

## PHY554: HOMEWORK 3

### 1. PROBLEM 1

From the lecture notes, we have

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

where the transfer matrix  $M(s_2, s_1)$  is

$$\begin{aligned} M(s_2, s_1) &= \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos \mu + \alpha_1 \sin \mu) & \sqrt{\beta_1 \beta_2} \sin \mu \\ -\frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin \mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1 \beta_2}} \cos \mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos \mu - \alpha_1 \sin \mu) \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos \mu & \sin \mu \\ -\sin \mu & \cos \mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix} \end{aligned}$$

We can solve  $X(s_2)$

$$X(s_2) = \sqrt{\frac{\beta_2}{\beta_1}}(\cos \mu + \alpha_1 \sin \mu) \cdot X(s_1) + \sqrt{\beta_1 \beta_2} \sin \mu \cdot X'(s_1)$$

If a particle is kicked at  $s_1$  by angle  $\theta$ , we have

$$\begin{pmatrix} X(s_2) + \Delta x_2 \\ X'(s_2) + \Delta x'_2 \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) + \theta \end{pmatrix}$$

We can solve  $X(s_2) + \Delta x_2$

$$X(s_2) + \Delta x_2 = \sqrt{\frac{\beta_2}{\beta_1}}(\cos \mu + \alpha_1 \sin \mu) \cdot X(s_1) + \sqrt{\beta_1 \beta_2} \sin \mu \cdot (X'(s_1) + \theta)$$

Taking difference between  $X(s_2)$  and  $X(s_2) + \Delta x_2$ , we have

$$\Delta x_2 = \theta \sqrt{\beta_1 \beta_2} \sin \mu$$

$\Delta x_2$  is proportion to  $\sqrt{\beta_1}$ , and  $\beta_1$  is the  $\beta$  function at the kicker location.

To obtain the maximum kicker strength, the kicker should be located in the position where  $\beta$  function reaches maximum. To obtain the minimum kicker strength, the kicker should be located in the position where  $\beta$  function reaches minimum.

### 2. PROBLEM 2

**2.1. Maximum.** Maximum betatron functions are located at center of QFs, so a FODO cell is arranged as

$$QF/2 \Rightarrow B \Rightarrow QD \Rightarrow B \Rightarrow QF/2$$

Assuming the quadrupoles are thin lens, the corresponding transfer matrix is

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix} \end{aligned}$$

The transfer matrix can also be written as

$$M = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix}$$

So we can solve that

$$\begin{aligned} \cos \Phi &= \frac{1}{2} \text{Tr}(M) \\ &= 1 - \frac{L_1^2}{2f^2} \end{aligned}$$

And with  $\cos \Phi = 1 - \sin^2 \frac{\Phi}{2}$ , we can solve that  $\sin \frac{\Phi}{2} = \frac{L_1}{2f}$ , and we have

$$\begin{aligned} \alpha &= 0 \\ \beta &= \frac{2L_1(1 + \frac{L_1}{2f})}{\sin \Phi} \\ &= \frac{2L_1(1 + \sin \frac{\Phi}{2})}{\sin \Phi} \end{aligned}$$

In this problem, number of FODO cells is  $n_{FODO} = 12$ , circumference is  $L = 180m$ , so  $L_1 = L/n_{FODO}/2 = 7.5m$ .

The betatron tunes are  $Q_x = 3.5, Q_y = 3.4$ , so the phase advance for each FODO cell should be

$$\begin{aligned} \Phi_x &= \frac{2\pi Q_x}{n_{FODO}} \\ &= \frac{7\pi}{12} \\ \Phi_y &= \frac{2\pi Q_y}{n_{FODO}} \\ &= \frac{6.8\pi}{12} \end{aligned}$$

Therefore

$$\begin{aligned} \beta_{x,max} &= \frac{2L_1(1 + \sin \frac{\Phi_x}{2})}{\sin \Phi_x} \\ &= 27.85m \\ \beta_{y,max} &= \frac{2L_1(1 + \sin \frac{\Phi_y}{2})}{\sin \Phi_y} \\ &= 27.25m \end{aligned}$$

2.2. **Minimum.** Minimum betatron functions are located at center of QDs, so a FODO cell is arranged as

$$QD/2 \Rightarrow B \Rightarrow QF \Rightarrow B \Rightarrow QD/2$$

Assuming the quadrupoles are thin lens, the corresponding transfer matrix is

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 - \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 + \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix} \end{aligned}$$

The transfer matrix can also be written as

$$M = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\gamma \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix}$$

So we can solve that

$$\begin{aligned} \cos \Phi &= \frac{1}{2} Tr(M) \\ &= 1 - \frac{L_1^2}{2f^2} \end{aligned}$$

And with  $\cos \Phi = 1 - \sin^2 \frac{\Phi}{2}$ , we can solve that  $\sin \frac{\Phi}{2} = \frac{L_1}{2f}$ , and we have

$$\begin{aligned} \alpha &= 0 \\ \beta &= \frac{2L_1(1 - \frac{L_1}{2f})}{\sin \Phi} \\ &= \frac{2L_1(1 - \sin \frac{\Phi}{2})}{\sin \Phi} \end{aligned}$$

In this problem, number of FODO cells is  $n_{FODO} = 12$ , circumference is  $L = 180m$ , so  $L_1 = L/n_{FODO}/2 = 7.5m$ .

The betatron tunes are  $Q_x = 3.5, Q_y = 3.4$ , so the phase advance for each FODO cell should be

$$\begin{aligned} \Phi_x &= \frac{2\pi Q_x}{n_{FODO}} \\ &= \frac{7\pi}{12} \\ \Phi_y &= \frac{2\pi Q_y}{n_{FODO}} \\ &= \frac{6.8\pi}{12} \end{aligned}$$

Therefore

$$\begin{aligned} \beta_{x,min} &= \frac{2L_1(1 - \sin \frac{\Phi_x}{2})}{\sin \Phi_x} \\ &= 3.21m \\ \beta_{y,min} &= \frac{2L_1(1 - \sin \frac{\Phi_y}{2})}{\sin \Phi_y} \\ &= 3.42m \end{aligned}$$

**2.3. Chamber size.** Now we have maximum of betatron functions and RMS beam emittance, we can calculate the RMS beam size

$$\begin{aligned}
 \sigma_x &= \sqrt{\beta_{x,max}\varepsilon} \\
 &= \sqrt{27.85 \cdot 1e-6} \\
 &= 5.28e-3m \\
 \sigma_y &= \sqrt{\beta_{y,max}\varepsilon} \\
 &= \sqrt{27.25 \cdot 1e-6} \\
 &= 5.22e-3m
 \end{aligned}$$

As we know, in a 1D normal distribution, integral of density function from  $-8\sigma$  to  $8\sigma$  will be more than 99%. So if we take  $8 \cdot \max(\sigma_x, \sigma_y)$  as the chamber radius, the vacuum chamber will be large enough to house such beam. So the vacuum chamber size should be at least  $4.224e-2m$  in radius or  $8.448e-2m$  in diameter.