \) is changing

Electron linac

\( \beta = \frac{v}{c} \sim 1 \)
Wave-form in 5-cell cavity
How $\beta=1$ RF linac works?

Example of 5-cell cavity

$t=0$

$t=1/4T_o$

$t=1/2T_o$

$t=3/4T_o$

$t=T_o$

$t=5/4T_o$

$t=3/2T_o$

$t=7/4T_o$

$t=2T_o$

$t=9/4T_o$

$t=5/2T_o$

Electrons are out
Simple things to remember

- Acceleration in DC electrostatic is limited to the difference in terminal potential (e.g. voltage between the ground and the cathode)
- RF linear accelerators (RF linacs or simply linacs) are not limited in beam energy
- In RF linacs, the coherent addition/subtraction of the energy gain from cell to cell happens by design: period of the electric field oscillation is matched to the travel time of electron between the cells.
- Accurate synchronization of RF linac is important task for any linear accelerator
A bit of EM and conducting media

\[ \vec{j} = \sigma \vec{E}; \]

- Assuming oscillating field we can use Coulomb gauge for EM field

\[ \vec{A} = \text{Re}\{\vec{A}(\vec{r})\exp(i\omega t)\}; \phi = 0; \]

\[ \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = \text{curl}\vec{A}. \]

\[ \left| \vec{H} \right| \propto \frac{(\alpha + i\beta)}{k_o} \left| \vec{E} \right| = \sqrt{1 + \frac{4\pi i\sigma}{\omega}} \left| \vec{E} \right| \]

\[ \sigma \to \infty \]
Boundary conditions

- We are considering oscillating EM fields in RF structures
- RF structures are built from highly conducting material, both to contain EM field inside and to provide low losses
- In first approximation we can consider an ideal boundary conditions and take finite conductivity as a perturbation later
- Q-factor: \( Q_{\text{room temp}} \sim 10^4 - 10^5 \), \( Q_{\text{SRF}} \sim 10^9 - 10^{10} \)

\[ \tilde{A} = \text{Re}\{ \tilde{A}(\vec{r}) \exp(i\omega t - \alpha t) \}; \]
\[ \alpha = \frac{2\pi\omega}{Q} \]

\[ \tilde{E} = \hat{n}(\hat{n}\tilde{E}) + \tilde{E}_//; \tilde{B} = \hat{n}(\hat{n}\tilde{B}) + \tilde{B}_//; \]
Waveguides

Rectangular

\[
\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \tilde{A} = 0; \quad \Delta \equiv \tilde{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}; \\
\tilde{A} = \text{Re}\left\{ \tilde{A}(\tilde{r}_\perp) \exp\left( i(\omega t - k_z z) \right) \right\}; \\
\tilde{\nabla}_\perp^2 \tilde{A} + \left( k_o^2 - k_z^2 \right) \tilde{A} = 0; \quad k_o = \frac{\omega}{c}.
\]

\[
\left( \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \tilde{A} = 0; \quad \Delta \equiv \tilde{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}; \\
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\tilde{\nabla}_\perp^2 \tilde{A} + \left( k_o^2 - k_z^2 \right) \tilde{A} = 0; \quad k_o = \frac{\omega}{c}.
\]

At the surfaces

\[
\left. \tilde{n} \times \tilde{E} \right|_s = 0; \quad \left. \tilde{n} \cdot \tilde{B} = 0 \rightarrow \right. E_z \left|_s = 0; \quad \left. \frac{\partial B_z}{\partial n} \right|_s = 0
\]

Circular
TE and TM waves

- There is simplification
  - The modes are divided into two types: TE (transverse electric) and TM (transverse magnetic)

\[
\begin{align*}
\vec{E} &= \vec{E}_z + \vec{E}_\perp; \quad \vec{B} = \vec{B}_z + \vec{B}_\perp; \quad \vec{A}_z \equiv \hat{z}\vec{A}_z; \\
\nabla \times \vec{E} &= ik_o \vec{B}; \quad \nabla \times \vec{B} = -ik_o \vec{E}; \quad \Rightarrow \\

& ik_z \vec{E}_\perp + ik_o \left[ \hat{z} \times \vec{B}_\perp \right] = \nabla_{\perp} \vec{E}_z; \\

& ik_z \vec{B}_\perp - ik_o \left[ \hat{z} \times \vec{E}_\perp \right] = \nabla_{\perp} \vec{B}_z; \\

\text{At the surfaces} \\
\n\vec{n} \times \vec{E}\bigg|_s = 0; \quad \vec{n} \cdot \vec{B} = 0 \rightarrow \vec{E}_z\bigg|_s = 0; \quad \frac{\partial \vec{B}_z}{\partial n}\bigg|_s = 0

TM: \vec{B}_z \equiv 0; \quad \vec{E}_z\bigg|_s = 0; \\

TE: \vec{E}_z \equiv 0; \quad \vec{B}_z\bigg|_s = 0;
\]

- Last two equations indicated that \(E_z\) and \(B_z\) fully determine transverse component of the EM field

- It means that we can always consider a linear combination of the fields with \(E_z = 0\) everywhere (TE) and \(B_z = 0\) everywhere (TM)

- Naturally, when we interested in accelerating particles, we will need TM mode with \(E_z \neq 0\).

\[
\vec{B}_\perp = \pm \frac{k_z}{k_o} \left[ \hat{z} \times \vec{E}_\perp \right] \quad \text{for both TE and TM modes}
\]

TM: \(\vec{E}_\perp = \nabla_{\perp} \psi_1(\vec{r}_\perp)\); TE: \(\vec{B}_\perp = \nabla_{\perp} \psi_2(\vec{r}_\perp)\);
Cut-off frequency

- EM field is a linear combination of modes with $E_z = 0$ everywhere (TE) and $B_z = 0$ everywhere (TM)

At the surfaces

$$\vec{n} \times \vec{E} \big|_s = 0; \quad \vec{n} \cdot \vec{B} = 0 \quad \Rightarrow \quad E_z \big|_s = 0; \quad \frac{\partial B_z}{\partial n} \big|_s = 0$$

$\vec{B}_\perp = \pm \frac{k_z}{k_o} \left[ \hat{z} \times \vec{E}_\perp \right]$ for both TE and TM modes

TM: $\vec{E}_\perp = \nabla_\perp \psi_1 (\vec{r}_\perp)$; TE: $\vec{B}_\perp = \nabla_\perp \psi_2 (\vec{r}_\perp)$;

$$\nabla_\perp^2 \psi + \left( k_o^2 - k_z^2 \right) \psi = 0 + \text{boundary conditions}$$

Different boundary conditions for TE and TM modes
In general case we need to find eigen function (modes)

Cut-off frequency

$$k_{z,\lambda}^2 = k_o^2 - \gamma_\lambda^2 > 0$$

Below cut-off

evanescent wave: $k_z = \pm i \sqrt{\omega_{\text{cut-off}}^2 - \omega^2} = \pm i \kappa_z$

Exp decay

$$\psi = \psi_0 e^{\pm \kappa_z z}$$
Cut-off frequency

Different boundary conditions for TE and TM modes

\[
TM : \psi \bigg|_s = 0; \quad TE: \quad \frac{\partial \psi}{\partial n} \bigg|_s = 0.
\]

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi + \gamma_{mn}^2 \psi = 0
\]

\[
\psi_{TE}^{mn} = \psi_o \cos k_m x \cos k_n x; \quad m + n \geq 1;
\]

\[
\psi_{TM}^{mn} = \psi_o \sin k_m x \sin k_n x; \quad m \geq 1; n \geq 1;
\]

\[
k_m = \pi \frac{m}{a}; \quad k_m = \pi \frac{n}{b}; \quad \gamma_{mn} = k_m^2 + k_n^2.
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \gamma_{mn}^2 \psi = 0
\]

\[
\psi_{mn} = \phi_{mn} (r)e^{in\theta} \Rightarrow r \frac{\partial}{\partial r} \left( r \frac{\partial \psi_{mn}}{\partial r} \right) + (r^2 \gamma_{mn}^2 - n) = 0
\]

\[
\phi_{mn} = J_n \left( \gamma_{mn} r \right)
\]

Lowes cut-off frequency

Rectangular

\[
TE : a > b; \quad m = 1; \quad n = 0; \quad \omega_{cut-off} = \frac{\pi c}{a};
\]

\[
TM : m = 1; \quad n = 1; \quad \omega_{cut-off} = \frac{\pi c}{a} \sqrt{1 + \frac{a^2}{b^2}}.
\]

Circular

\[
TM : J_0 \left( \gamma_{01} R \right) = 0 \rightarrow \gamma_{01} \equiv \frac{2.40483...}{R}; \quad \omega_{cut-off} \equiv \frac{2.40 c}{R};
\]

\[
TE : J_1 \left( \gamma_{11} R \right) = 0 \rightarrow \gamma_{11} \equiv \frac{1.84118...}{R}; \quad \omega_{cut-off} \equiv \frac{1.84 c}{R}.
\]
Modes in rectangular waveguide
RF cavities are designed to confine the EM field inside: It means that they operate at frequency below cut-off of the beam-pipes attached to them.
RF Cavity Modes: 
the lowest accelerating is TM\(_{010}\) mode

- Fields in the cavity are solutions of the equation
- Subject to the boundary conditions
- Two extra surfaces (z=0 and z=d): but this is no problem for TM mode
- An infinite number of solutions (eigen modes) belong to two families of modes with different field structure and eigen frequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency TM\(_{010}\) mode is used.
- For the pillbox cavity w/o beam tubes
- Note that frequency does not depend of the cavity length! But only its radius.

\[
\begin{align*}
E_z &= E_0 J_0 \left( \frac{2.405r}{R} \right) e^{i\omega t} \\
H_\phi &= -iE_0 J_1 \left( \frac{2.405r}{R} \right) e^{i\omega t} \\
\omega_{010} &= \frac{2.405c}{R}, \quad \lambda_{010} = 2.61R
\end{align*}
\]
Fundamental and high order modes (HOMs)

Eigenmodes in a Pill-box cavity

\[ TM : \varphi_{mnl} = J_n(\gamma_{mn}r)\cos k_{z,l}z; \quad k_{z,l} = l\frac{\pi}{d}; \quad J_n(\gamma_{mn}R) = 0; \]
\[ \omega_{res} = c\sqrt{\gamma_{mn}^2 + l^2 \frac{\pi^2}{d^2}}; \quad l = 0,1,2,\ldots \]

\[ TE : \varphi_{mnl} = J_n(\kappa_{mn}r)\sin k_{z,l}z; \quad k_{z,l} = l\frac{\pi}{d}; \quad J'_n(\kappa_{mn}R) = 0; \]
\[ \omega_{res} = c\sqrt{\kappa_{mn}^2 + l^2 \frac{\pi^2}{d^2}}; \quad l = 1,2,\ldots \]
Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

\[ V_c = \left| \int_{-\infty}^{\infty} E_z (\rho = 0, z) e^{i\omega_0 z / \beta c} \, dz \right| \]

For the pillbox cavity one can integrate this analytically:

\[ V_c = E_0 \left| \int_{0}^{d} e^{i\omega_0 z / \beta c} \, dz \right| = E_0 d \frac{\sin \left( \frac{\omega_0 d}{2 \beta c} \right)}{\frac{\omega_0 d}{2 \beta c}} = E_0 d \cdot T \]

where \( T \) is the transit time factor.

To get maximum acceleration:

\[ T_{\text{transit}} = t_{\text{exit}} - t_{\text{enter}} = \frac{T_0}{2} \Rightarrow d = \beta \lambda / 2 \Rightarrow V_c = \frac{2}{\pi} E_0 d \]

Thus for the pillbox cavity \( T = 2/p \).

The accelerating field \( E_{\text{acc}} \) is defined as \( E_{\text{acc}} = V_c / d \). Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be \( d = bl/2 \). This works OK for multi-cell cavities, but poorly for single-cell ones.
Multicell cavities: coupled oscillators

- Several cells can be connected together to form a multicell cavity.
- Coupling of $\text{TM}_{010}$ modes of the individual cells via the iris (primarily electric field) causes them to split:
Multicell cavities (2)

- The split mode forms a passband of closely spaced modes equal in number to the number of cells.

- The width of the passband is determined by the strength of the cell-to-cell coupling $k$ and the frequency of the $n$-th mode can be calculated from the dispersion formula

$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k \left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

where $N$ is the number of cells, $n = 1 \ldots N$ is the mode number.
Multicell cavities (2)

- Figure shows an example of calculated eigenmodes amplitudes in a 9-cell TESLA cavity compared to the measured amplitude profiles. Also shown are the calculated and measured eigenfrequencies.

- A longer cavity with more cells has more modes in the same frequency range, hence the reduction in frequency difference between adjacent modes. The number of cells is usually a result of the accelerating structure optimization.

- The accelerating mode for SC cavities is usually the \( p \)-mode, which has the highest frequency for electrically coupled structures.

- The same considerations are true for HOMs.
How $\beta=1$ RF accelerator works?
In pictures
How $\beta=1$ RF accelerator works?  
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What we learned

• Resonant modes in a cavity resonator belong to two families: TE and TM.
• There is an infinite number of resonant modes.
• The lowest frequency TM mode is usually used for acceleration.
• All other modes (HOMs) are considered parasitic as they can harm the beam.
• Several figures of merits are used to characterize accelerating cavities: main are accelerating voltage, transit time and Q-factor.
• In a multi-cell cavity every mode splits into a pass-band.
• The number of modes in each pass-band is equal to the number of cavity cells.
• The width of the pass-band is determined by the cell-to-cell coupling.
• Accelerating cavities operate at frequency below the cut-off frequency of vacuum pipes connected to them. The RF field decay exponentially along the pipes and reduces to a negligible level at length ~ few beam-pipe radii (assuming $R \ll \lambda_{RF}$).
• Coaxial lines and rectangular waveguides are commonly used in RF systems for power delivery to cavities.

• Homework will be posted on the website this evening: due on October 12, 2016