PHY 554
Fundamentals of Accelerator Physics

Lecture 12: Synchrotron Radiation Sources

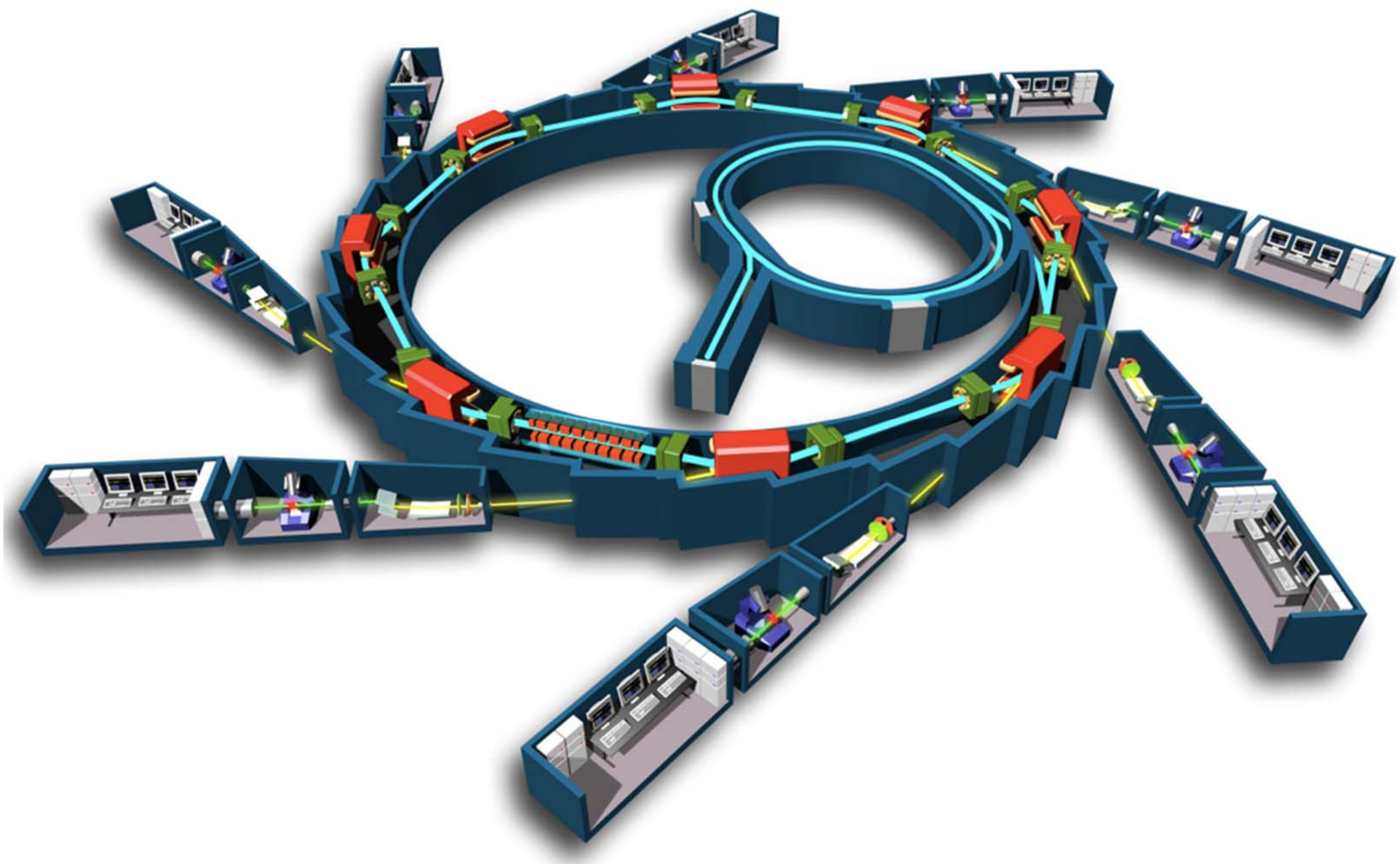
G. Wang

There is a large number of dedicated courses on Synchrotron Radiation Sources and Their Applications. If you are interested in this topic, I would strongly recommend lectures given by Prof. D.T. Attwood at UC Berkeley, https://people.eecs.berkeley.edu/~attwood/srms//

Detailed derivation can be found from 'Soft X-ray and Extreme Ultraviolet Radiation' by D. Attwood, chapter 5.

SR Light Sources

- To generate IR, UV and X-ray radiation
  - From dipoles, undulators/wigglers



VERY POPULAR SCIENTIFIC TOOL: With thousands of users

LIGHT INTERACTS with the MATTER

Light sources around the world

SR Light Sources Worldwide

MAX IV, 3 GeV, Sweden

NSLS II, 3 GeV, BNL, USA

Diamond, 3 GeV, England

ALBA, 3 GeV, Spain

SR Light Sources Worldwide

SSRF, China, 3.5 GeV

Soleil, France, 2.75 GeV

SLS, Switzerland, 2.4 GeV

PLS, Korea, 3 GeV

Australian Synchrotron, 3 GeV

BESSY II, Germany, 1.7 GeV

NSRRC, Taiwan, 3GeV

Indus II, India, 2.5GeV

SESAME, Jordan......

What matters

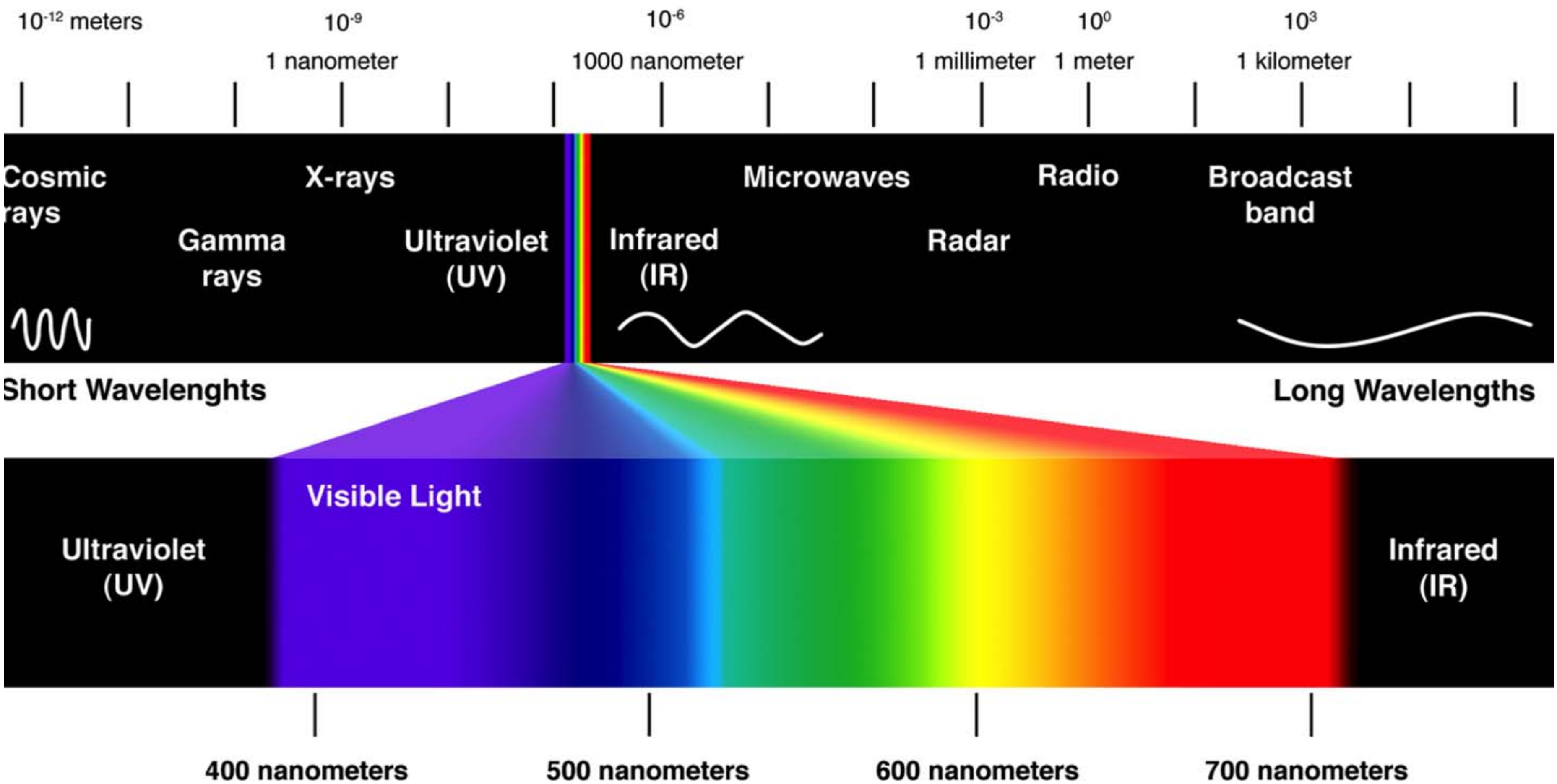
$$E_{ph} = \hbar\omega = \hbar c \frac{2\pi}{\lambda};$$

$$E_{ph}[keV] \approx \frac{12.4}{\lambda[\AA]};$$

$$E_{ph}[eV] \approx \frac{1.24}{\lambda[\mu m]}$$

- Rarely there is interest just a radiation power
- Typically people are interested in specific energy of photons (wavelength of radiation)

1 Å $=10^{-10}$ m (0.1 nm or 100 pm), 12.4 keV photons

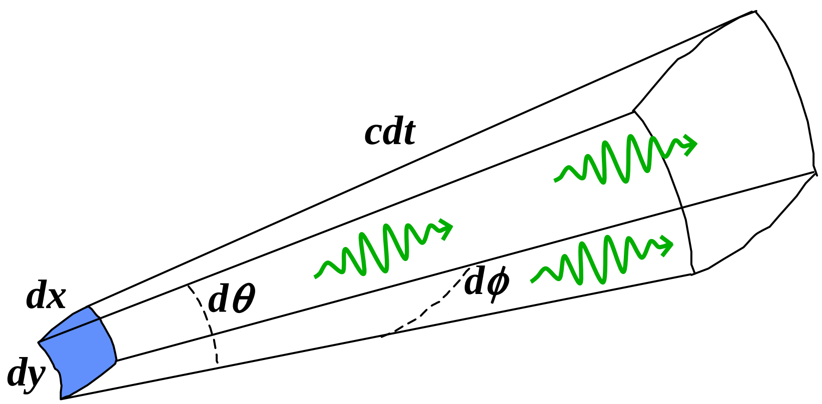


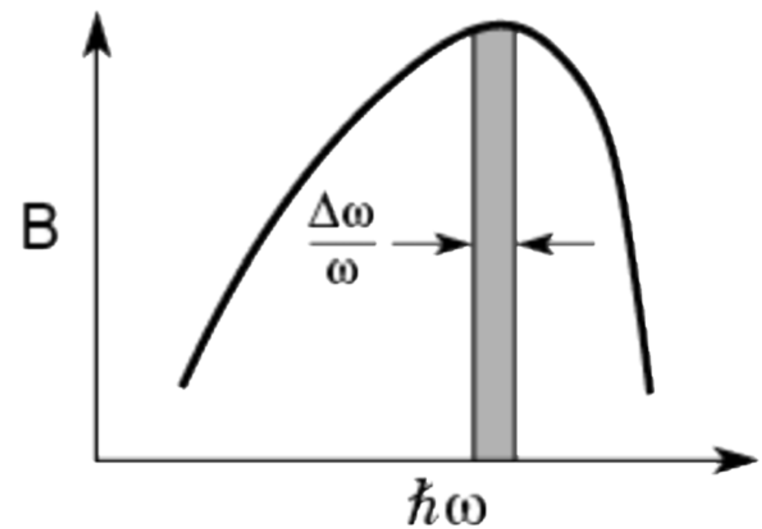
3 eV 2.5 eV 2 eV 1.75 eV

Figures of merit of light source

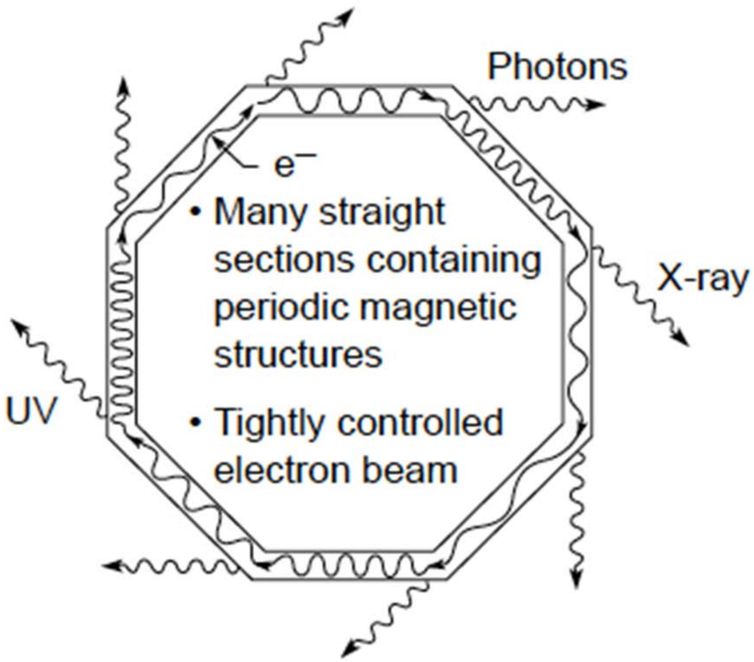
- Spectral photon flux $$\dot{N}_\omega = \frac{d^2N}{dt(d\omega/\omega)}$$

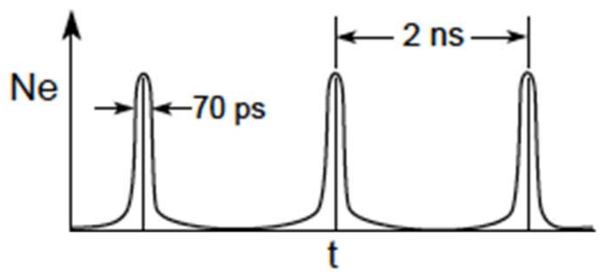
- Spectral brightness of the source $$B = \frac{d^4N}{dt\,d\Omega\,dA(d\omega/\omega)}$$

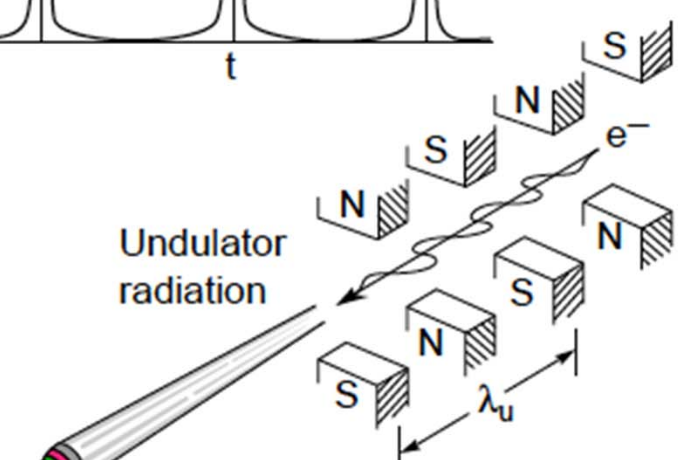




Sources of Spontaneous Radiation







Courtesy of D. Attwood

Bending Magnet:

$$\hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m}$$

$$P = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\gamma^4}{\rho^2}$$

Wiggler:

$$\hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m}$$

$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right)$$

$$P_T = \frac{\pi e K^2 \gamma^2 I N}{3\epsilon_0 \lambda_u}$$

Undulator:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

$$K = \frac{eB_0\lambda_u}{2\pi mc}$$

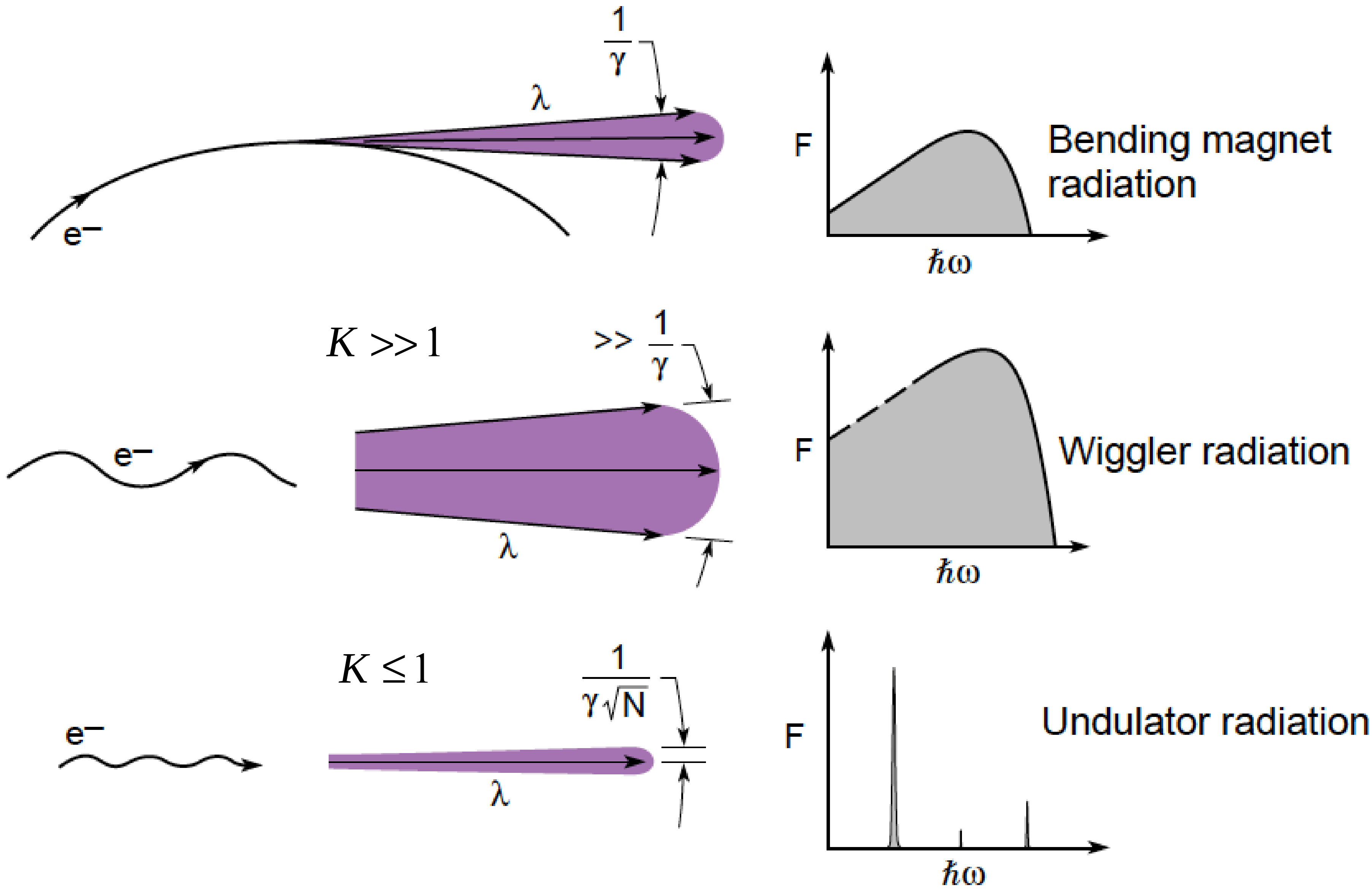
$$\theta_{\text{cen}} = \frac{1}{\gamma^* \sqrt{N}}$$

$$\left.\frac{\Delta\lambda}{\lambda}\right|_{\text{cen}} = \frac{1}{N}$$

$$\bar{P}_{\text{cen}} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u}\frac{K^2}{\left(1 + \frac{K^2}{2}\right)^2} f(K)$$

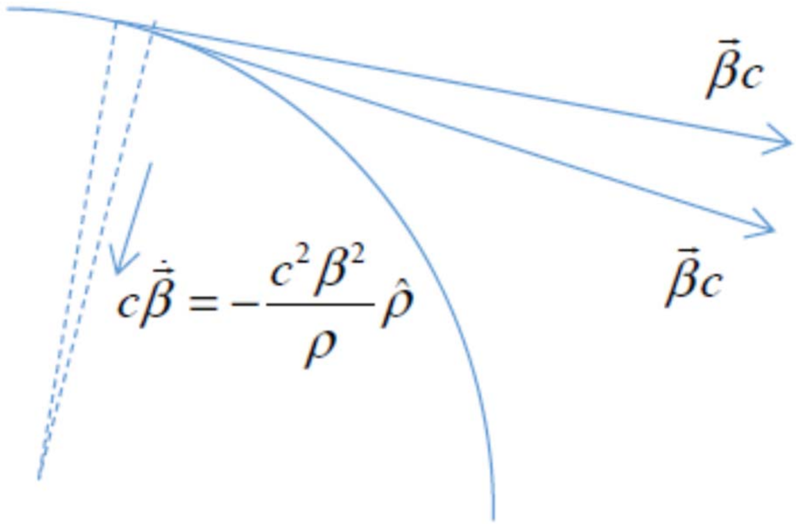
SI units

Comparison of angular spread and radiation bandwidth of different synchrotron radiation sources.



Courtesy of D. Attwood

Circular orbit



$$\dot{a} = -\frac{v^2}{\rho}\hat{\rho} \Rightarrow \dot{\vec{\beta}} = -\frac{\beta^2 c}{\rho}\hat{\rho}$$

$$P(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\dot{\beta}^2\left(1-\beta^2\right) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2c\beta^4\gamma^4}{3\rho^2}$$

For a storage ring, the energy loss per turn:

$$U_0 = \int_C P(t_r)\,dt = \frac{1}{\beta c}\int_C P(t_r)\,ds = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\beta^3\gamma^4}{3}\int_C \frac{1}{\rho^2}ds$$

If all dipoles in the storage ring has the same bending radius (iso-magnetic case):

$$U_0 = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\beta^3\gamma^4}{3}\frac{2\pi\rho}{\rho^2} = \frac{e^2\beta^3\gamma^4}{3\varepsilon_0\rho}$$

Power radiated by a beam of average current I_b:

$$P_{beam} = U_0\frac{I_b}{e} = \frac{e\beta^3\gamma^4}{3\varepsilon_0\rho}I_b$$

Energy spectrum V

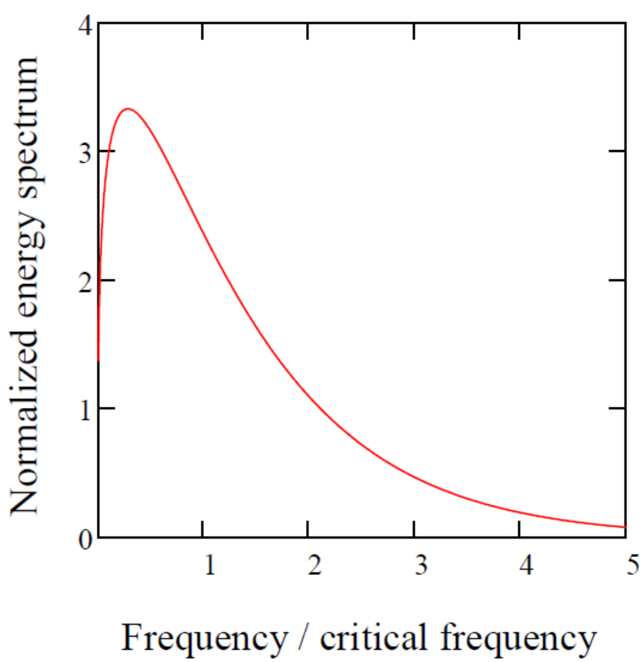
- The total energy spectrum is obtained by integrating over the solid angle:

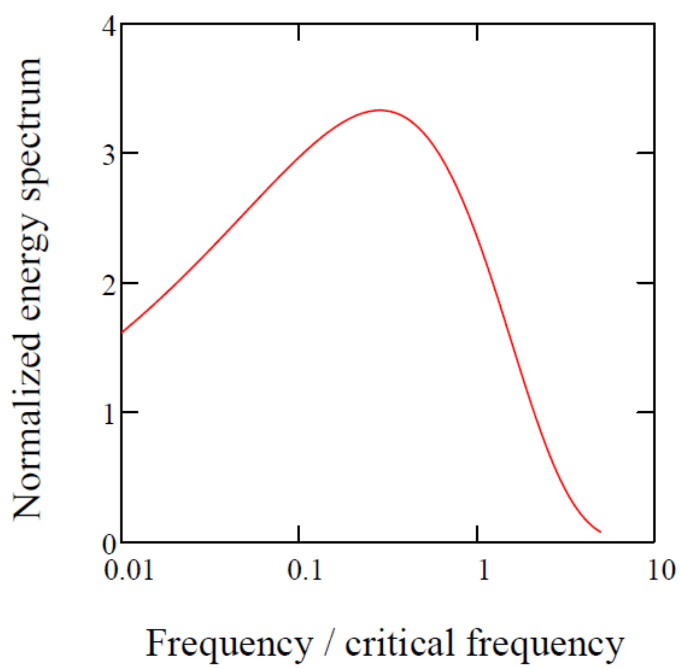
Critical frequency

$$\omega_c = \frac{3}{2}\gamma^3 \frac{c}{\rho} \approx \frac{3}{2}\gamma^3 \omega_0$$

$$\frac{dW}{d\omega} = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d^2 I(\omega)}{d\omega d\Omega} \cos\theta \, d\theta = \frac{2\pi}{\gamma} \int_{-\gamma\frac{\pi}{2}}^{\gamma\frac{\pi}{2}} \frac{d^2 I(\omega)}{d\omega d\Omega} d(\gamma\theta)$$

$$\approx \frac{1}{4\pi\varepsilon_0} \frac{3e^2\gamma}{2\pi c} \frac{\omega^2}{\omega_c^2} \int_{-\infty}^{\infty} \left(1+y^2\right)^2 \left\{ \frac{y^2}{\left(1+y^2\right)} K_{\frac{1}{3}}^{\ 2}\left(\frac{\omega}{2\omega_c} \left(1+y^2\right)^{\frac{3}{2}} \right) + K_{\frac{2}{3}}^{\ 2}\left(\frac{\omega}{2\omega_c} \left(1+y^2\right)^{\frac{3}{2}} \right) \right\} dy$$

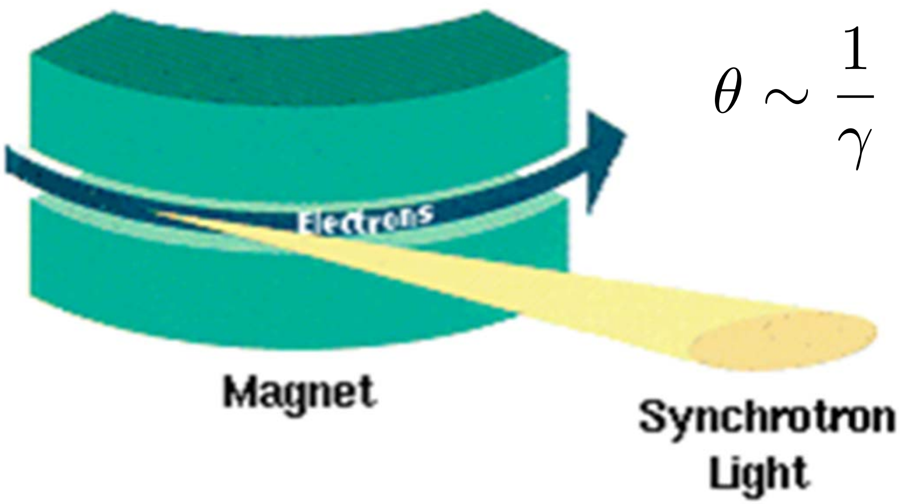




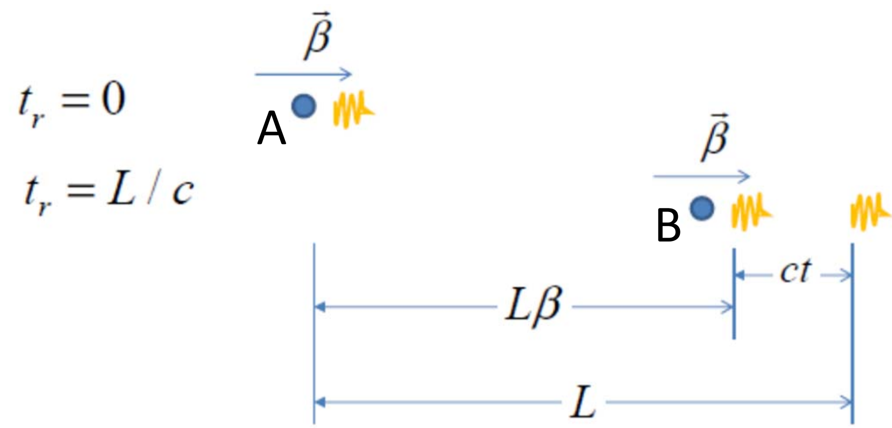
A more concise and popular expression for the energy spectrum:

$$\frac{dW}{d\omega} = \frac{1}{4\pi\varepsilon_0} \sqrt{3} \frac{e^2\gamma}{c} \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{\frac{5}{3}}(x) \, dx$$

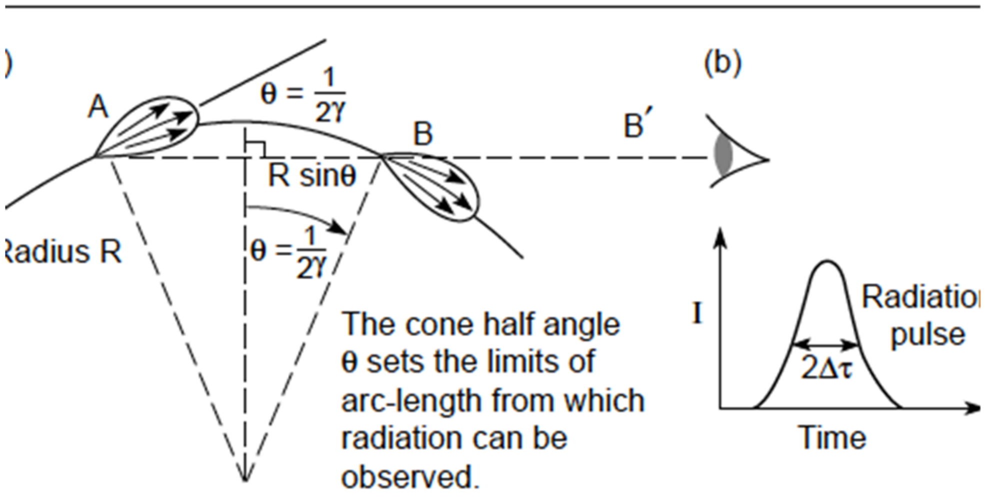
SR from Bending Magnet: simple considerations



$$\theta \sim \frac{1}{\gamma}$$



$$L\beta = \frac{R}{\gamma} \Rightarrow L = \frac{R}{\beta\gamma}$$



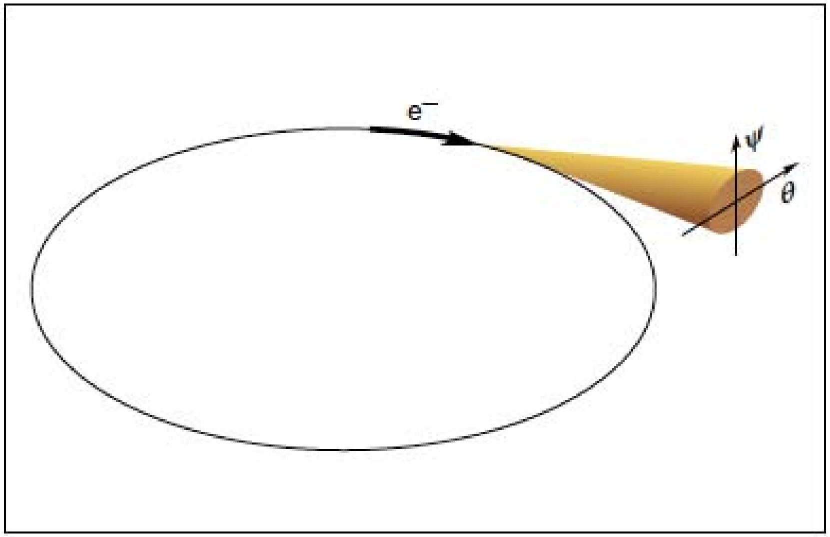
$$2\Delta\tau = \frac{ct}{c} = \frac{L}{c}(1-\beta) \approx \frac{L}{2\gamma^2 c} = \frac{R}{2\beta c\gamma^3} \approx \frac{R}{2\gamma^3 c}$$

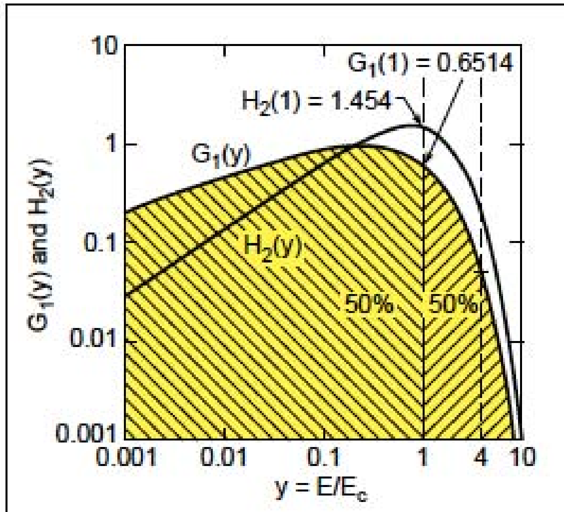
$$\Delta\omega \sim \frac{1}{\Delta\tau} \sim \frac{\gamma^3 c}{R}$$

$$\epsilon_c \equiv \hbar\omega_c = \frac{3}{2}\frac{\hbar c\gamma^3}{\rho}$$

Courtesy of D. Attwood

SR from bending magnet (dipole magnet)





y	$G_1(y)$	$H_2(y)$
0.0010	2.131×10^{-1}	2.910×10^{-2}
0.0100	4.450×10^{-1}	1.348×10^{-1}
0.1000	8.182×10^{-1}	6.025×10^{-1}
0.3000	9.177×10^{-1}	1.111×10^{0}
0.5000	8.708×10^{-1}	1.356×10^{0}
0.7000	7.879×10^{-1}	1.458×10^{0}
1.000	6.514×10^{-1}	1.454×10^{0}
3.000	1.286×10^{-1}	5.195×10^{-1}
5.000	2.125×10^{-2}	1.131×10^{-1}
7.000	3.308×10^{-3}	2.107×10^{-2}
10.00	1.922×10^{-4}	1.478×10^{-3}

$$\frac{m\gamma c^2}{\rho} = ecB \Rightarrow \rho = \frac{m\gamma c}{eB}$$

$$E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \qquad (5.7a)$$

$$E_c(\text{keV}) = 0.6650 E_e^2(\text{GeV}) B(\text{T}) \qquad (5.7b)$$

$$\gamma = \frac{E_e}{mc^2} = 1957 E_e(\text{GeV}) \qquad (5.5)$$

On axis photon flux:

$$\left.\frac{d^3 F_B}{d\theta\, d\psi\, d\omega/\omega}\right|_{\psi=0} = 1.33 \times 10^{13} E_e^2(\text{GeV}) I(\text{A}) H_2(E/E_c) \frac{\text{photons/s}}{\text{mrad}^2 \cdot (0.1\%\ \text{BW})} \qquad (5.6)$$

$$H_2(y) = y^2 K_{2/3}^2(y/2)$$

Integrated photon flux:

$$\frac{d^2 F_B}{d\theta\, d\omega/\omega} = 2.46 \times 10^{13} E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad} \cdot (0.1\%\text{BW})} \qquad (5.8)$$

$$G_1(y) = y \int_y^\infty K_{5/3}(y')\, dy'$$

*The critical photon energy is that for which half the radiated power is in higher energy photons and half is in lower energy photons.

Courtesy of D. Attwood

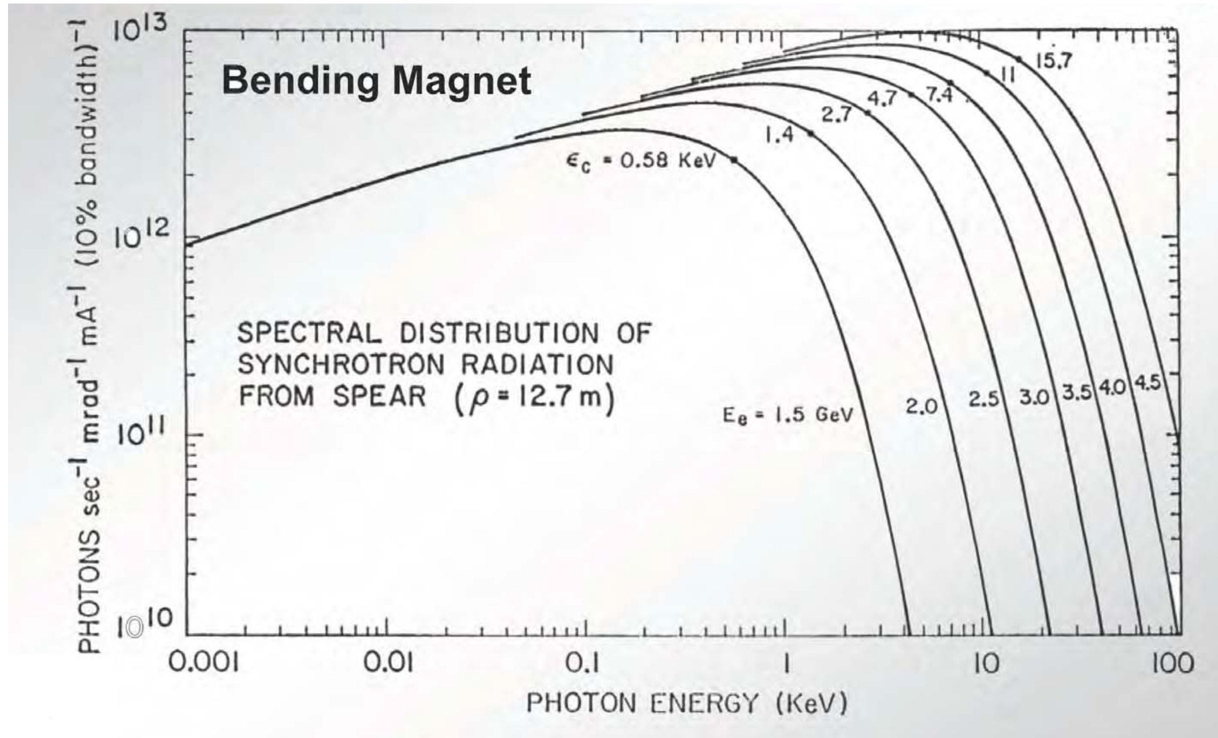
Bending magnet radiation spectrum

*Stanford Positron Electron Accelerating Ring

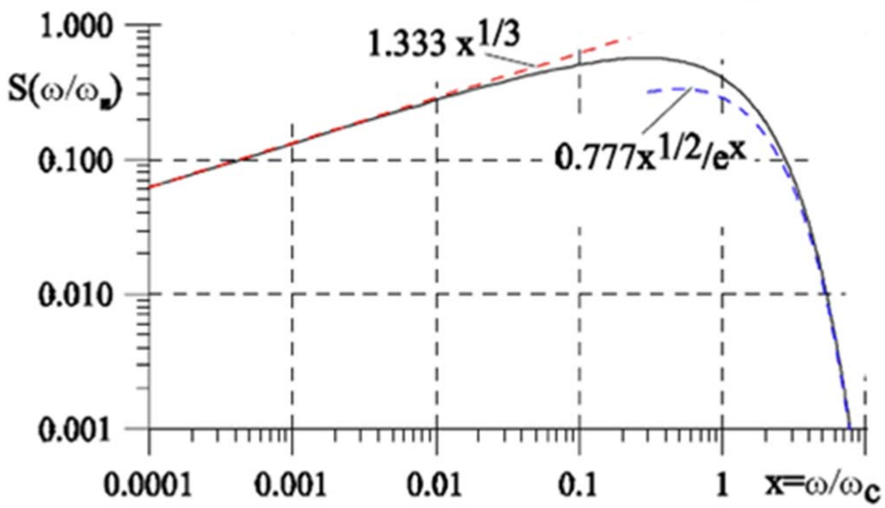
$$\frac{dW}{d\omega} = \frac{1}{4\pi\varepsilon_0}\sqrt{3}\frac{e^2\gamma}{c}\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{\frac{5}{3}}(x)\,dx$$

$$\frac{dW}{d\omega} = \frac{2e^2\gamma}{9\varepsilon_0 c} S\left(\frac{\omega}{\omega_c}\right)$$

$$S(y) = \frac{9\sqrt{3}}{8\pi} y\int_y^{\infty} K_{5/3}(x)\,dx$$

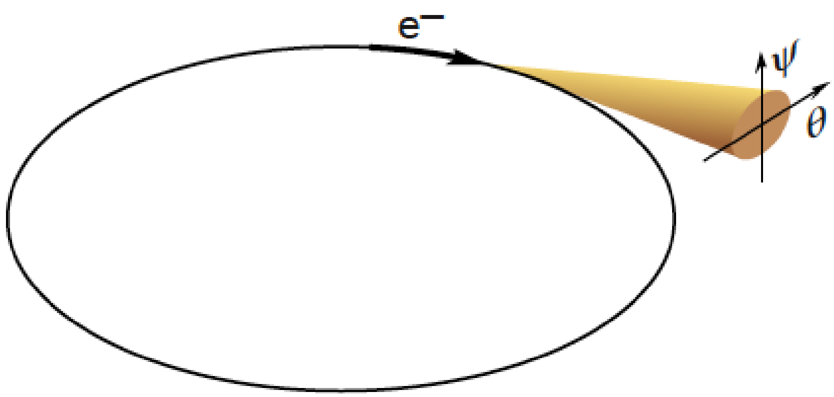


Frequency distribution of the radiated photon flux



Frequency distribution of the radiated energy

Summary of radiation generated with Bending magnets



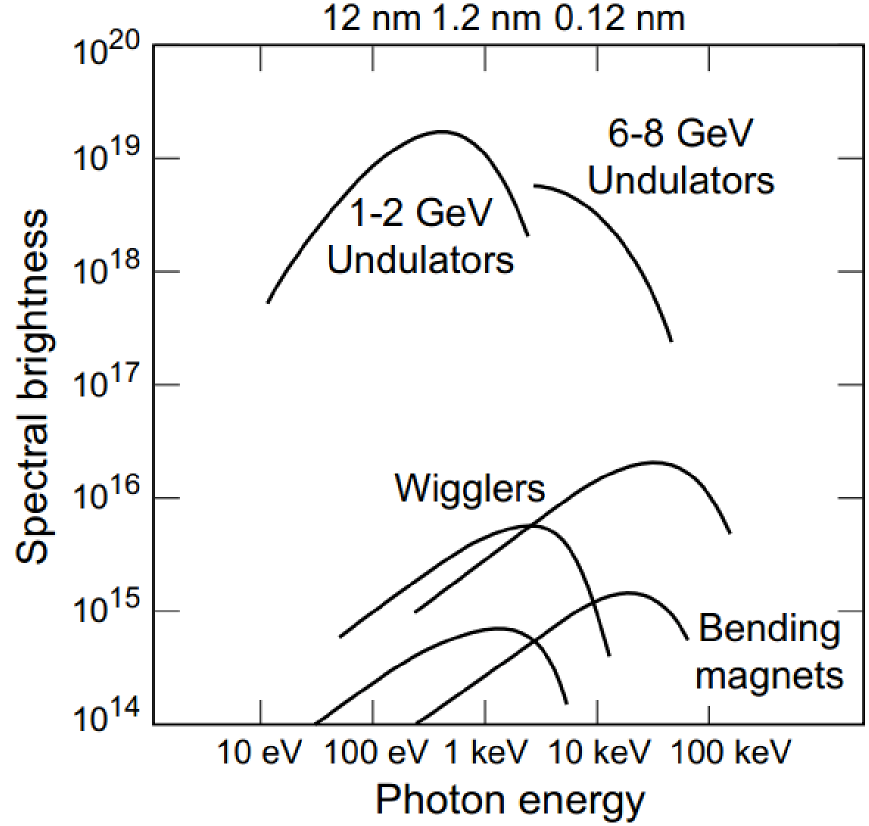
$$E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \qquad (5.7a)$$

$$E_c(\mathrm{keV}) = 0.6650 E_e^2(\mathrm{GeV}) B(\mathrm{T}) \qquad (5.7b)$$

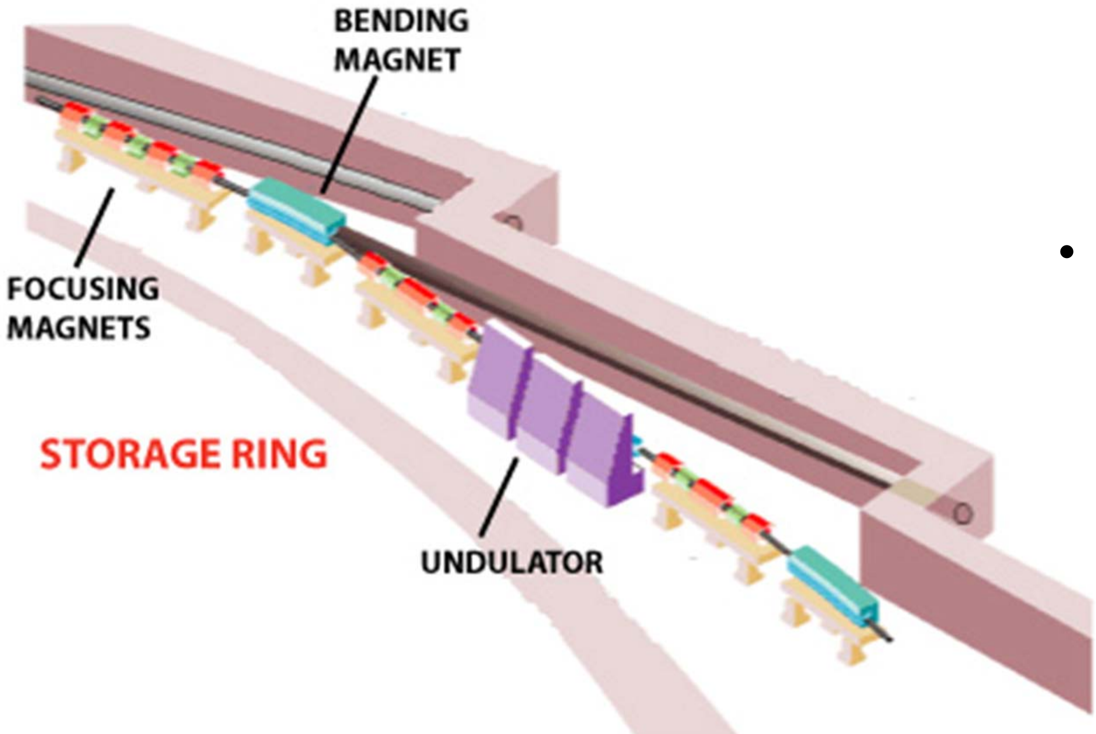
$$\frac{d^2 F_B}{d\theta\, d\omega/\omega} = 2.46 \times 10^{13} E_e(\mathrm{GeV}) I(\mathrm{A}) G_1(E/E_c) \frac{\mathrm{photons/s}}{\mathrm{mrad} \cdot (0.1\%\mathrm{BW})} \qquad (5.8)$$

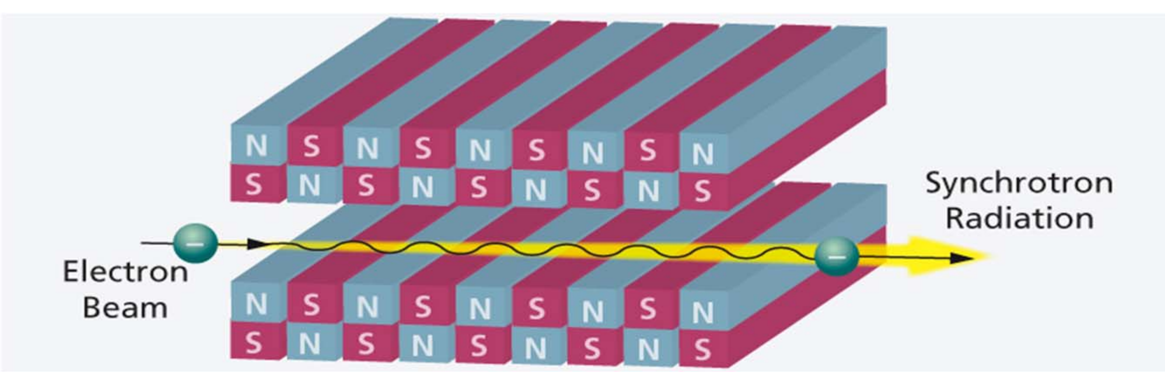
Advantages:
- covers broad spectral range
- least expensive
- most accessable

Disadvantages:
- limited coverage of hard x-rays
- not as bright as undulator



Undulator/Wiggler





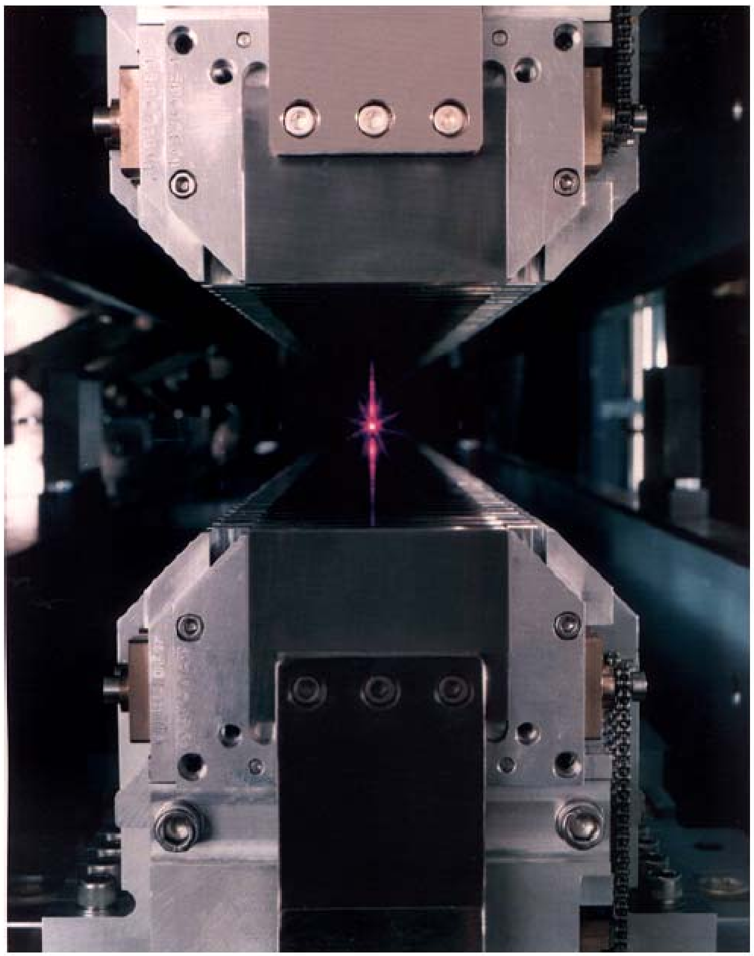
- In addition to the SR from dipoles, modern light sources has many long straight section with zero dispersion function. They frequently used for undulators and wigglers.
- Undulators and wigglers collect radiation from multiple poles: the difference is in coherence of generated radiation

Example: NSLS II
# of DBA cells: 30
# of 5m straights: 15
# of 8m straights: 15

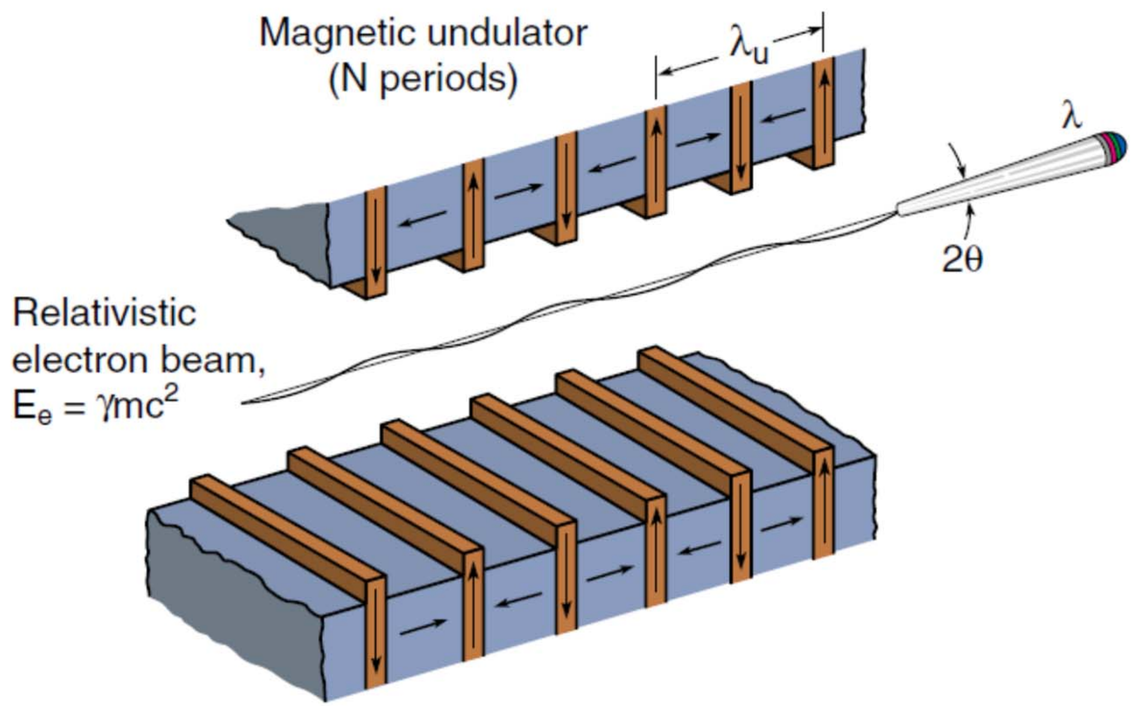
How wiggler/undulator looks like

Helical wiggler for CeC PoP

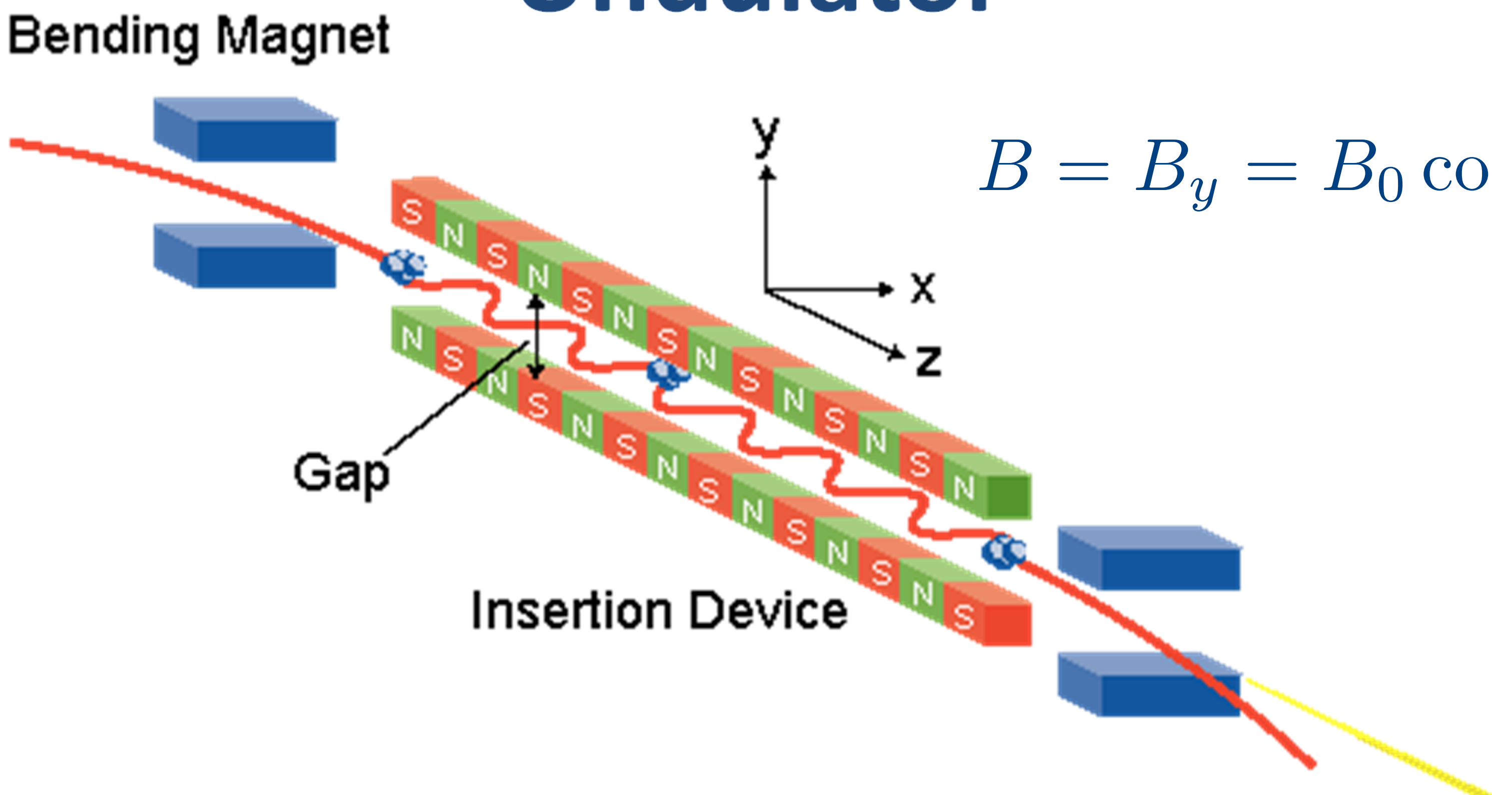
Plannar undulator for Advanced Light Source at LBNL



ALS U5 undulator, beamline 7.0, N = 89, λ_u = 50 mm



Electron Motion inside planar Undulator



$$B = B_y = B_0 \cos \frac{2\pi z}{\lambda_u}$$

$$v_x = \frac{eB_0\lambda_u}{2\pi m\gamma} \sin \frac{2\pi z}{\lambda_u} \equiv \frac{Kc}{\gamma} \sin \frac{2\pi z}{\lambda_u}$$

$$K = \frac{eB_0\lambda_u}{2\pi mc} = 0.934 B_0[T]\lambda_u[cm]$$

← Undulator parameter

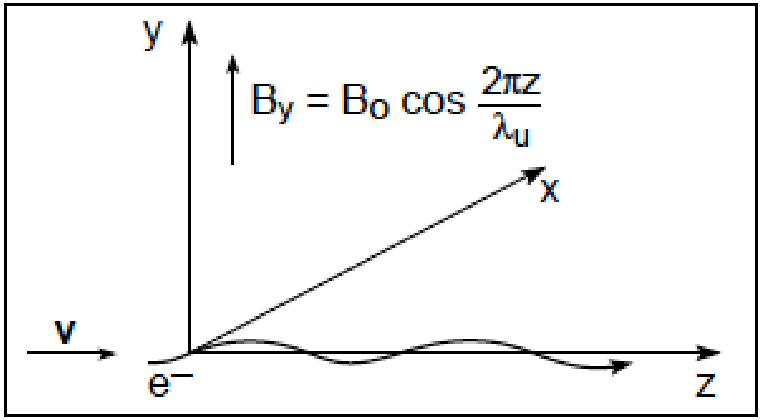
Equation of motion for electrons

Magnetic fields in the periodic undulator cause the electrons to oscillate and thus radiate. These magnetic fields also slow the electrons axial (z) velocity somewhat, reducing both the Lorentz contraction and the Doppler shift, so that the observed radiation wavelength is not quite so short. The force equation for an electron is

$$\frac{d\mathbf{p}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (5.16)$$

where $\mathbf{p} = \gamma m\mathbf{v}$ is the momentum. The radiated fields are relatively weak so that

$$\frac{d\mathbf{p}}{dt} \simeq -e(\mathbf{v} \times \mathbf{B})$$



Taking to first order $v \simeq v_z$, motion in the x-direction is

$$m\gamma \frac{dv_x}{dt} = +ev_z B_y$$

$$m\gamma \frac{dv_x}{dt} = e\frac{dz}{dt} \cdot B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right) \qquad (0 \leq z \leq N\lambda_u)$$

$$m\gamma\, dv_x = e\, dz\, B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right)$$

Courtesy of D. Attwood

Transverse velocity of electrons

$$m\gamma\, d\mathrm{v}_x = e\, dz\, B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right)$$

integrating both sides

$$m\gamma \mathrm{v}_x = e B_0 \frac{\lambda_u}{2\pi} \int \cos\left(\frac{2\pi z}{\lambda_u}\right) \cdot d\left(\frac{2\pi z}{\lambda_u}\right)$$

$$m\gamma \mathrm{v}_x = \frac{e B_0 \lambda_u}{2\pi} \sin\left(\frac{2\pi z}{\lambda_u}\right) \tag{5.17}$$

$$\mathrm{v}_x = \frac{Kc}{\gamma} \sin\left(\frac{2\pi z}{\lambda_u}\right) \tag{5.19}$$

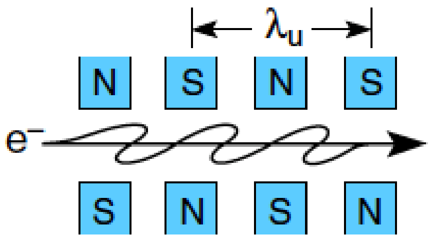
$$K \equiv \frac{e B_0 \lambda_u}{2\pi mc} = 0.9337 B_0(\mathrm{T}) \lambda_u(\mathrm{cm}) \tag{5.18}$$

is the non-dimensional "magnetic deflection parameter."
The "deflection angle", θ, is

$$\theta = \frac{\mathrm{v_x}}{\mathrm{v_z}} \simeq \frac{\mathrm{v_x}}{c} = \frac{K}{\gamma} \sin k_u z$$

Wavelength of undulator radiation

Laboratory Frame of Reference

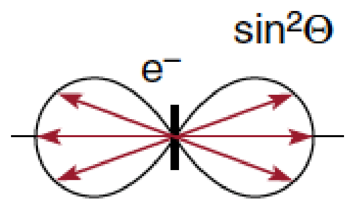


$E = \gamma mc^2$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

N = # periods

Frame of Moving e⁻

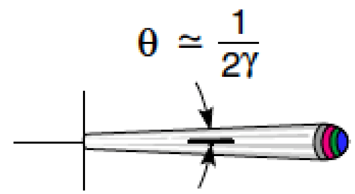


e⁻ radiates at the Lorentz contracted wavelength:

$$\lambda' = \frac{\lambda_u}{\gamma}$$

$$a_x' \approx \frac{K\omega_u'^2}{k_u \gamma} \cos(\omega_u' t')$$

Frame of Observer



Doppler shortened wavelength on axis:

$$\lambda = \lambda' \gamma (1 - \beta \cos\theta)$$

$$\lambda = \frac{\lambda_u}{2\gamma^2}(1 + \gamma^2\theta^2)$$

Accounting for transverse motion due to the periodic magnetic field:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

where $K = eB_0\lambda_u / 2\pi mc$

Courtesy of D. Attwood

Longitudinal velocity of electrons

In a magnetic field γ is a constant; to first order the electron neither gains nor looses energy.

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v_x^2+v_z^2}{c^2}}}$$

thus

$$\frac{v_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{v_x^2}{c^2} \qquad (5.22)$$

$$\frac{v_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{K^2}{\gamma^2}\sin^2\left(\frac{2\pi z}{\lambda_u}\right)$$

Taking the square root, to first order in the small parameter K/γ

$$\frac{v_z}{c} = 1 - \frac{1}{2\gamma^2} - \frac{K^2}{2\gamma^2}\sin^2\left(\frac{2\pi z}{\lambda_u}\right) \qquad (5.23a)$$

Using the double angle formula $\sin^2 k_u z = (1 - \cos 2k_u z)/2$, where $k_u = 2\pi/\lambda_u$,

$$\frac{v_z}{c} = 1 - \frac{1+K^2/2}{2\gamma^2} + \frac{K^2}{4\gamma^2}\cos\left(2\cdot\frac{2\pi z}{\lambda_u}\right)$$

Reduced axial velocity | A double frequency component of the motion

The first two terms show the reduced axial velocity due to the finite magnetic field (K). The last term indicates the presence of harmonic motion, and thus harmonic frequencies of radiation.

Correction to radiation wavelength

Averaging the z-component of velocity over a full cycle (or N full cycles) gives

$$\frac{\bar{v}_z}{c} = 1 - \frac{1 + K^2/2}{2\gamma^2} \qquad (5.25)$$

We can use this to define an effective Lorentz factor γ^* in the axial direction

$$\gamma^* \equiv \frac{\gamma}{\sqrt{1 + K^2/2}} \qquad (5.26)$$

As a consequence, the observed wavelength in the laboratory frame of reference is modified from Eq. (5.12), taking the form

$$\lambda = \frac{\lambda_u}{2\gamma^{*2}}(1 + \gamma^{*2}\theta^2)$$

that is, the Lorentz contraction and relativistic Doppler shift now involve γ^* rather than γ

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\left(1 + \frac{\gamma^2}{1 + K^2/2}\theta^2\right)$$

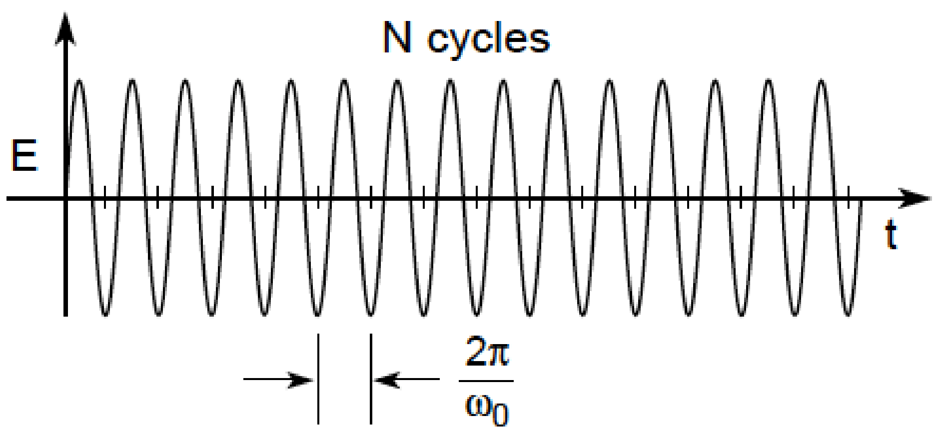
$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)} \qquad (5.28)$$

where $K \equiv e B_o \lambda_u/2\pi mc$. This is the *undulator equation*, which describes the generation of short (x-ray) wavelength radiation by relativistic electrons traversing a periodic magnet structure, accounting for magnetic tuning (K) and off-axis ($\gamma\theta$) radiation. In practical units

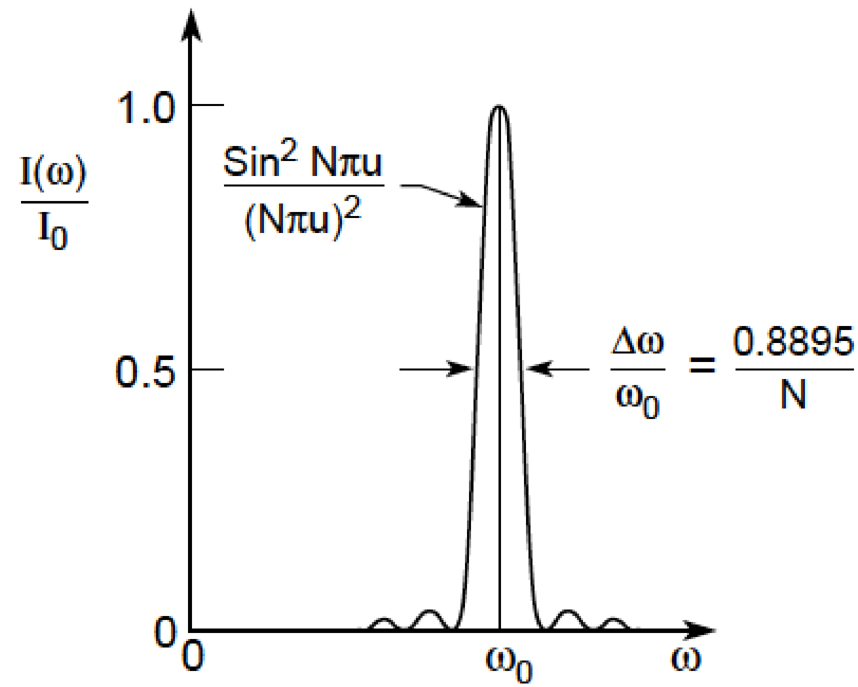
$$\lambda(\text{nm}) = \frac{1.306\lambda_u(\text{cm})\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}{E_e^2(\text{GeV})} \qquad (5.29a)$$

Spectral Bandwidth of Undulator Radiation from a Single Electron

Radiated Wavetrain



Spectral Distribution

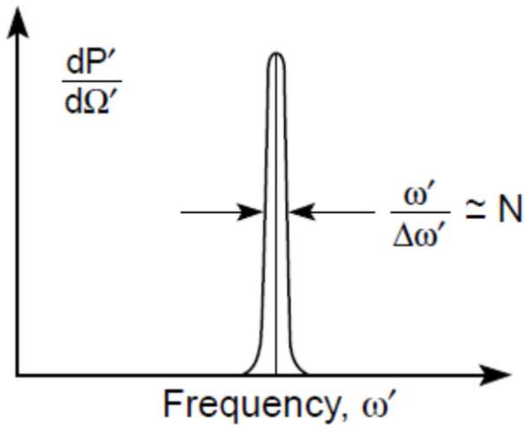


Bandwidth broadening due to Doppler shift for off-axis radiations

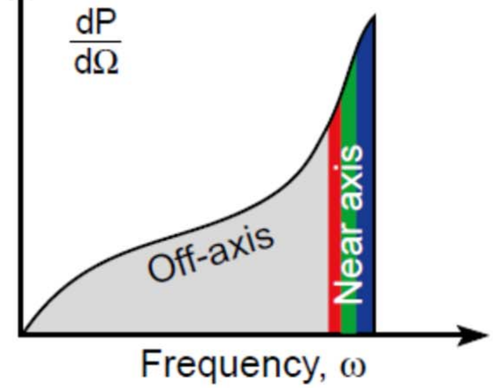
$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

- The low frequency tail in spectrum can be removed by a pinhole.

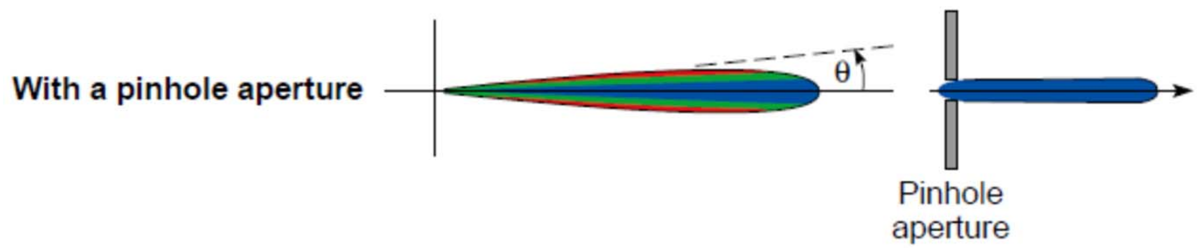
- A natural choice of opening angle of radiation is '**central cone**', i.e. the broadening due to off-axis Doppler shift equals to the bandwidth due to finite duration of radiation train, 1/N



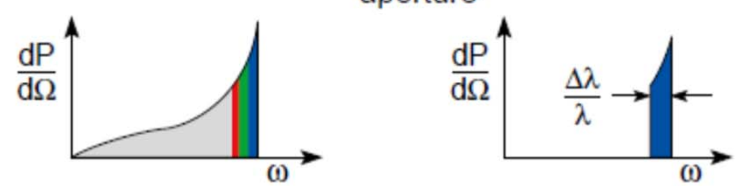
Execution of N electron oscillations produces a transform-limited spectral bandwidth, $\Delta\omega'/\omega' = 1/N$.



The Doppler frequency shift has a strong angle dependence, leading to lower photon energies off-axis.

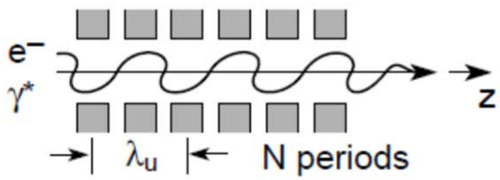


$$\left.\begin{aligned} \frac{\Delta\lambda}{\lambda} &= \frac{\lambda(\theta_{cen}) - \lambda(0)}{\lambda(0)} = \frac{\frac{\lambda_u}{2}\theta_{cen}^{\;2}}{\frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)} = \frac{\gamma^2\theta_{cen}^{\;2}}{1+\frac{K^2}{2}} \\ \frac{\Delta\lambda}{\lambda} &= \frac{1}{N} \end{aligned}\right\} \Rightarrow \theta_{cen} = \sqrt{\frac{1}{N\gamma^2}\left(1+\frac{K^2}{2}\right)} = \frac{1}{\gamma^*\sqrt{N}}$$



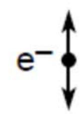
Calculating Power in the Central Radiation Cone: Using the well known "dipole radiation" formula by transforming to the frame of reference moving with the electrons

x, z, t laboratory frame of reference

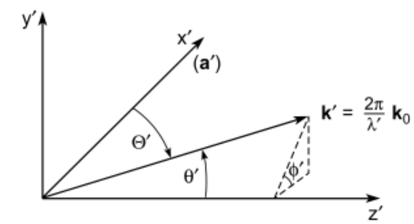


Lorentz transformation

x′, z′, t′ frame of reference moving with the average velocity of the electron



$\lambda'_u = \dfrac{\lambda_u}{\gamma^*}$



x′, z′, t′ motion
$a'(t')$ acceleration

$$a_x' \approx \frac{K\omega_u'^2}{k_u\gamma}\cos(\omega_u' t')$$

$$\omega_u' = \gamma^* k_u c = \frac{\gamma k_u c}{(1+K^2/2)^{1/2}}$$

$$\sin^2\Theta' = 1 - \sin^2\theta'\cos^2\phi'$$

Determine *x, z, t* motion:

$$\frac{d\mathbf{p}}{dt} = -e\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$m\gamma\frac{dv_x}{dt} = e\,\frac{dz}{dt}B_0\cos\frac{2\pi z}{\lambda_u}$$

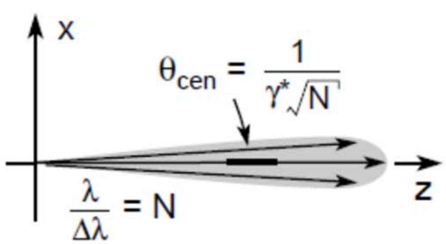
$$v_x(t);\ a_x(t) = \cdots$$

$$v_z(t);\ a_z(t) = \cdots$$

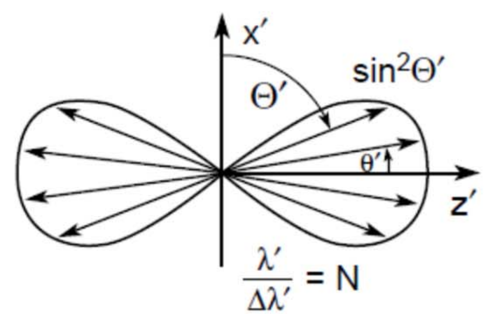
Dipole radiation:

$$\frac{dP'}{d\Omega'} = \frac{e^2 a'^2 \sin^2\Theta'}{16\pi^2\epsilon_0 c^3}$$

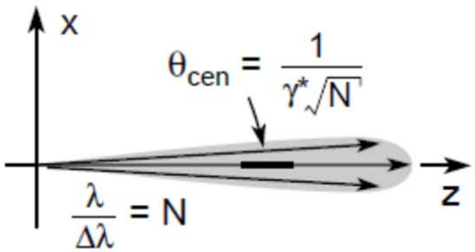
$$\frac{dP'}{d\Omega'} = \frac{e^2 c\gamma^2}{4\epsilon_0\lambda_u^2}\,\frac{K^2}{(1+K^2/2)^2}\,(1-\sin^2\theta'\cos^2\phi')\cos^2\omega_u' t'$$



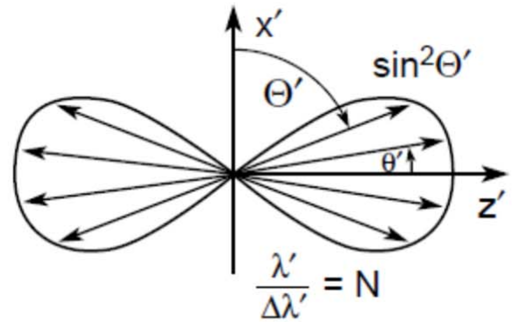
Lorentz transformation



Calculating Power in the Central Radiation Cone: Using the well known “dipole radiation” formula by transforming to the frame of reference moving with the electrons (cont.)



Lorentz transformation



$$\frac{dP}{d\Omega} = 8\gamma^{*2}\frac{dP'}{d\Omega'}$$

$$\frac{dP'}{d\Omega'} = \frac{e^2 c\gamma^2}{4\epsilon_0\lambda_u^2}\frac{K^2}{(1+K^2/2)^2}(1-\sin^2\theta'\cos^2\phi')\cos^2\omega'_u t'$$

$$\frac{d\bar{P}}{d\Omega} = \frac{e^2 c\gamma^4}{\epsilon_0\lambda_u^2}\frac{K^2}{(1+K^2/2)^3}\begin{cases} K \le 1 \\ \theta \le \theta_{cen}\end{cases}$$

$$\Delta\Omega_{cen} = \pi\theta_{cen}^2 = \pi/\gamma^{*2} N$$

$$\bar{P}_{cen} = \frac{\pi e^2 c\gamma^2}{\epsilon_0\lambda_u^2 N}\frac{K^2}{(1+K^2/2)^2}$$

N_e uncorrelated electrons:

$$N_e = IL/ec,\ L = N\lambda_u$$

$$\bar{P}_{cen} = \frac{\pi e\gamma^2 I}{\epsilon_0\lambda_u}\frac{K^2}{(1+K^2/2)^2}$$

Detailed derivation can be found in ‘Soft X-ray and Extreme Ultraviolet Radiation’ by D. Attwood, chapter 5.

Corrections to $\bar{P}_{cen}$ for Finite K

Our formula for calculated power in the central radiation cone ($\theta_{cen} = 1/\gamma^* \sqrt{N}$, $\Delta\lambda/\lambda = 1/N$)

$$\bar{P}_{cen} \simeq \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2} \qquad (5.39)$$

is strictly valid for $K \ll 1$. This restriction is due to our neglect of K^2 terms in the axial velocity v_z. The $\bar{P}_{cen}$ formula, however, indicates a peak power at $K = \sqrt{2}$, suggesting that we explore extension of this very useful analytic result to somewhat higher K values. Kim* has studied undulator radiation for arbitrary K and finds an additional multiplicative factor, $f(K)$, which accounts for energy transfer to higher harmonics:

$$\bar{P}_{cen} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2} f(K) \qquad (5.41a)$$

where

$$f(K) = [J_0(x) - J_1(x)]^2 \qquad (5.40a)$$

and

$$x = K^2/4(1 + K^2/2) \quad \leftarrow \quad x = \frac{K^2}{4\left(1 + \frac{K^2}{2}\right)}$$

$$f(K) = 1 - x - \frac{x^2}{4} + \frac{3x^3}{8} + \cdots \qquad (5.40b)$$

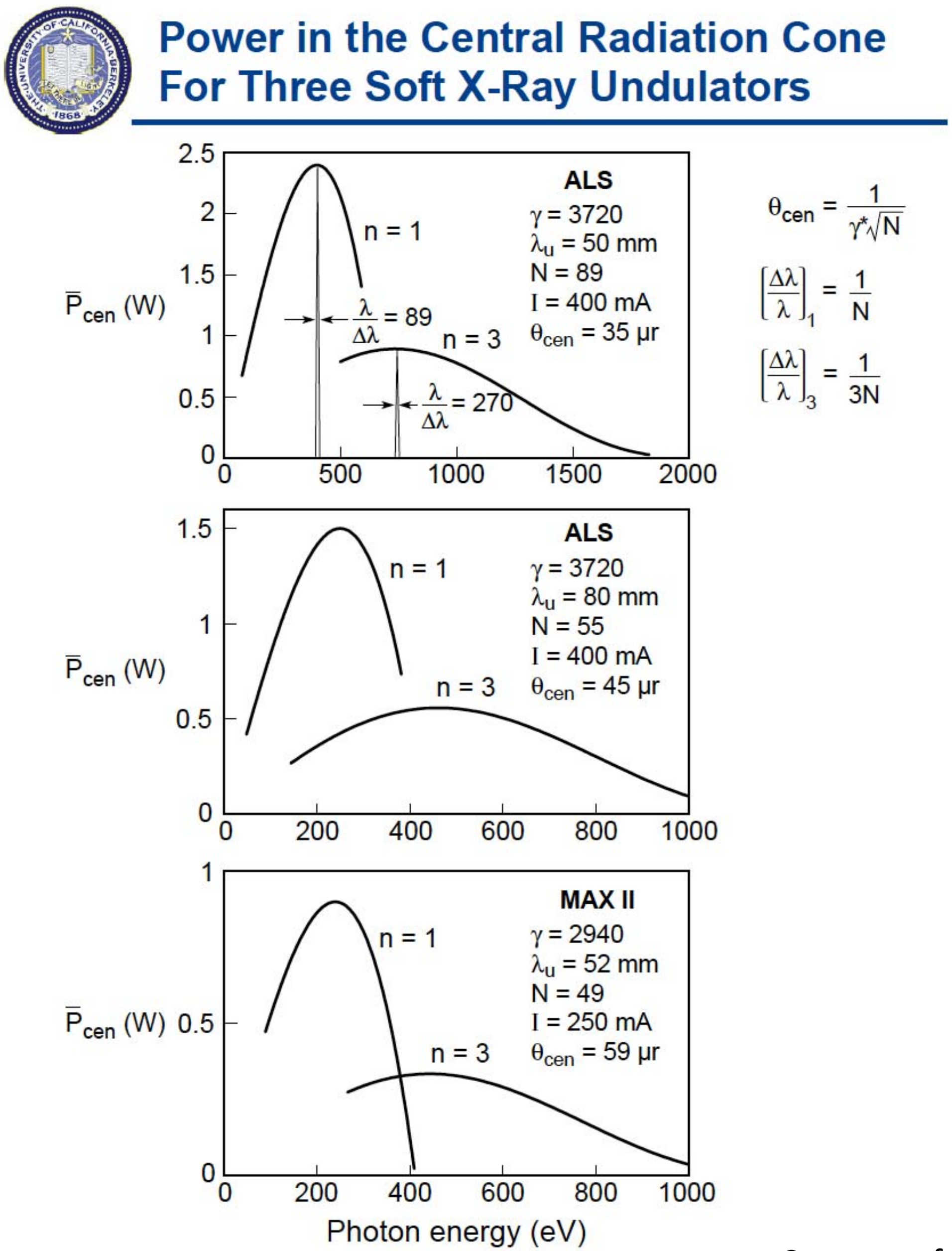
K	x	$f(K)$
0	0	1.000
0.5	0.0556	0.944
1.0	0.1667	0.828
$\sqrt{2}$	0.2500	0.740
1.5	0.2647	0.725
2.0	0.3333	0.653
2.5	0.3788	0.606

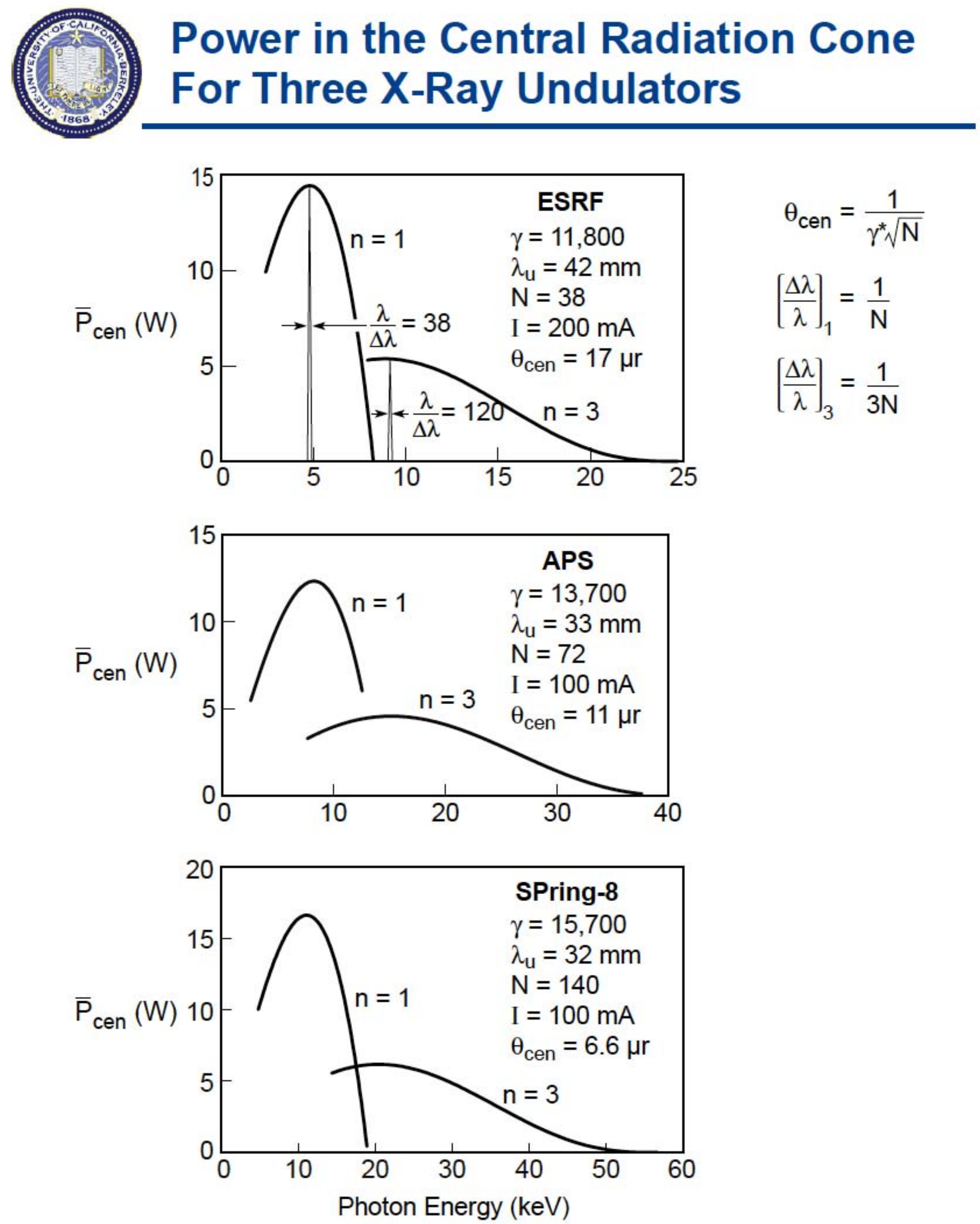
* K.-J. Kim, "Characteristics of Synchrotron Radiation", pp. 565-632 in *Physics of Particle Accelerators* (AIP, New York, 1989), M. Month and M. Dienes, Editors.

Also see: P.J. Duke, *Synchrotron Radiation* (Oxford Univ. Press, UK, 2000).
A. Hofmann, "The Physics of Synchrotron Radiation" (Cambridge Univ. Press, 2004).

Some examples for various light sources

(Tuning curves, i.e. change K to change wavelength)



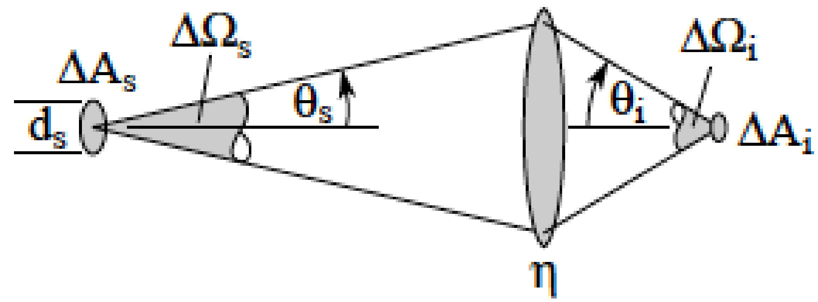


Courtesy of D. Attwood

Brightness and Spectral Brightness

Brightness is defined as radiated power per unit area and per unit solid angle at the source:

$$B = \frac{\Delta P}{\Delta A \cdot \Delta \Omega} \qquad (5.57)$$

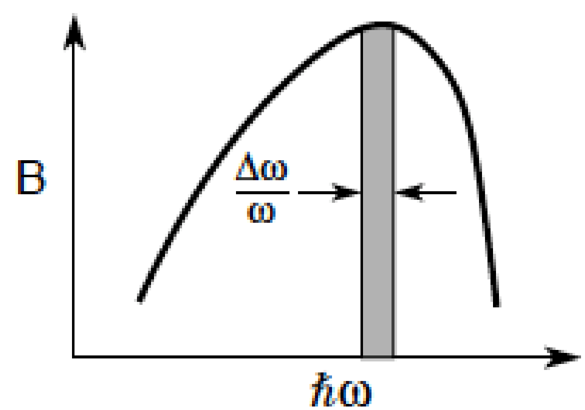


Brightness is a conserved quantity in perfect optical systems, and thus is useful in designing beamlines and synchrotron radiation experiments which involve focusing to small areas.

Perfect optical system:
$\Delta A_s \cdot \Delta \Omega_s = \Delta A_i \cdot \Delta \Omega_i$; $\eta = 100\%$

Spectral brightness is that portion of the brightness lying within a relative spectral bandwidth $\Delta\omega/\omega$:

$$B_{\Delta\omega/\omega} = \frac{\Delta P}{\Delta A \cdot \Delta \Omega \cdot \Delta\omega/\omega} \qquad (5.58)$$



Courtesy of D. Attwood

How electron beam parameters affects spectrum bandwidth?

$$\omega = \frac{4\pi c \gamma_e^2}{\lambda_u \left(1 + \frac{K^2}{2} + \gamma_e^2 \theta^2\right)}$$

$$\frac{\Delta E_{ph}}{E_{ph}} = \frac{2\Delta\gamma_e}{\gamma_e} \le \frac{1}{N} \Rightarrow \frac{\Delta\gamma_e}{\gamma_e} < \frac{1}{2N}$$

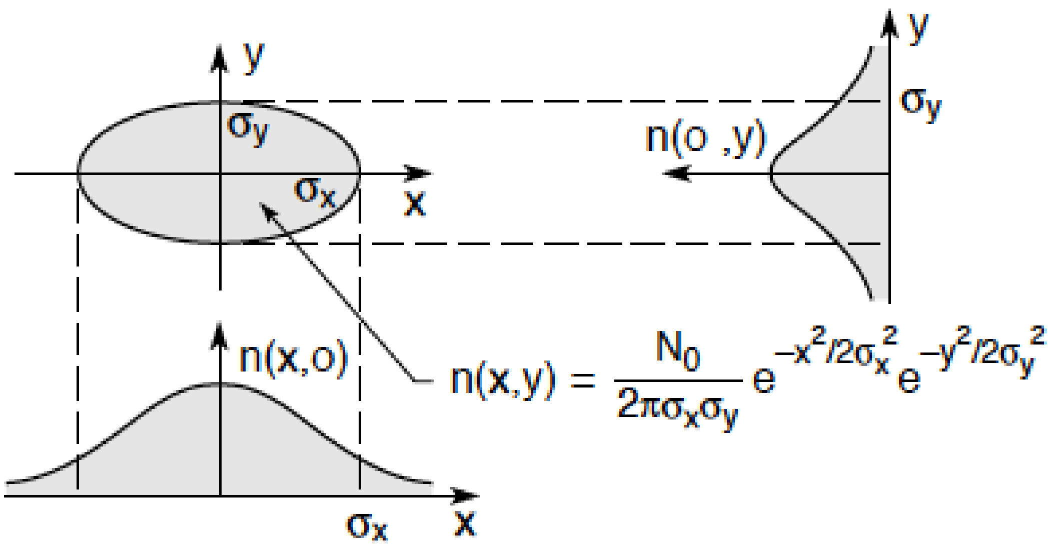
To avoid significant bandwidth broadening due to electron beam quality, energy spread of electrons should be smaller than 1/N, which can be easily satisfied for modern machine.

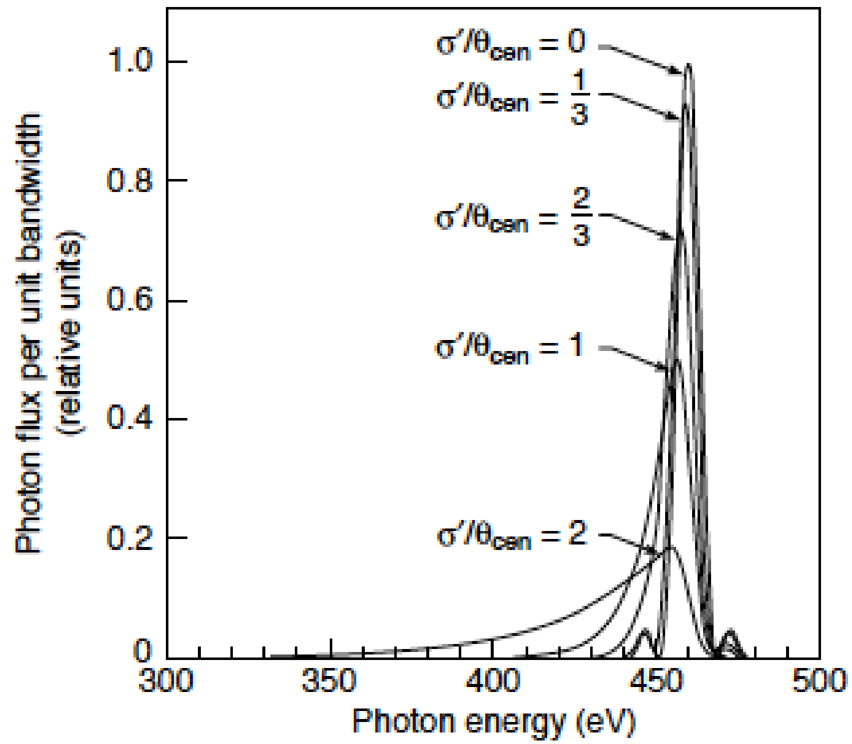
$$\frac{\Delta E_{ph}}{E_{ph}} = \gamma^{*2}\theta^2 \le \frac{1}{N} \Rightarrow \sigma'_{x,y} \le \frac{1}{\gamma^* \sqrt{N}} \Rightarrow \sigma'_{x,y} \le \theta_{cen}$$

Angular spread of electrons should be smaller than the central cone.

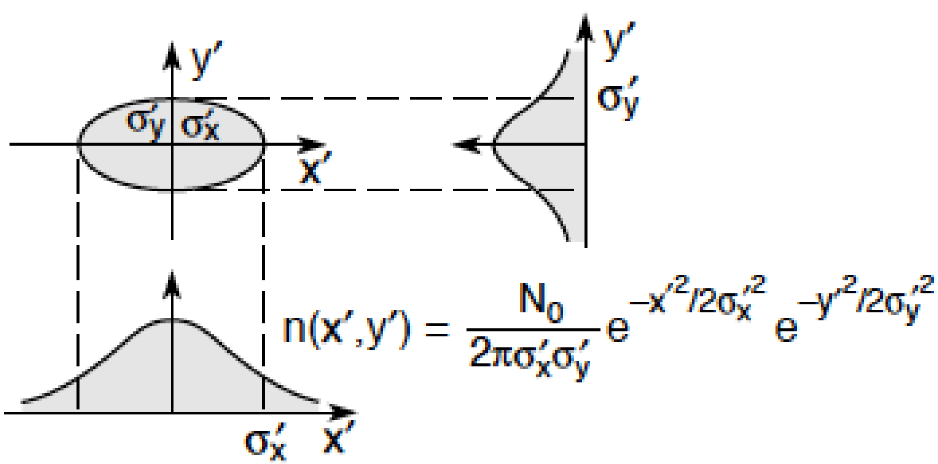
What defines Brightness?

Beam size (σ)





Beam angular divergence (σ′)



Preserving the spectral line shape of undulator radiation requires

$$\sigma'^2 \ll \theta_{cen}^2 \qquad \theta_{cen} \simeq \frac{1}{\gamma^* \sqrt{N}} \qquad (5.55b)$$

Define effective, or total central cone half-angles

$$\theta_{Tx} = \sqrt{\theta_{cen}^2 + \sigma_x'^2} \quad \text{and} \quad \theta_{Ty} = \sqrt{\theta_{cen}^2 + \sigma_y'^2} \qquad (5.56)$$

Courtesy of D. Attwood

Spectral Brightness of Undulator Radiation

The Synchrotron radiation community prefers to express spectral brightness in units of photons/sec, rather than power, and has standardized on a relative spectral bandwidth of $\Delta\omega/\omega = 10^{-3}$, or 0.1% BW. To obtain a relationship for spectral brightness of undulator radiation we can use our expression for $\bar{P}_{cen}$, radiated into a solid angle $\Delta\Omega = \pi\theta_{cen}^2 = \pi\theta_{Tx}\theta_{Ty}$, from an elliptically shaped source area of $\Delta A = \pi\sigma_x\sigma_y$, and within a relative spectral bandwidth $\Delta\omega/\omega = 1/N$. Defining the photon flux in the central radiation cone as

$$\bar{F}_{cen} = \frac{\bar{P}_{cen}}{\hbar\omega/\text{photon}}$$

$$\bar{P}_{cen} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2} f(K)$$

$$\bar{B}_{\Delta\omega/\omega} = \frac{\bar{F}_{cen}}{\Delta A \cdot \Delta\Omega \cdot N^{-1}} = \frac{\bar{F}_{cen} \cdot (N/1000)}{\Delta A \cdot \Delta\Omega \cdot (0.1\%\text{BW})} \qquad (5.60)$$

on-axis

$$\bar{B}_{\Delta\omega/\omega}(0) = \frac{\bar{F}_{cen} \cdot (N/1000)}{2\pi^2 \sigma_x \sigma_y \theta_{Tx} \theta_{Ty} (0.1\%\text{BW})} \qquad (5.64)$$

$$\theta_{cen} \simeq \frac{1}{\gamma^* \sqrt{N}}$$

or

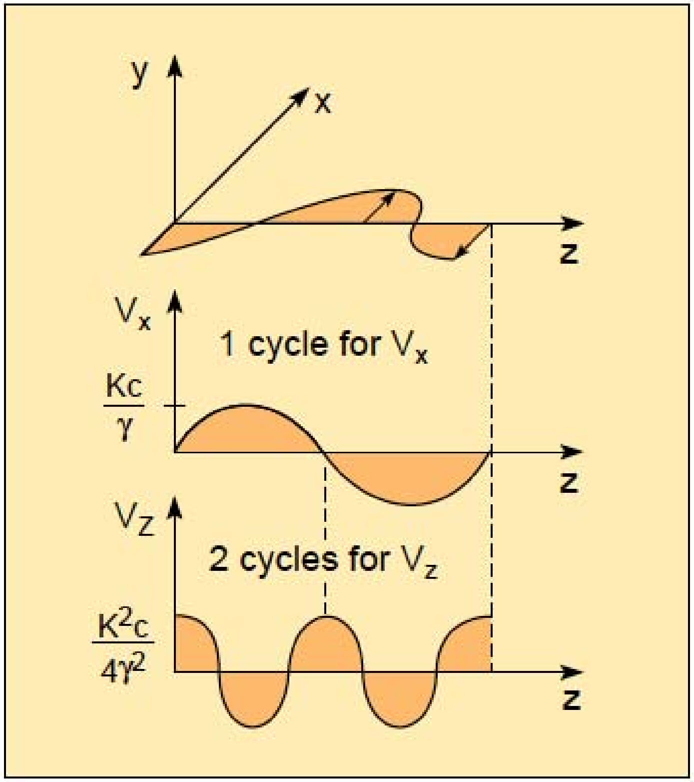
$$\bar{B}_{\Delta\omega/\omega}(0) = \frac{7.25 \times 10^6 \gamma^2 N^2 I(\text{A})}{\sigma_x(\text{mm})\sigma_y(\text{mm}) \left(1 + \frac{\sigma_x'^2}{\theta_{cen}^2}\right)^{1/2} \left(1 + \frac{\sigma_y'^2}{\theta_{cen}^2}\right)^{1/2}} \cdot \frac{K^2 f(K)}{\left(1 + K^2/2\right)^2} \frac{\text{photons/s}}{\text{mm}^2\text{mrad}^2(0.1\%\text{BW})} \qquad (5.65)$$

Assumes $\sigma'^2 << \theta_{cen}^2$. Note the N^2 factor.

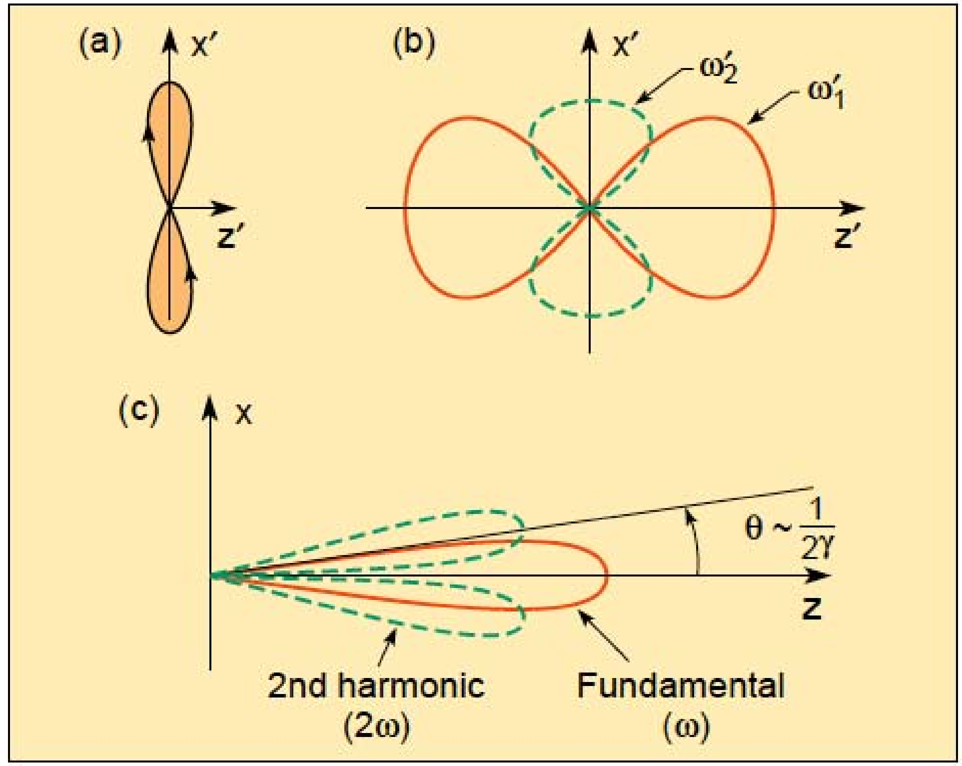
*One N from angular spread and the other from band width

Comments on Undulator Harmonics

First and second harmonic motions



Radiation patterns in the electron and laboratory frames



$$v_x = \frac{Kc}{\gamma} \sin\left(\frac{2\pi z}{\lambda_u}\right)$$

$$v_z = c\left[1 - \frac{1 + K^2/2}{2\gamma^2} + \frac{K^2}{4\gamma^2}\cos(2k_u z)\right]$$

$$\lambda_n = \frac{\lambda_u}{2\gamma^2 n}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right) \tag{5.30}$$

$$\left(\frac{\Delta\lambda}{\lambda}\right)_n = \frac{1}{nN} \tag{5.31}$$

How odd harmonics appear in radiation

Recall that the axial velocity has a double frequency component

$$v_z = c\left[1 - \frac{1 + K^2/2}{2\gamma^2} + \frac{K^2}{4\gamma^2}\cos(2k_u z)\right]$$

which in the frame of reference moving with the electrons, gives

$$z'(t') \simeq \frac{K^2}{8k'_u}\sin 2\omega'_u t' \qquad (5.70)$$

where $k'_u = \gamma^* k_u$ and $\omega'_u = \gamma^* \omega_u$. The transverse motion in this frame is

$$x'(t') \simeq -\frac{K}{k_u \gamma}\cos \omega_u \gamma^* \left(t' + \frac{z'}{c}\right)$$

To a higher degree of accuracy, we now keep the z'/c term

$$x'(t') \simeq -\frac{K}{k'_u}\cos\left(\omega'_u t' + \frac{K^2}{8}\sin 2\omega'_u t'\right) \qquad (5.71)$$

for small K

$$x'(t') \simeq -\frac{1}{k'_u}\left[K\cos \omega'_u t' + \frac{K^3}{16}\cos 3\omega'_u t'\right] \qquad (5.72)$$

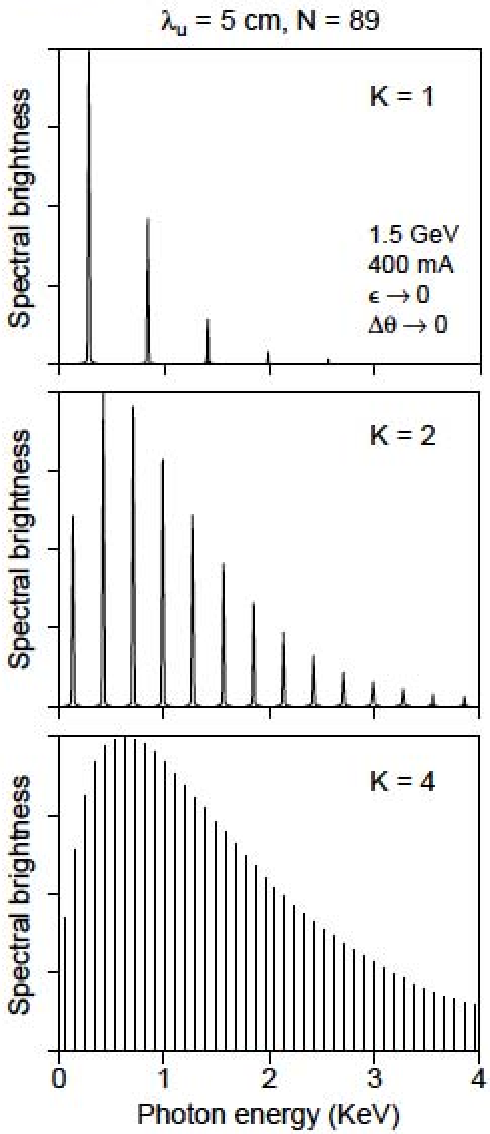
Taking second derivatives to find acceleration, and squaring $|a'(t')|^2$

$$\frac{dP'}{d\Omega'} \propto n^4 K^{2n}$$

$$\frac{dP'}{d\Omega'} = \frac{e^2 a'^2 \sin^2 \Theta'}{16\pi^2 \epsilon_0 c^3}$$

Thus harmonics grow very rapidly for K > 1.

The Transition from Undulator Radiation (K ≲ 1) to Wiggler Radiation (K >> 1)



(Courtesy of K.-J. Kim)

Undulator radiation (K ≲ 1)

- Narrow spectral lines
- High spectral brightness
- Partial coherence

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

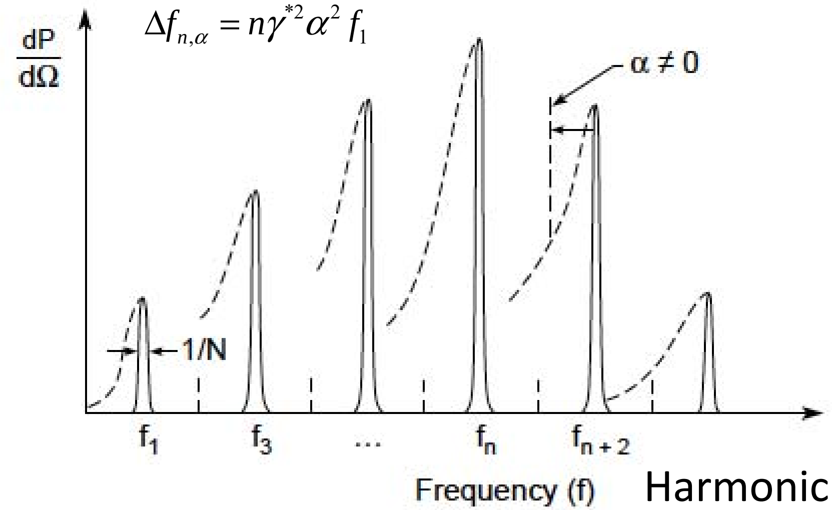
$$K = \frac{eB_o\lambda_u}{2\pi mc}$$

Wiggler radiation (K >> 1)

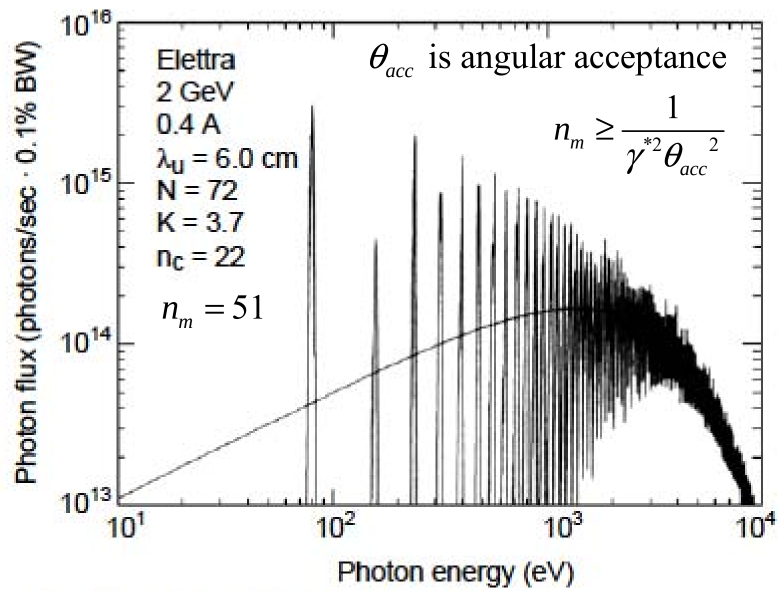
- Higher photon energies
- Spectral continuum
- Higher photon flux (2N)

$$\hbar\omega_c = \frac{3}{2}\frac{\hbar\gamma^2 eB_o}{m}$$

$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right)$$

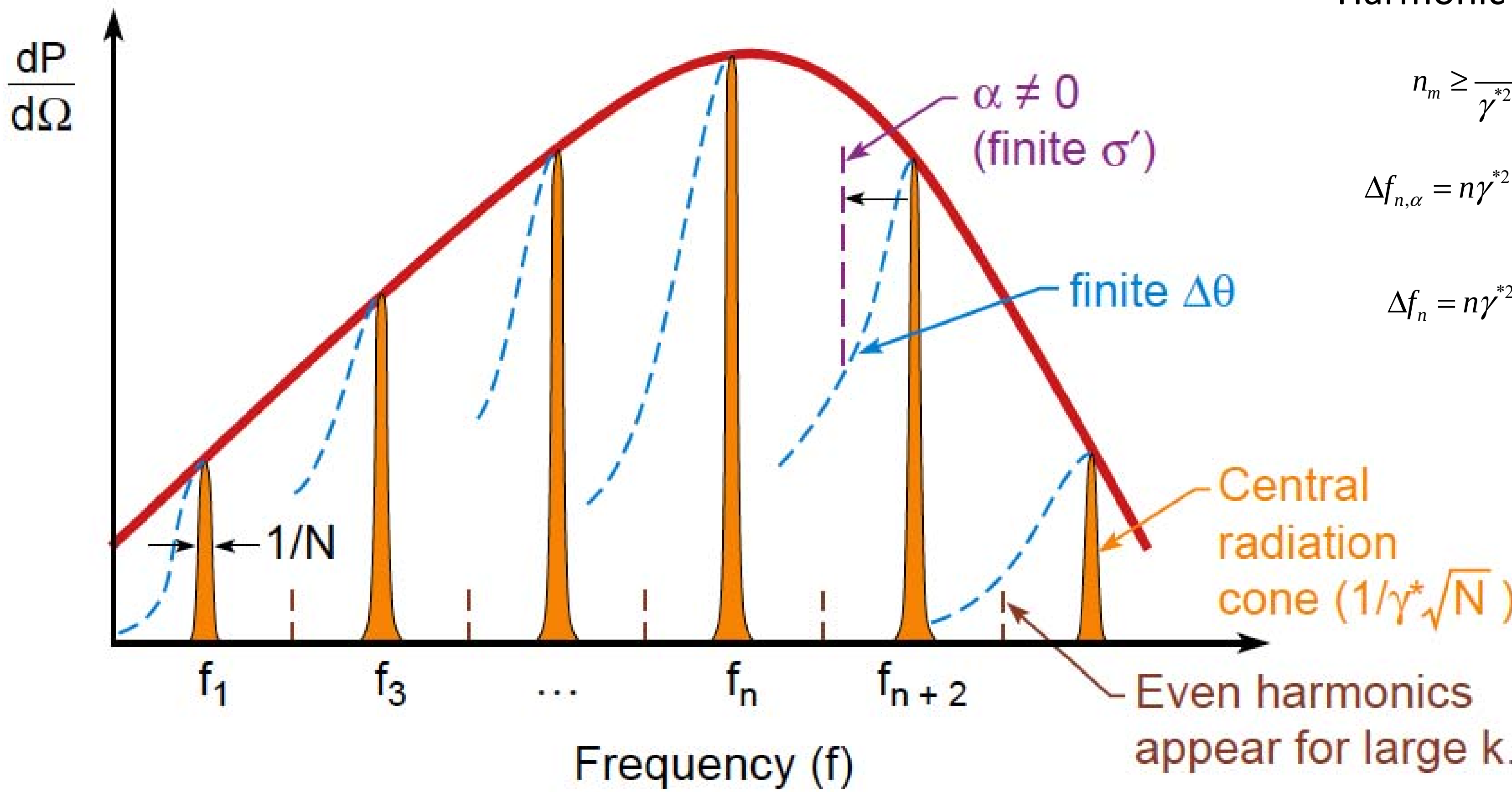


Harmonic merging



(Courtesy of R.P. Walker and B. Diviacco)

For Very Large K >> 1, and Large Dq, a Continuum Emerges



Harmonic merging

$$n_m \geq \frac{1}{\gamma^{*2}\theta_{acc}^{\;2}}$$

$$\Delta f_{n,\alpha} = n\gamma^{*2}\alpha^2 f_1$$

$$\Delta f_n = n\gamma^{*2}\sigma'^2 f_1$$

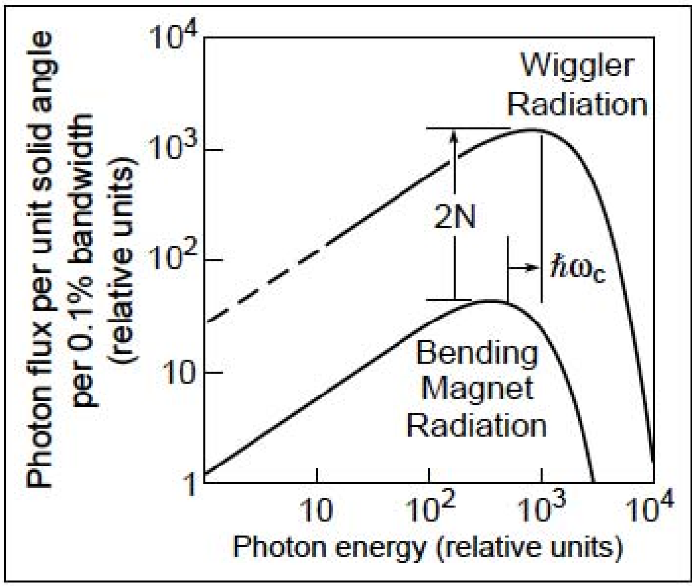
Wiggler Radiation

At very high K >> 1, the radiated energy appears in very high harmonics, and at rather large horizontal angles $\theta \simeq \pm K/\gamma$ (eq. 5.21). Because the emission angles are large, one tends to use larger collection angles, which tends to spectrally merge nearby harmonics. The result is a continuum at very high photon energies, similar to that of bending magnet radiation, but increased by 2N (the number of magnet pole pieces).

$$E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \quad ; \quad n_c = \frac{3K}{4}\left(1+\frac{K^2}{2}\right) \qquad (5.7a \,\&\, 82)$$

$$\left.\frac{d^2F}{d\theta\, d\Psi\, d\omega/\omega}\right|_0 = 2.65\times 10^{13} N E_e^2(\text{GeV}) I(\text{A}) H_2(E/E_c) \frac{\text{photons/s}}{\text{mrad}^2(0.1\%\text{BW})} \qquad (5.86)$$

$$\frac{d^2F}{d\theta\, d\omega/\omega} = 4.92\times 10^{13} N E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad}\cdot(0.1\%\text{BW})} \qquad (5.87)$$



Critical harmonic number

$$\omega_c = \frac{3eB\gamma^2}{2m} = \frac{3\pi c\gamma^2 K}{\lambda_u} \qquad K = \frac{eB\lambda_u}{2\pi mc}$$

$$\omega_1 = \frac{2\pi c}{\lambda_1} = \frac{4\pi c\gamma^2}{\lambda_u\left(1+K^2/2\right)}$$

$$n_c = \frac{\omega_c}{\omega_1} = \frac{3K}{4}\left(1+K^2/2\right)$$

What are the Relative Merits?

Bending magnet radiation

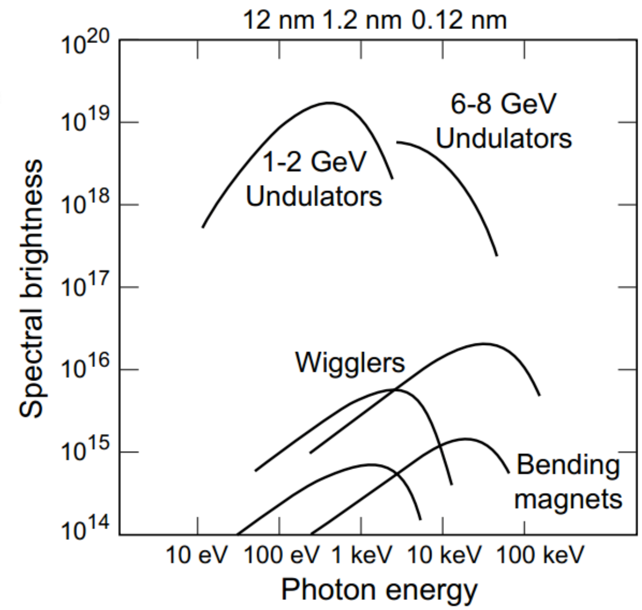
- Broad spectrum
- Good photon flux
- No heat load
- Less expensive
- Easier access

Wiggler radiation

- Higher photon energies
- More photon flux
- Expensive magnet structure
- Expensive cooled optics
- Less access

Undulator radiation

- Brighter radiation
- Smaller spot size
- Partial coherence
- Expensive
- Less access



Intense synchrotron radiation from high power wiggler sources has long been a difficult high-heat-load problem to the design of properly cooled x-ray optics

SUMMARY

- SR has a wide variety of applications
- Light sources are mostly storage ring based
- Bending magnet SR is broad band, high power, but not very bright when compared to undulator radiation.
- Undulator radiation is brightest among radiation sources: its spectral brightness is proportional to square of the number of periods.
- Undulators can produce also very bright radiation on harmonics.
- Wiggler is an undulator with very large field whose harmonics are overlapped (because of the electron beam parameters!) and its power and brightness is proportional to number of periods
- Ultimately, electron beam parameters (beam current, emittances and energy spread) are determining performance of the light sources

Typical parameters for various Synchrotron light source facilities

Facility	ALS	ELETTRA	Australian Synchrotron	APS
Electron energy	1.90 GeV	2.0 GeV	3.0 GeV	7.00 GeV
γ	3720	3910	5871	13,700
Current (mA)	400	300	200	100
Circumference (m)	197	259	216	1100
RF frequency (MHz)	500	500	500	352
Pulse duration (FWHM) (ps)	35-70	37	~100	100
Bending Magnet Radiation:				
Bending magnet field (T)	1.27	1.2	1.31	0.599
Critical photon energy (keV)	3.05	3.2	7.84	19.5
Critical photon wavelength	0.407 nm	0.39 nm	1.58 Å	0.636 Å
Bending magnet sources	24	12	28	35
Undulator Radiation:				
Number of straight sections	12	12	14	40
Undulator period (typical) (cm)	5.00	5.6	22.0	3.30
Number of periods	89	81	80	72
Photon energy ($K = 1$, $n = 1$)	457 eV	452 eV	2.59 keV	9.40 keV
Photon wavelength ($K = 1$, $n = 1$)	2.71 nm	2.74 nm	0.478 nm	1.32 Å
Tuning range ($n = 1$)	230-620 eV	2.0-6.7 nm	0.319-0.835 nm	3.5-12 keV
Tuning range ($n = 3$)	690-1800 eV	0.68-2.2 nm	0.106-0.278 nm	10-38 keV
Central cone half-angle ($K = 1$)	35 µrad	35 µrad	23 µrad	11 µrad
Power in central cone ($K = 1$, $n = 1$) (W)	2.3	1.7	6.6	12
Flux in central cone (photons/s)	3.1×10^{16}	2.3×10^{16}	1.6×10^{16}	7.9×10^{15}
σ_x, σ_y (µm)	260, 16	255, 23	320, 16	320, 50
σ'_x, σ'_y (µrad)	23, 3.9	31, 9	34, 6	23, 7
Brightness ($K = 1$, $n = 1$)[a] [(photons/s)/mm^2 · mrad^2 · (0.1%BW)]	2.3×10^{19}	9.9×10^{18}	1.3×10^{19}	5.9×10^{18}
Total power ($K = 1$, all n, all θ) (W)	83	126	476	350
Other undulator periods (cm)	3.65, 8.00, 10.0	8.0, 12.5	6.8, 18.3	2.70, 5.50, 12.8
Wiggler Radiation:				
Wiggler period (typical) (cm)	16.0	14.0	6.1	8.5
Number of periods	19	30	30	28
Magnetic field (maximum) (T)	2.1	1.5	1.9	1.0
K (maximum)	32	19.6	12	7.9
Critical photon energy (keV)	5.1	4.0	11.4 keV	33
Critical photon wavelength	0.24 nm	0.31 nm	0.11 nm	0.38 Å
Total power (max. K) (kW)	13	7.2	9.3	7.4

[a]Using Eq. (5.65). See comments following Eq. (5.64) for the case where $\sigma'_{x,y} \simeq \theta_{cen}$.