PHY 564
Advanced Accelerator Physics
Lecture 9
Linear accelerators and RF systems for storage rings:

this is a recap of RF lectures in the into-AP

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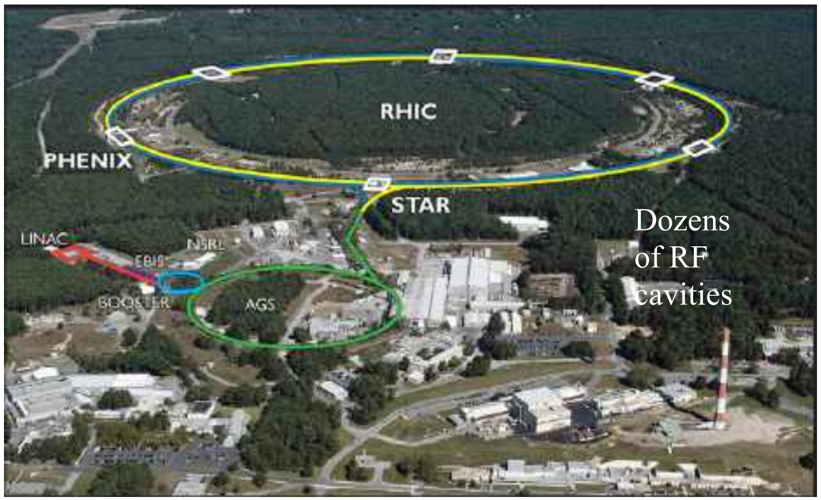
Acknowledgement

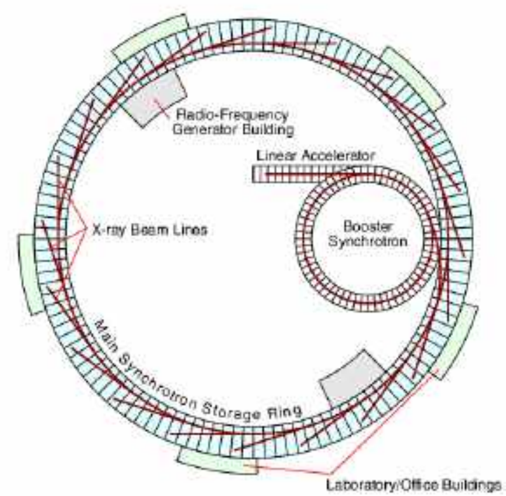
I used some materials from courses on RF and Superconducting RF (SRF) accelerators taught by Prof. S. Belomestnykh at SBU/BNL and USPAS, which can be found on the following websites:

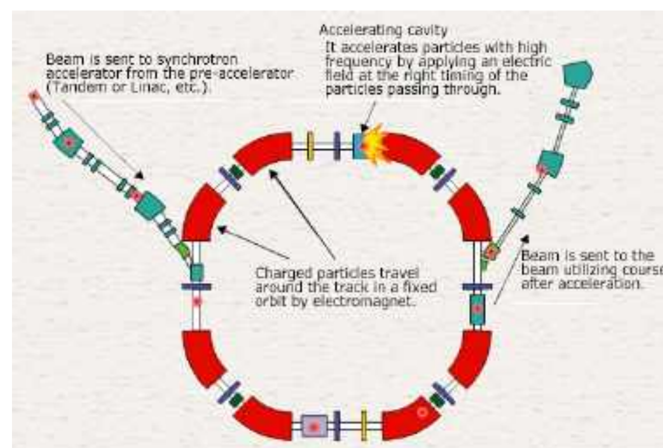
http://case.physics.stonybrook.edu/index.php/Courses:_P554_Fundamentals_of_Accelerator_Physics,_Spring_2014
http://case.physics.stonybrook.edu/index.php/CASE:Courses
https://sites.google.com/site/srfsbu11/
http://uspas.fnal.gov/materials/materials-table.shtml

Today we will focus our attention other important elements for accelerating particles – linear accelerators and RF cavities. There is a lot of material , majority of which is to introduce definitions and jargon used by people working with RF and SRF accelerator.

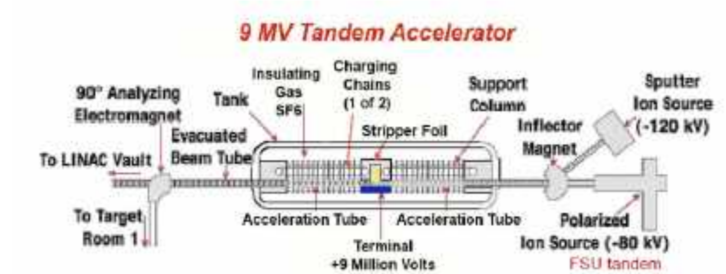
In other words, next time you would hear words “transit time factor” or “shunt impedance of linac”, you would know what they mean (or at least can locate their definitions)

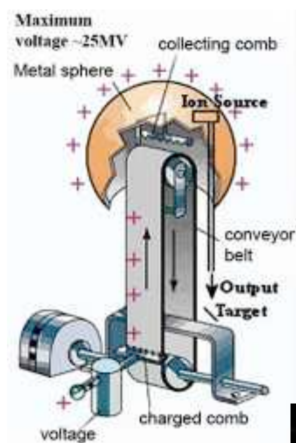


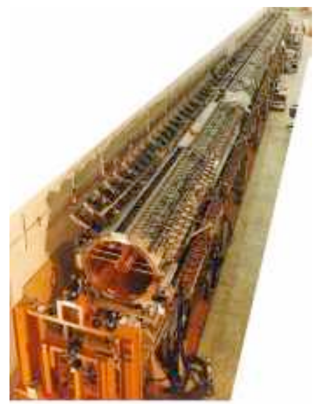


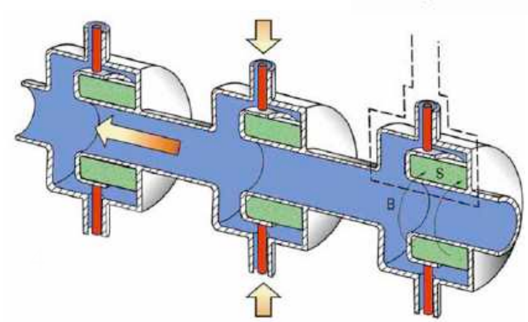


Linear accelerators: from electrostatic to RF







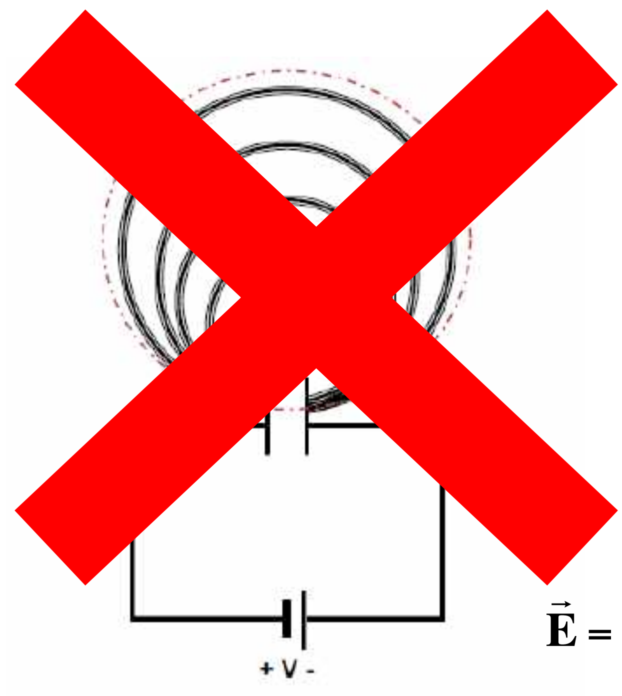


Can one gain the energy again and again by passing through a DC accelerating gap?

Electrostatic: what is the limit ?

Maxwell equations and energy conservation law!

$$\Delta E = e\oint \vec{\mathbf{E}}\cdot d\vec{l} = -\frac{e}{c}\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}}\cdot d\vec{s}\right)$$



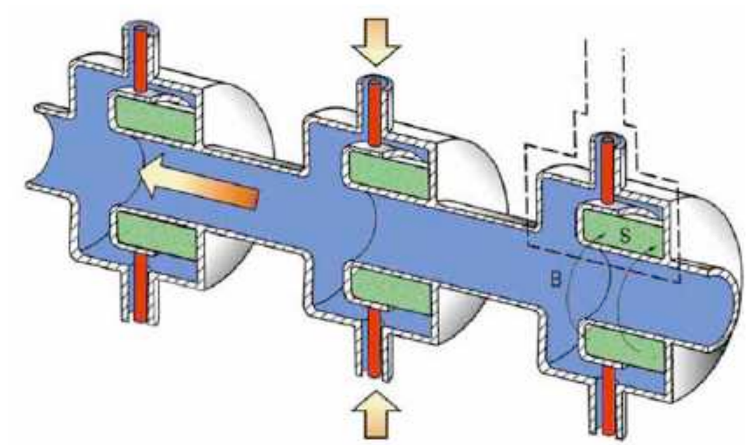
DC

$$\Delta E = e\oint \vec{\mathbf{E}}\cdot d\vec{l} = 0$$

$$\vec{\mathbf{E}} = -\vec{\nabla}\varphi \quad \rightarrow \quad E(\vec{r}) = E(0) - e\varphi(\vec{r})$$

Can not cheat the Maxwell equations

Induction linacs: linear betatrons

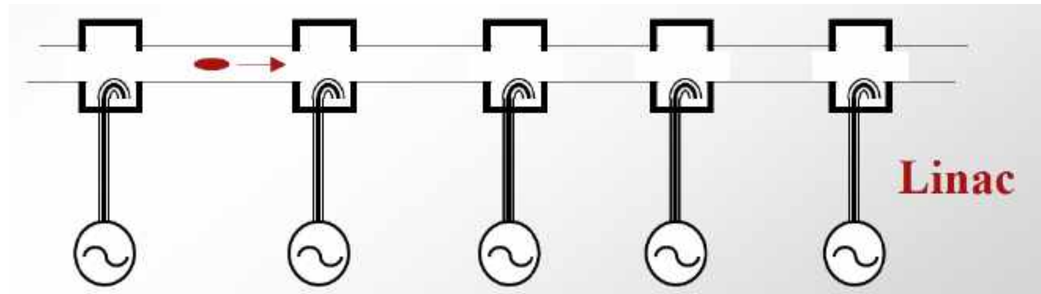


$$\oint \vec{\mathbf{E}} \cdot d\vec{l} = -\frac{\partial}{\partial t}\left(\int \vec{\mathbf{H}} \cdot d\vec{s}\right)$$

- Useful for high power and high current beams
- Have limited accelerating field
- By nature are pulsed, with relatedly low rep-rate (kHz)

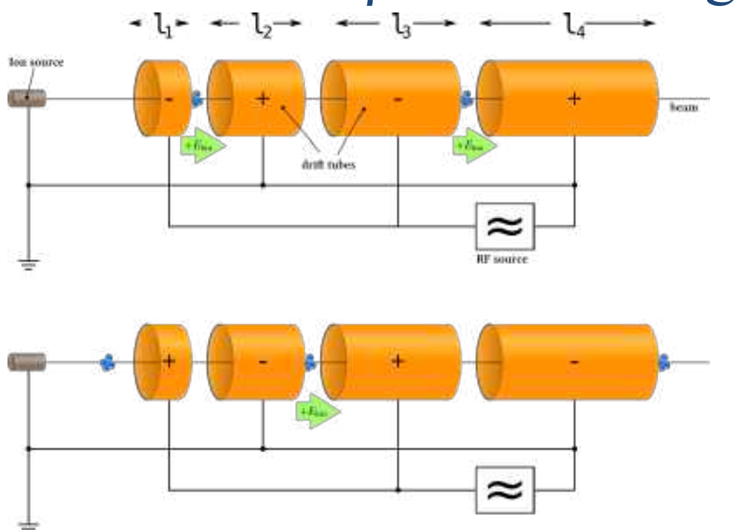
How RF accelerator works

- It has oscillating (typically sinusoidal in time) longitudinal (along the particle's trajectory) eclectic field
- It also has longitudinal structure (cells) which alternates the direction of the filed
- When particle propagates through the RF accelerator, the field direction in each cell is synchronized with the particle arrival and the effect from all cells is added coherently

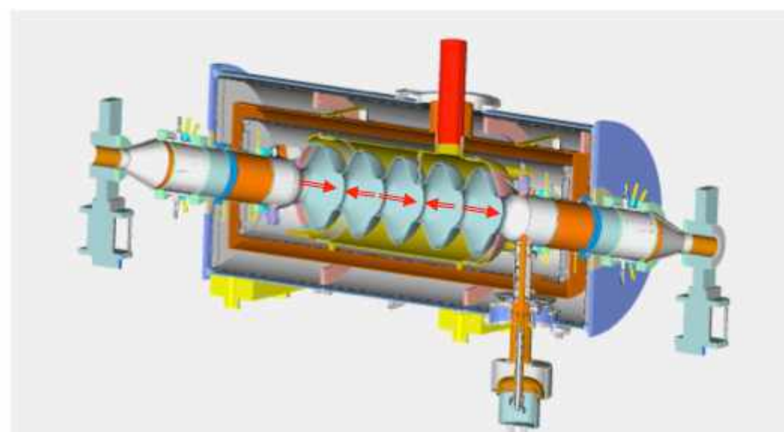


$$\frac{dE}{dt} = e\vec{\mathbf{E}} \cdot \vec{\mathrm{v}} \quad \rightarrow \mathrm{sign}(\vec{\mathbf{E}} \cdot \vec{\mathrm{v}}) = \mathrm{const}$$

Wideröe's linac: β=v/c is changing

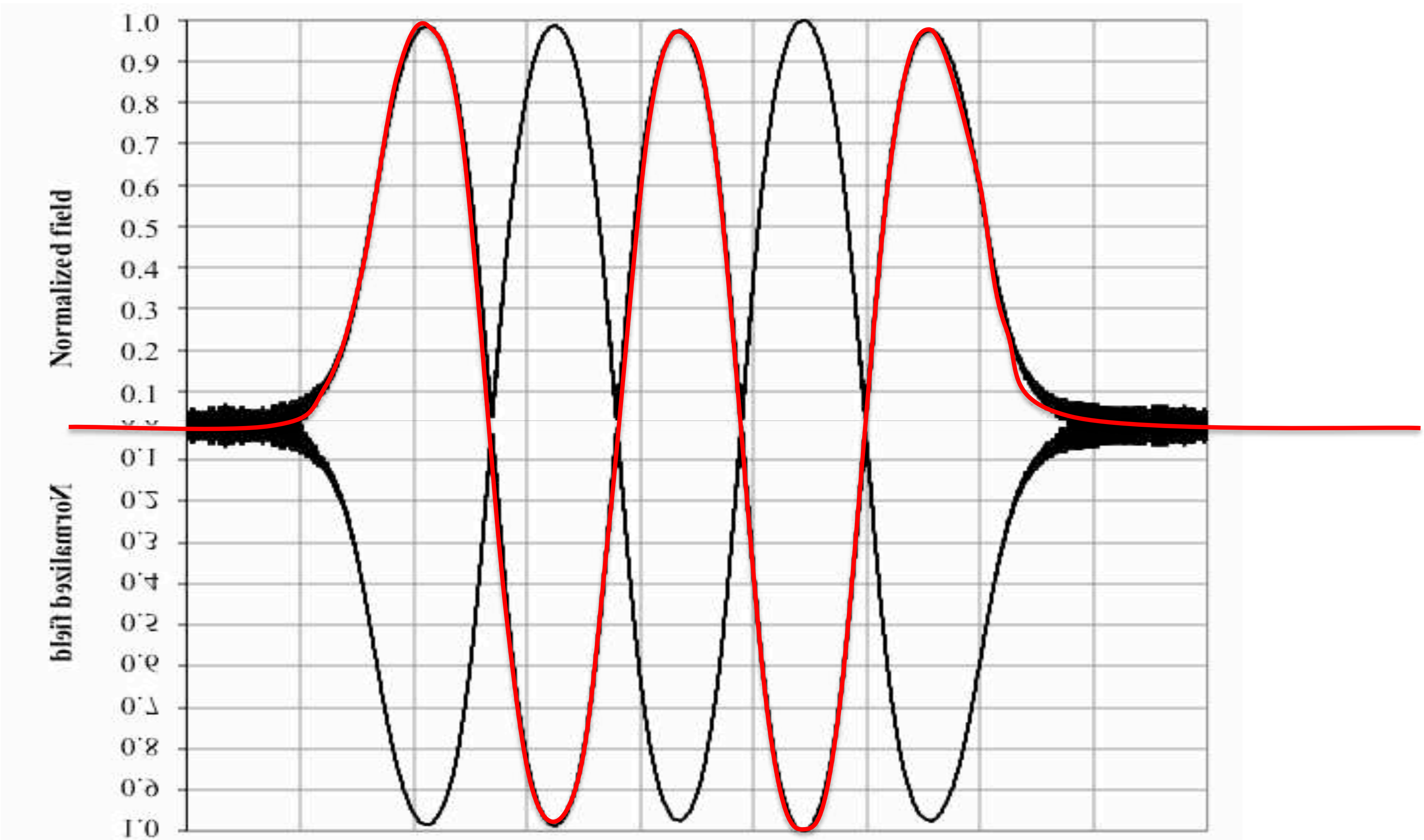


Electron linac

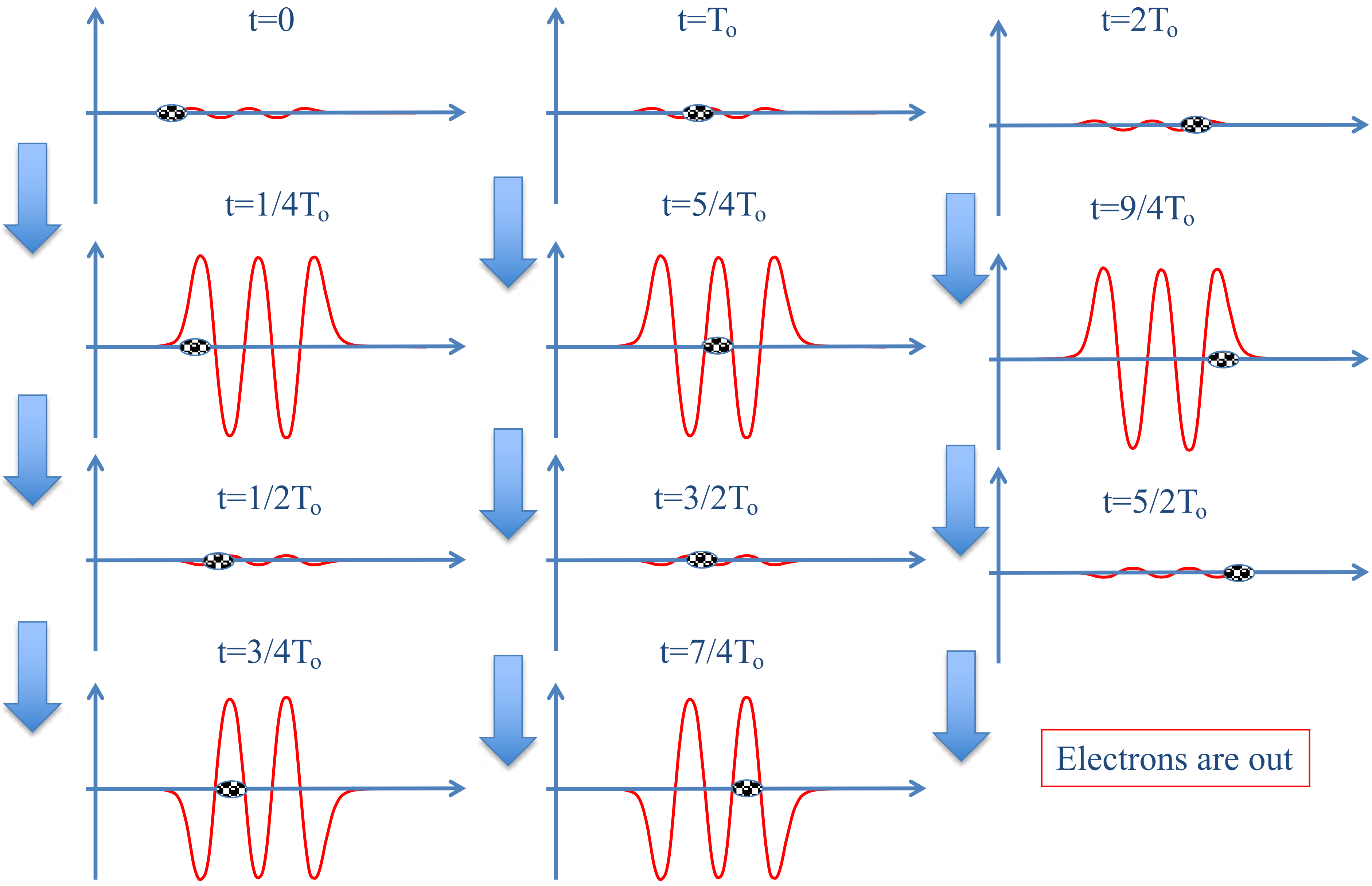


β=v/c ~1

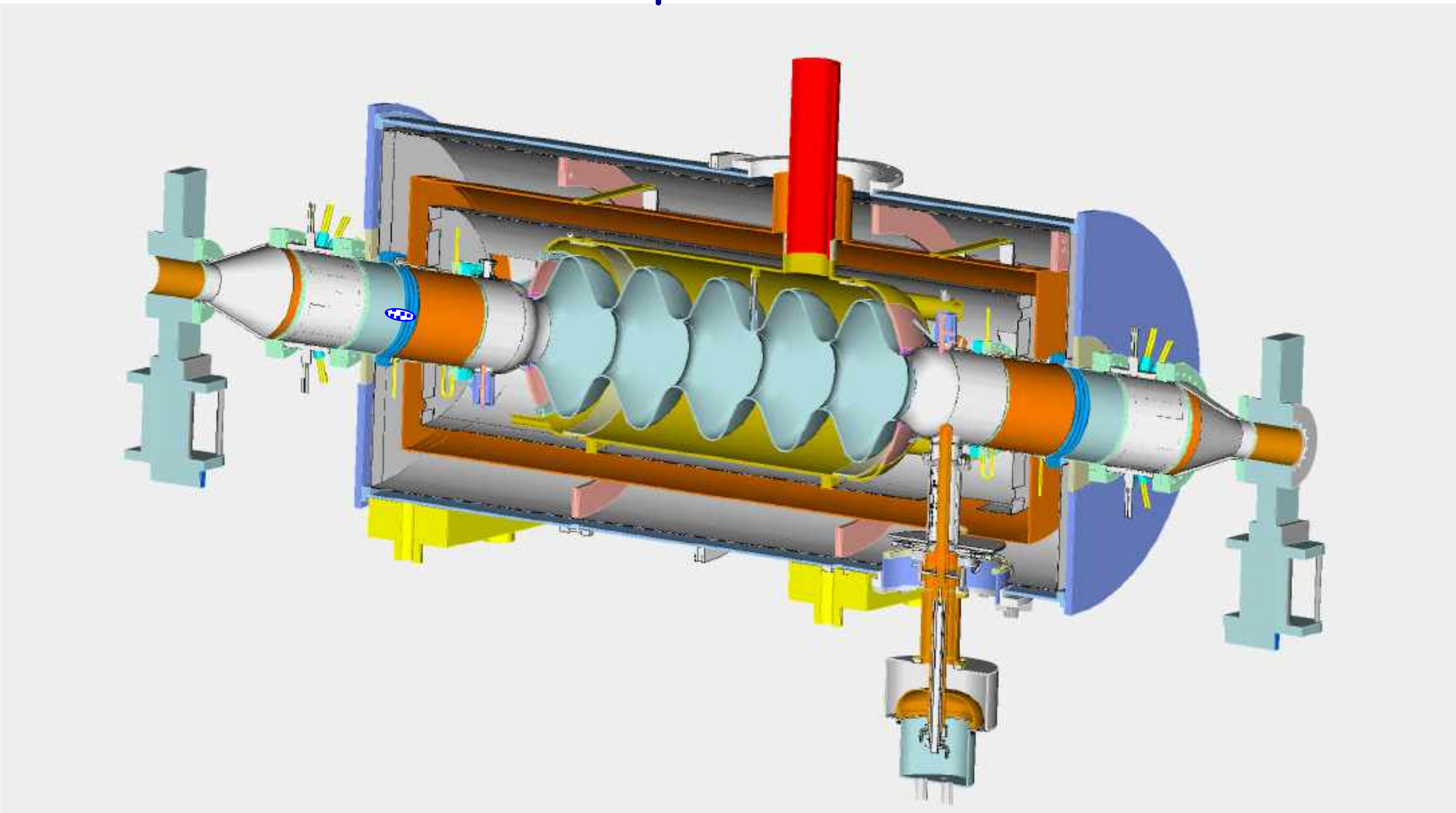
Wave-form in 5-cell cavity



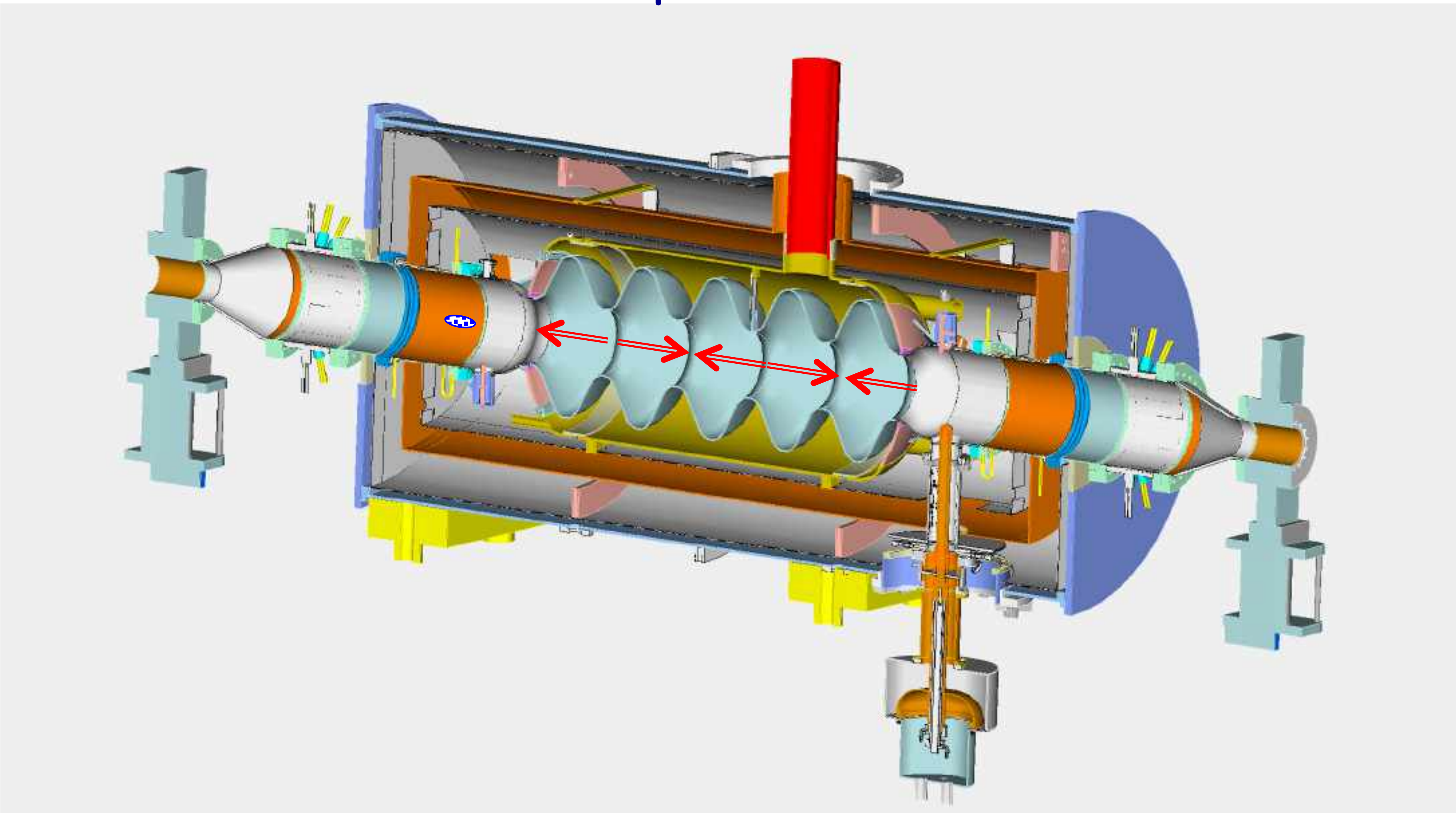
How β=1 RF linac works?
Example of 5-cell cavity



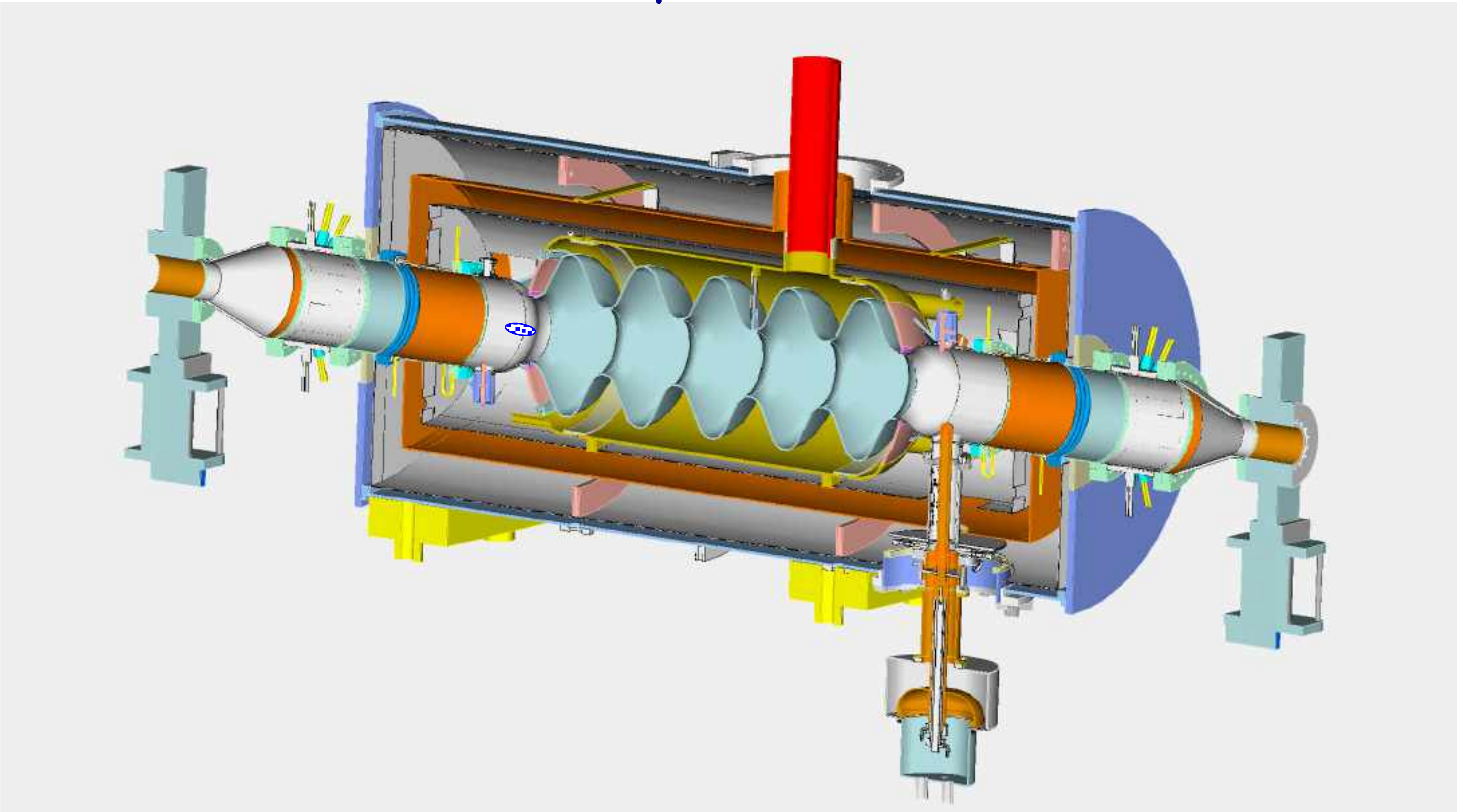
How β=1 RF accelerator works? In pictures



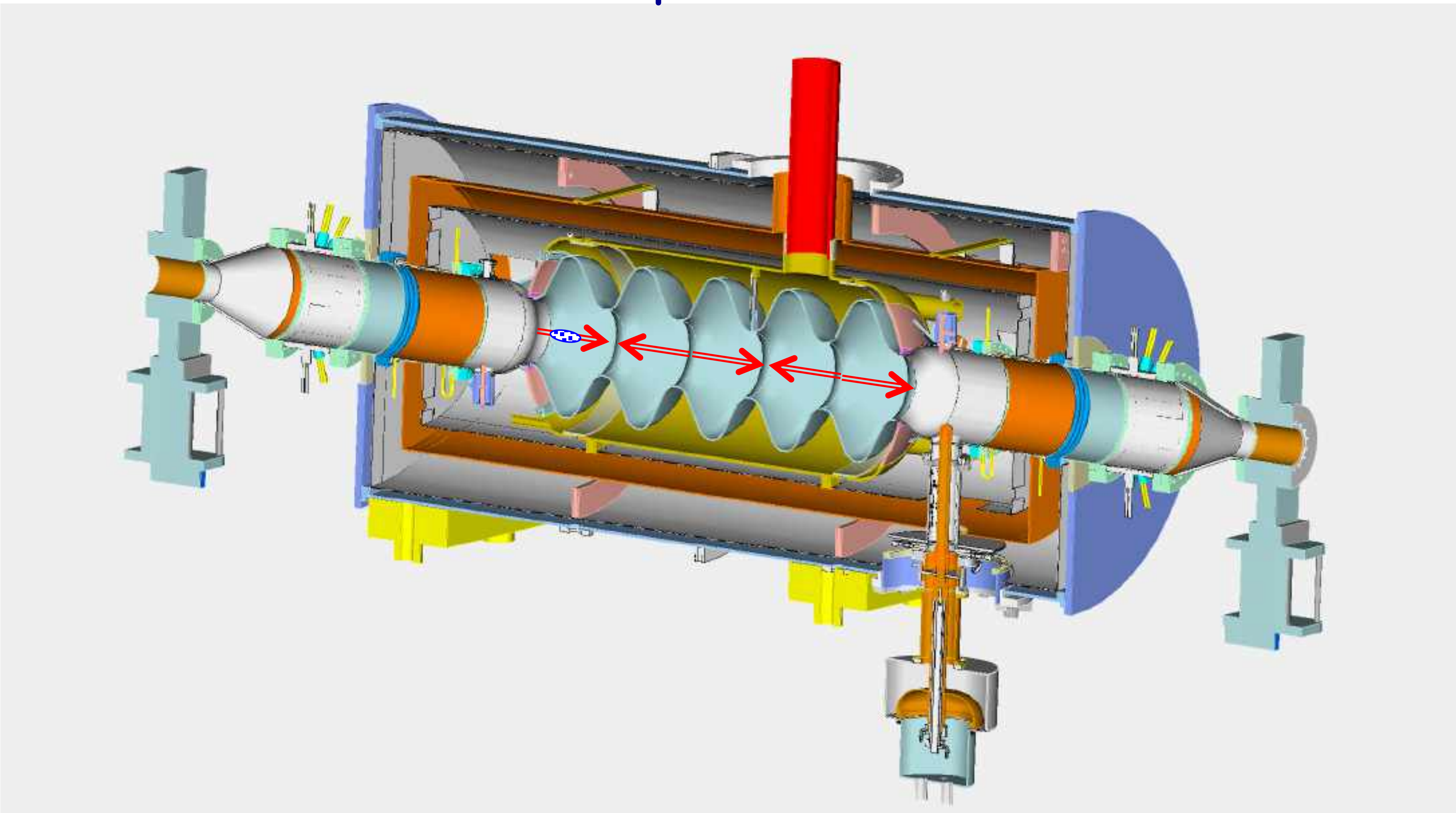
How β=1 RF accelerator works?
In pictures



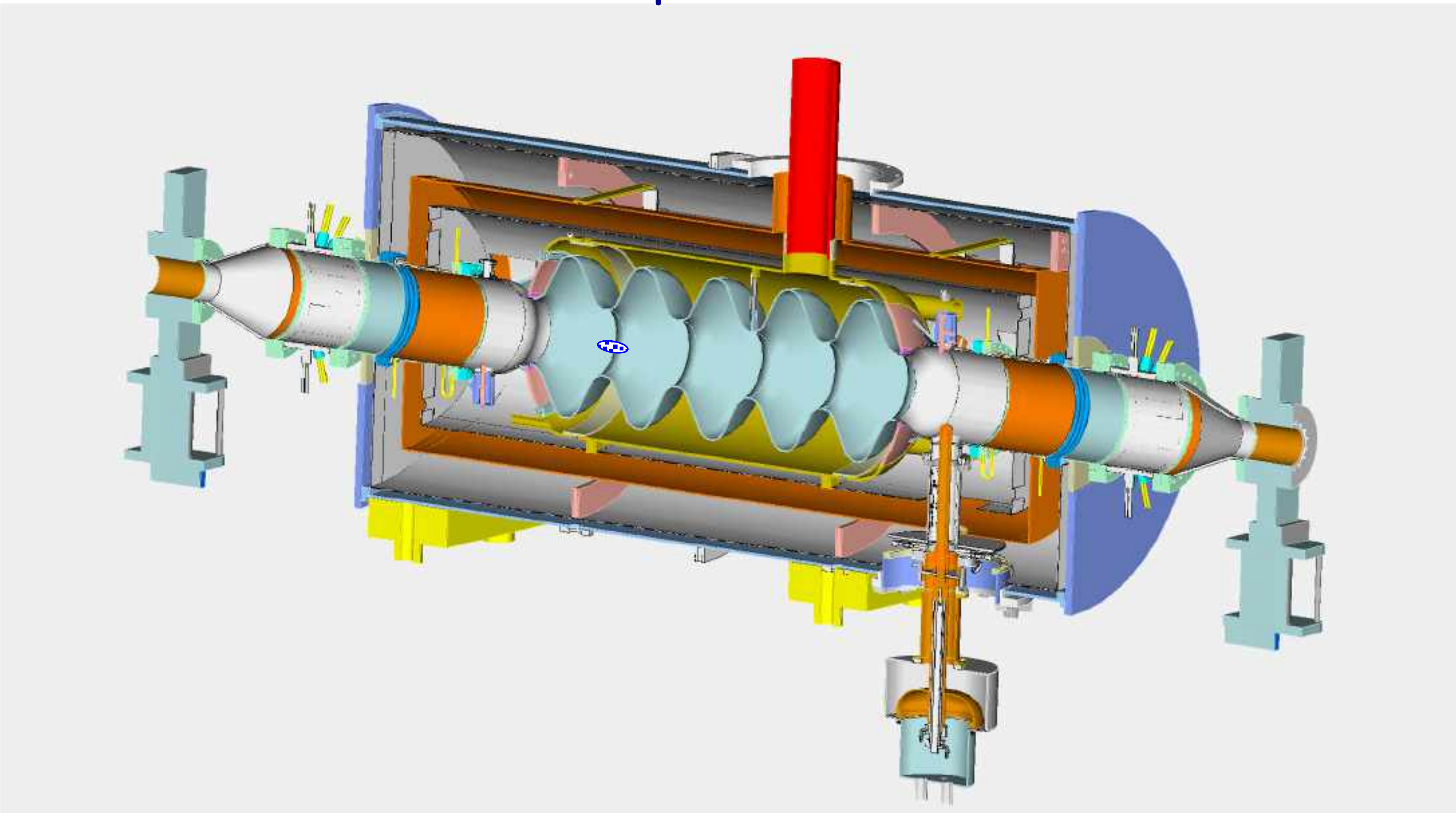
How β=1 RF accelerator works?
In pictures



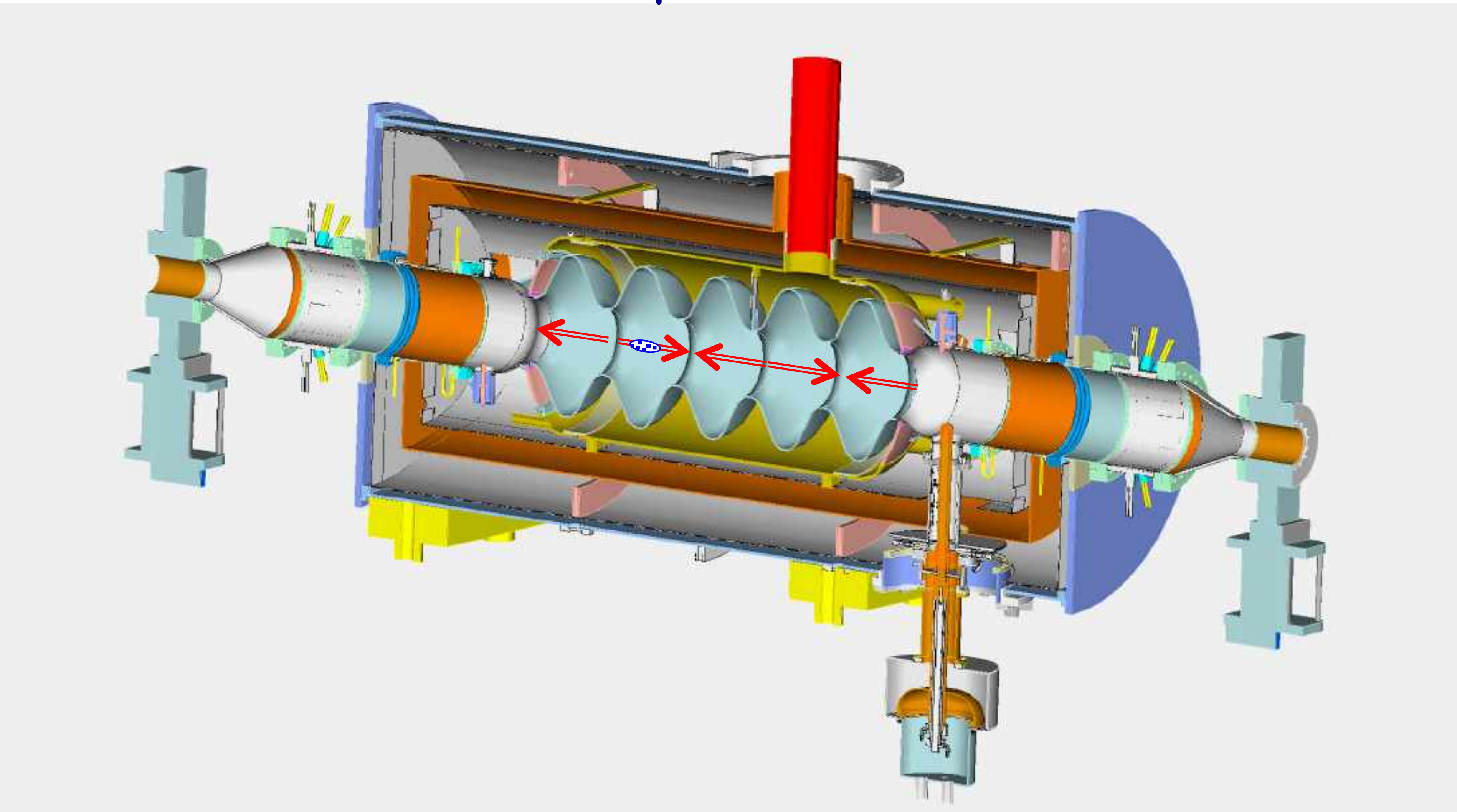
How β=1 RF accelerator works?
In pictures



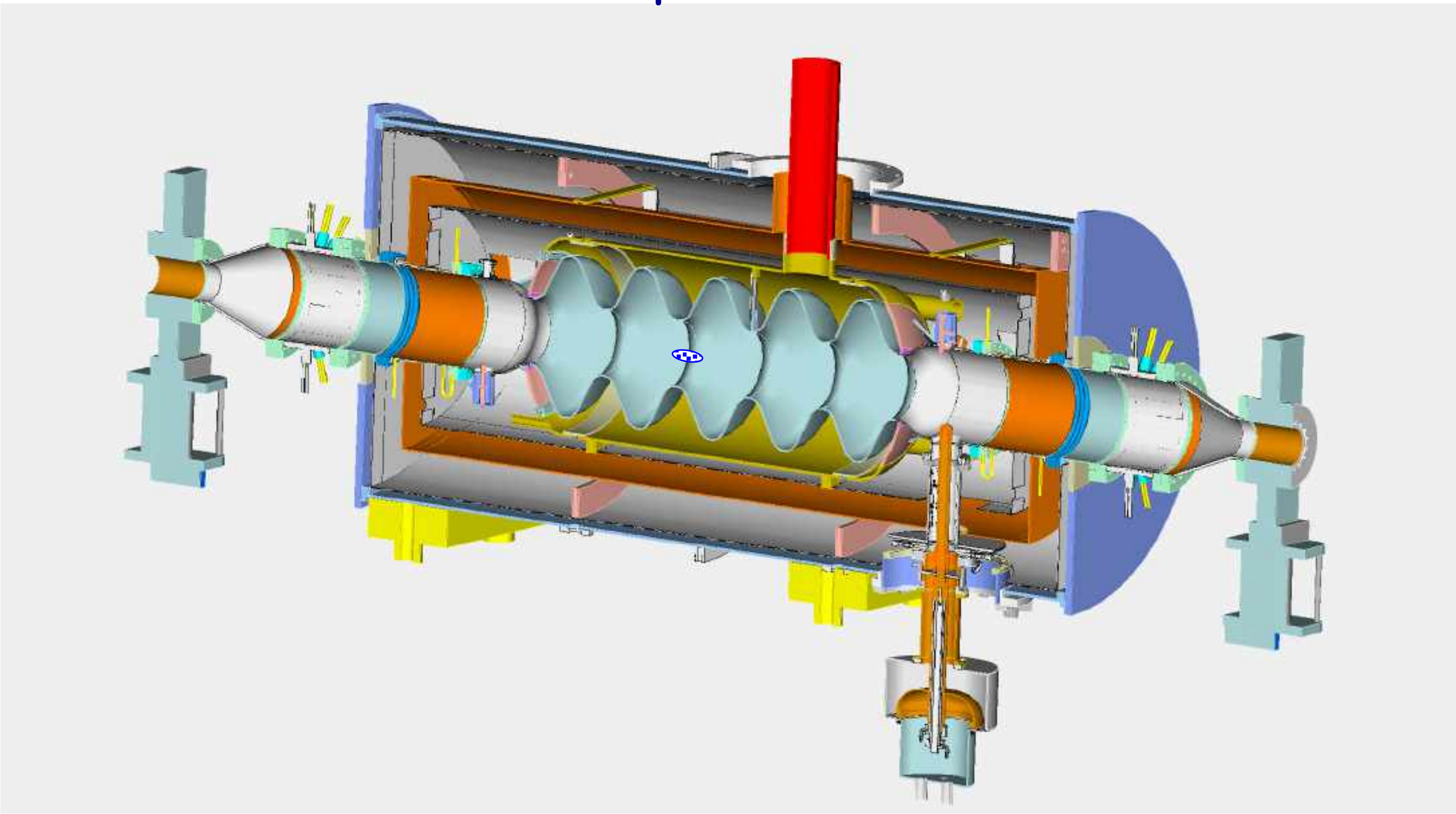
How β=1 RF accelerator works? In pictures



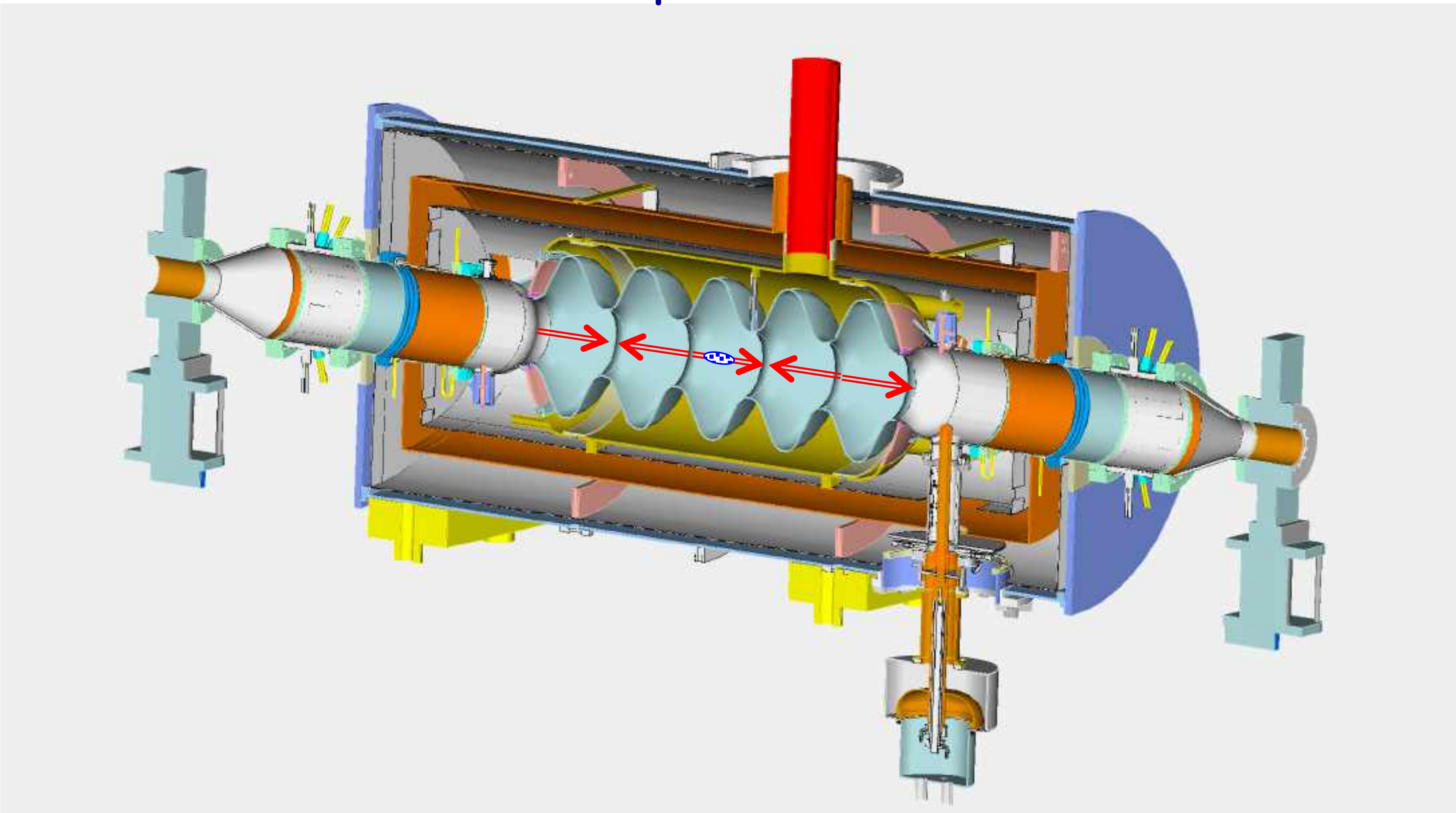
How β=1 RF accelerator works?
In pictures



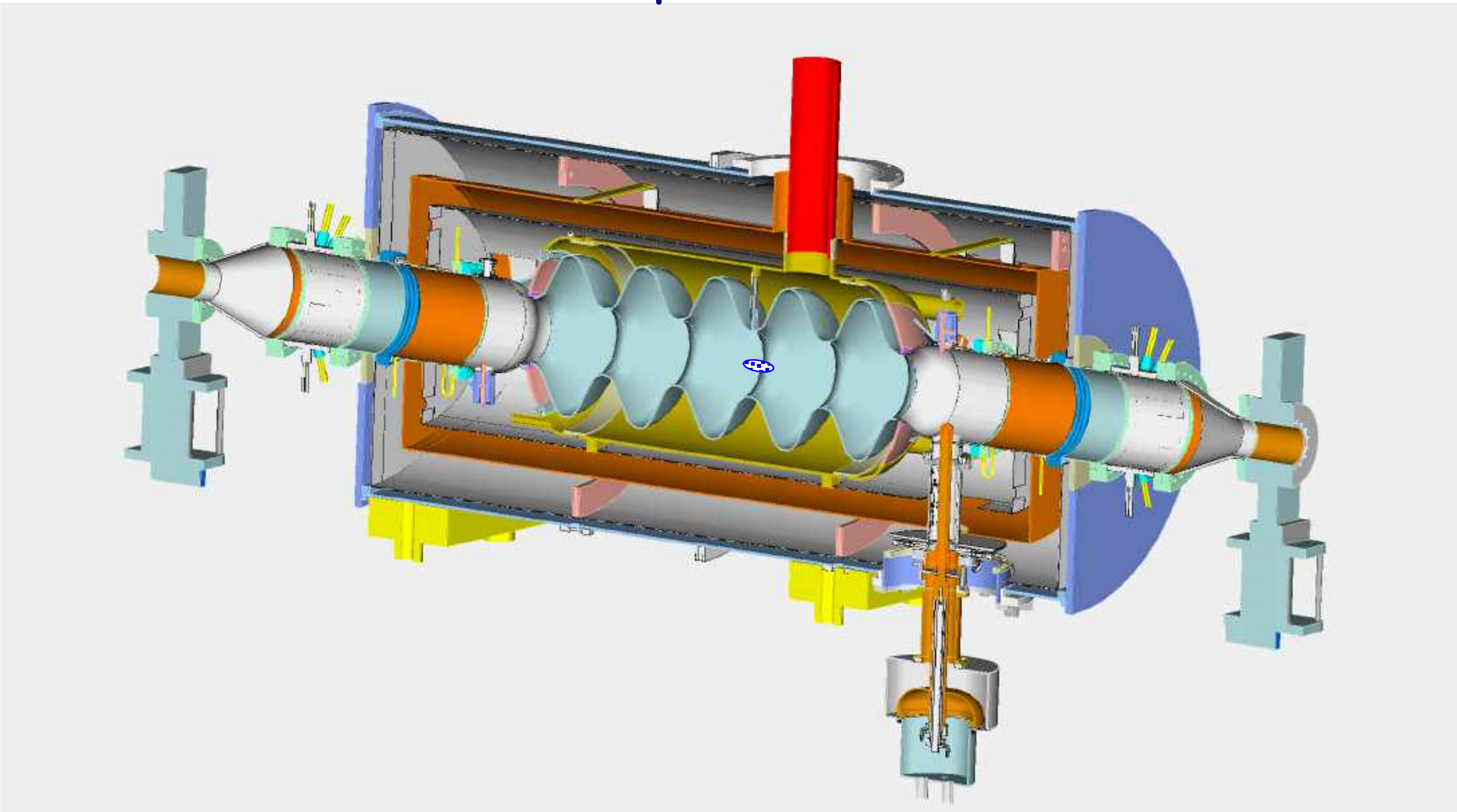
How β=1 RF accelerator works?
In pictures



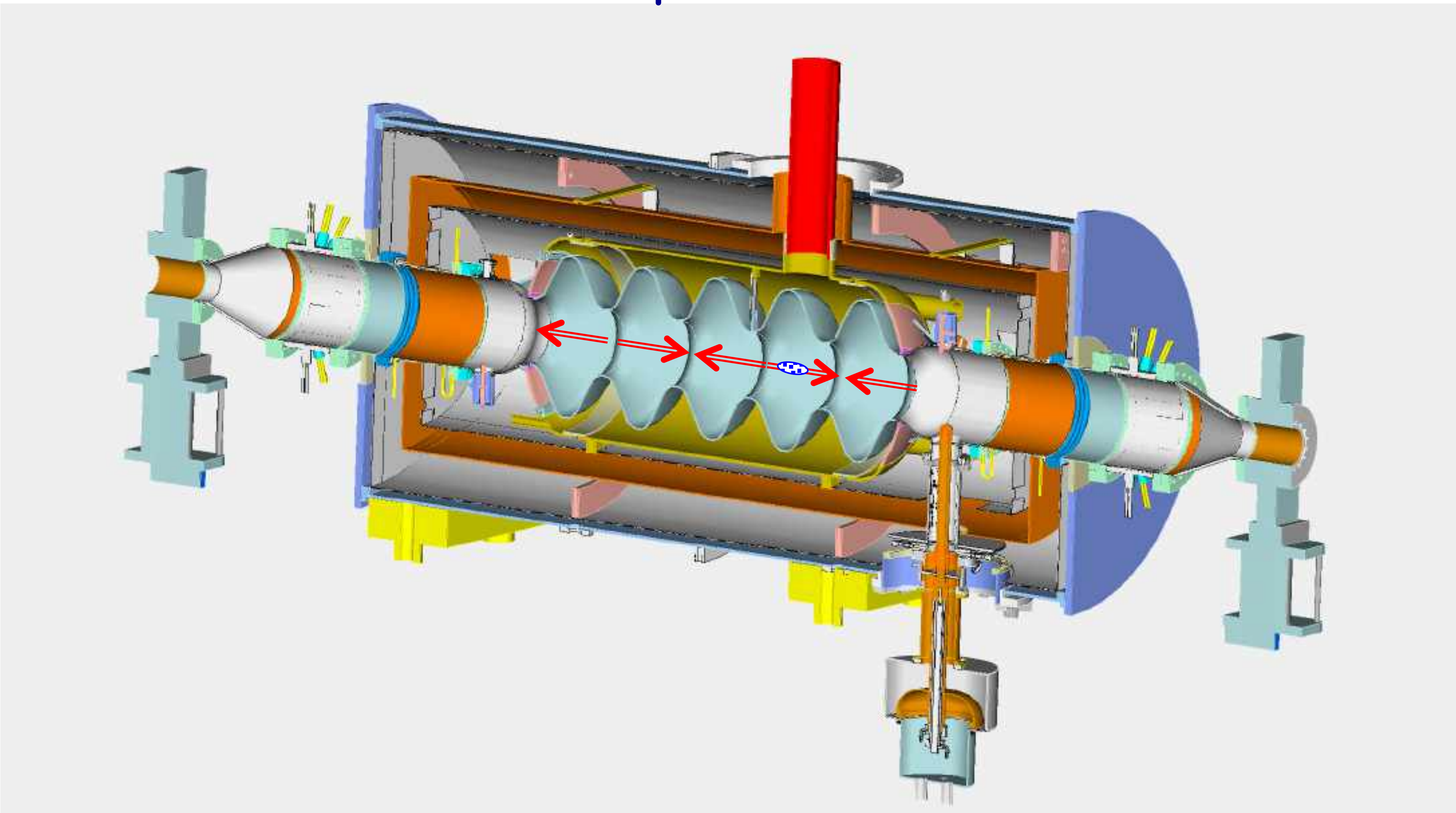
How β=1 RF accelerator works?
In pictures



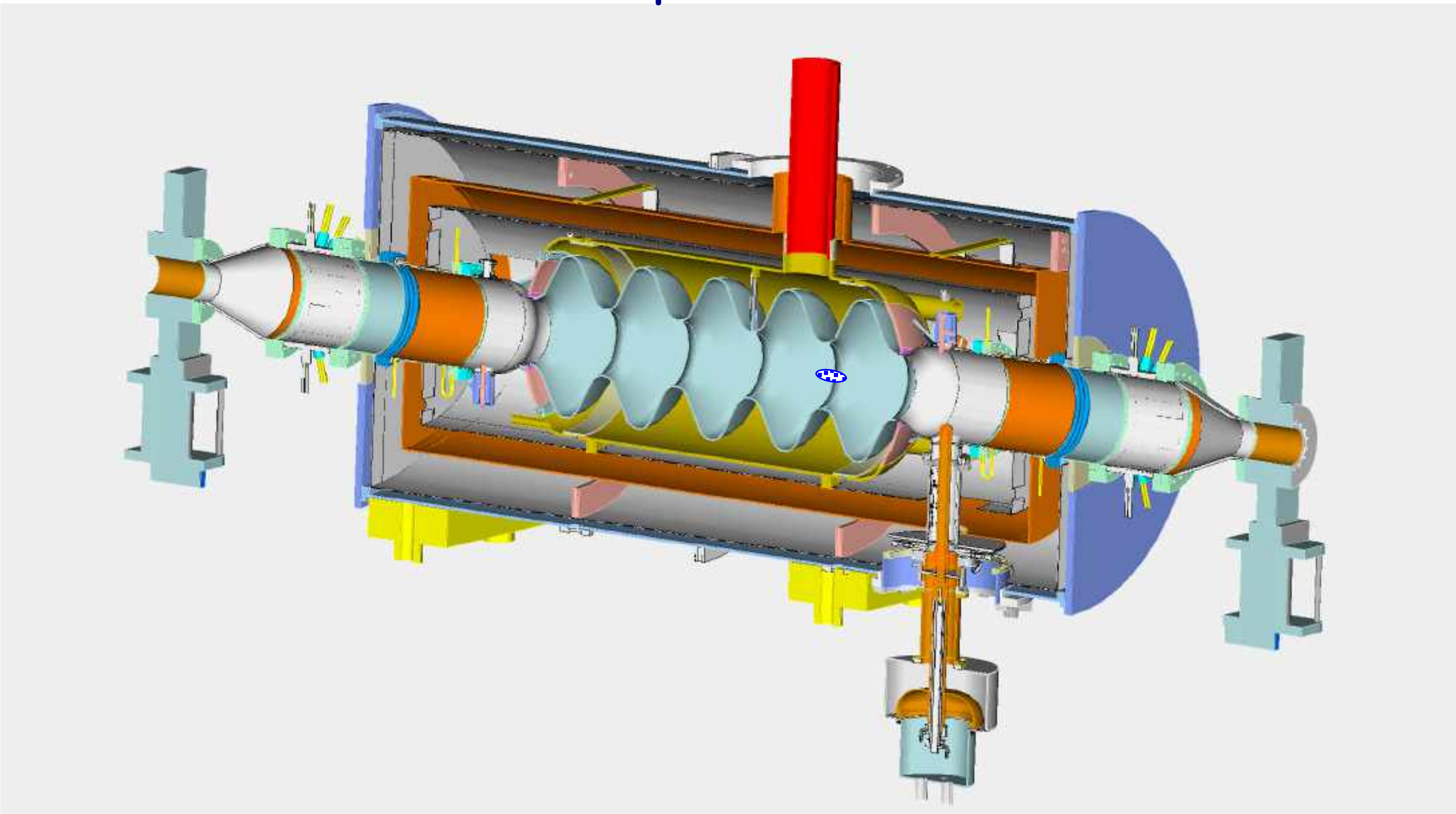
How β=1 RF accelerator works?
In pictures



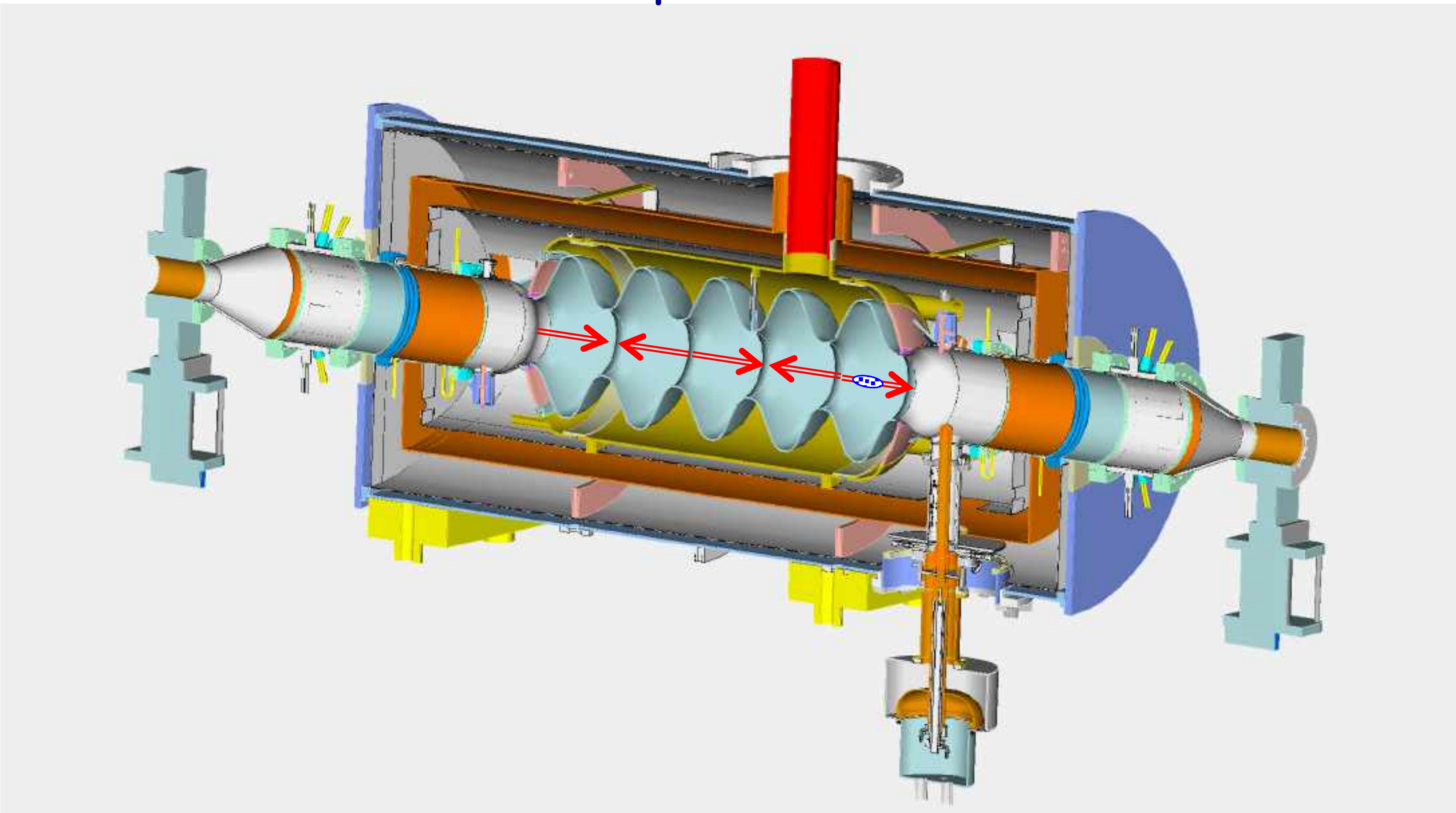
How β=1 RF accelerator works?
In pictures



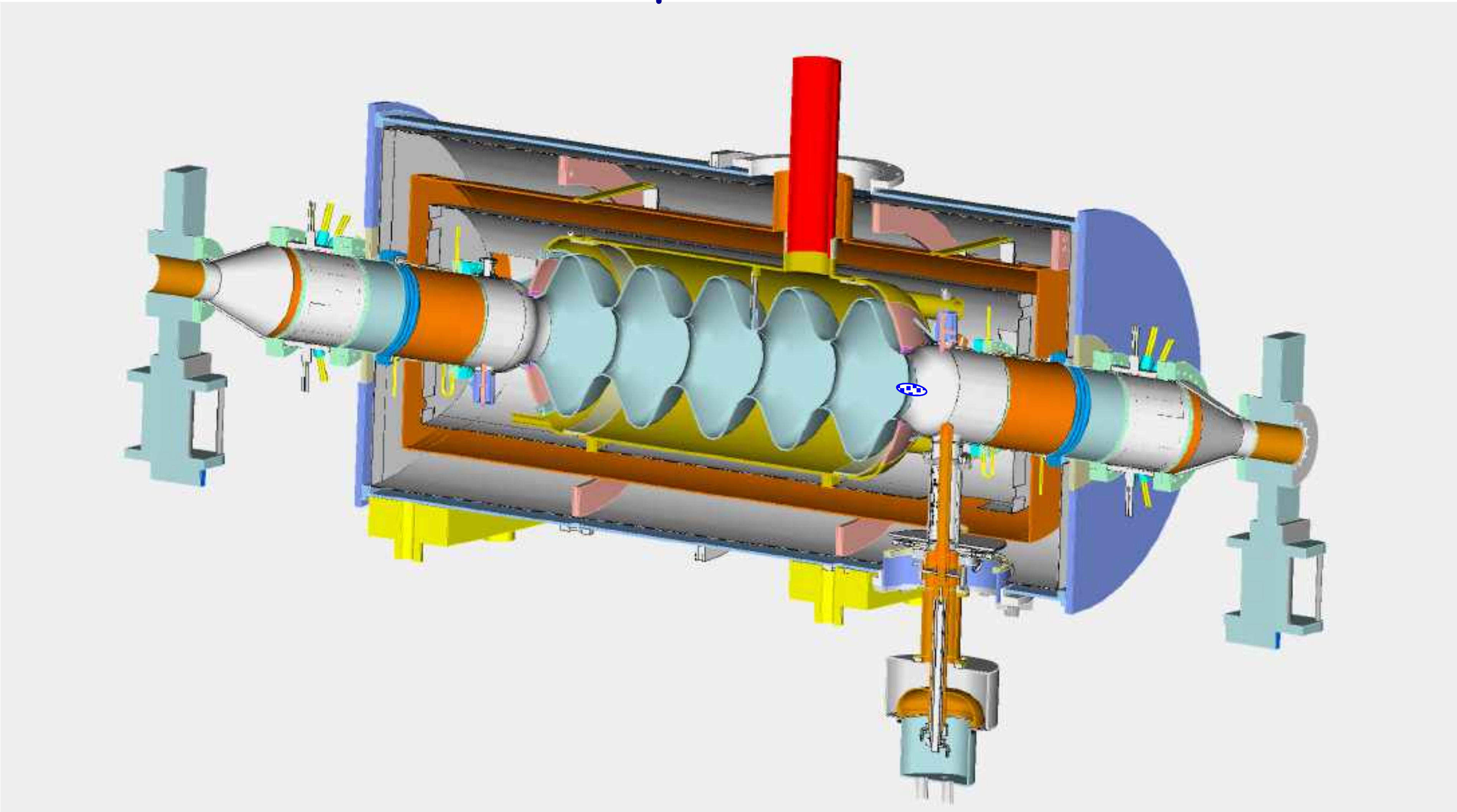
How β=1 RF accelerator works?
In pictures



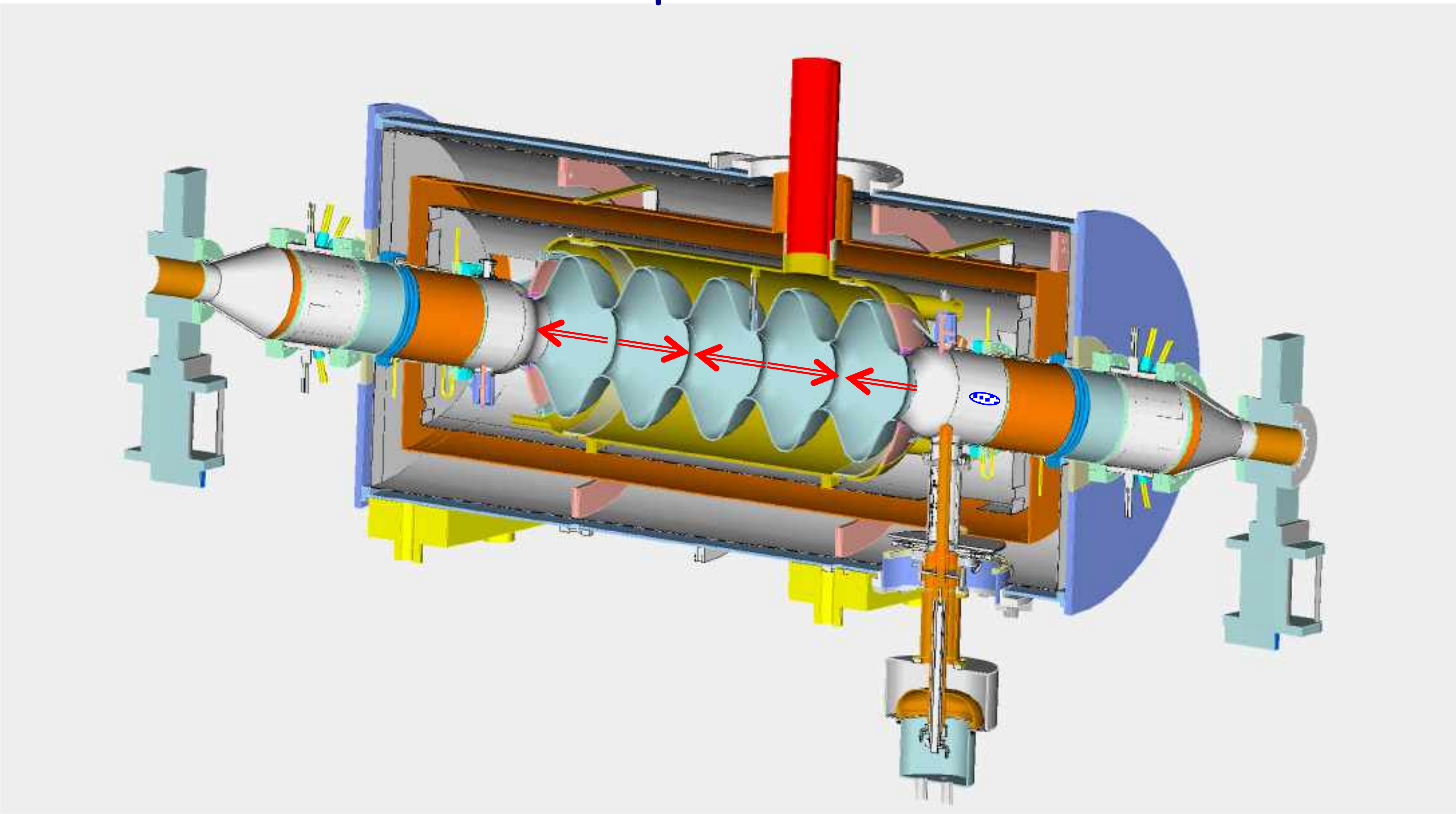
How β=1 RF accelerator works?
In pictures



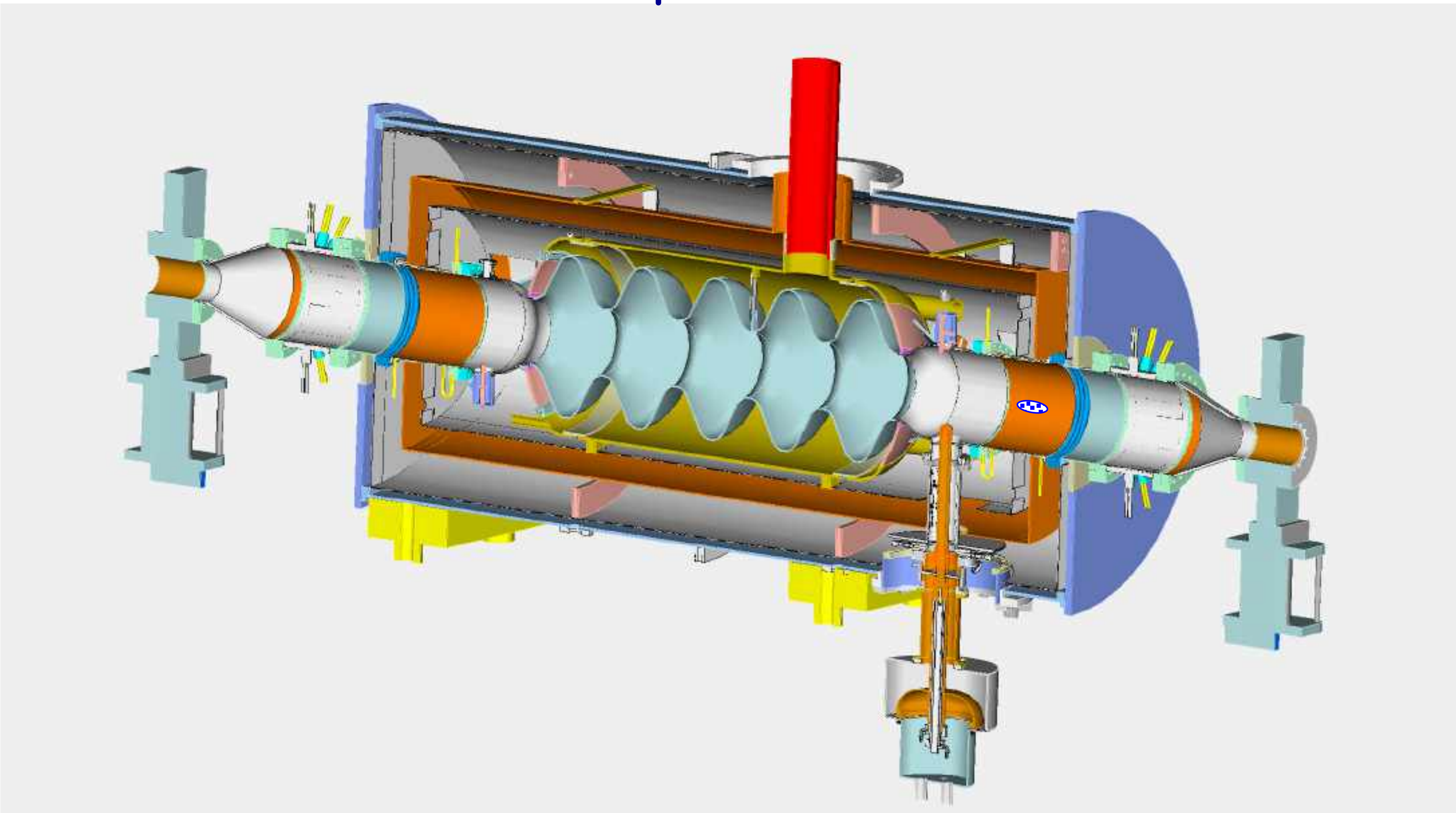
How β=1 RF accelerator works?
In pictures



How β=1 RF accelerator works?
In pictures



How β=1 RF accelerator works?
In pictures



Simple things to remember

- Acceleration in DC electrostatic is limited to the difference in terminal potential (e.g. voltage between the ground and the cathode)
- RF linear accelerators (RF linacs or simply linacs) are not limited in beam energy
- In RF linacs, the coherent addition/subtraction of the energy gain from cell to cell happens by design: period of the electric field oscillation is matched to the travel time of electron between the cells.
- Accurate synchronization of RF linac is important task for any linear accelerator

A bit of EM and conducting media

$$\vec{j} = \sigma \vec{E};$$

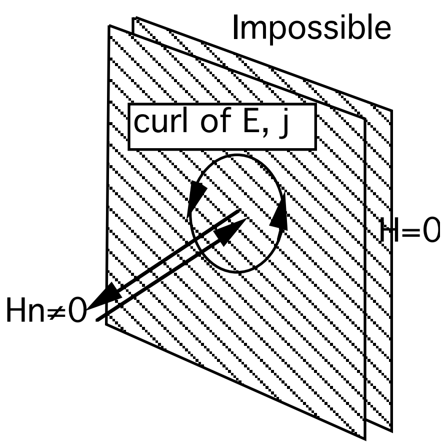
- Assuming oscillating field we can use Coulomb gauge for EM field

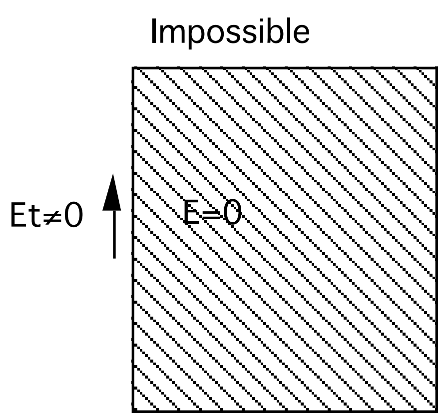
$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t)\right\}; \varphi = 0;$$

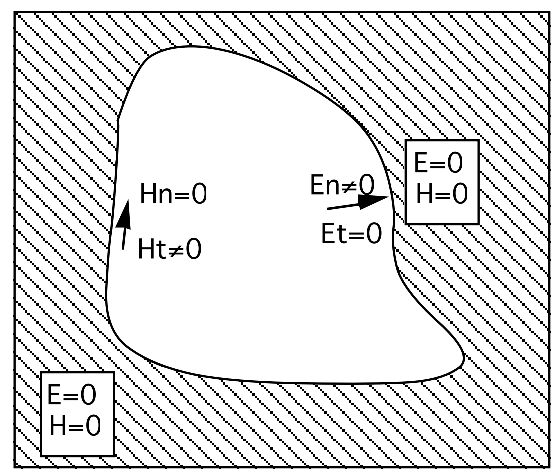
$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = curl\vec{A}.$$

$$\left|\vec{H}\right| \propto \left|\frac{(\alpha + i\beta)}{k_o}\right|\left|\vec{E}\right| = \left|\sqrt{1 + \frac{4\pi i\sigma}{\omega}}\right|\left|\vec{E}\right|$$

$$\sigma \to \infty$$





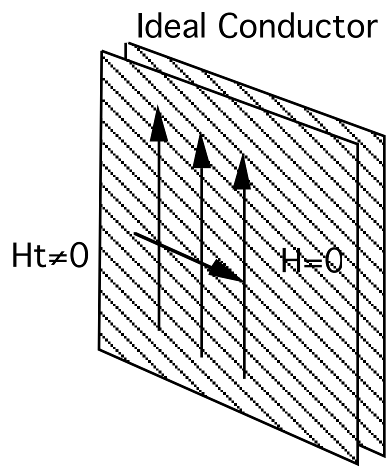


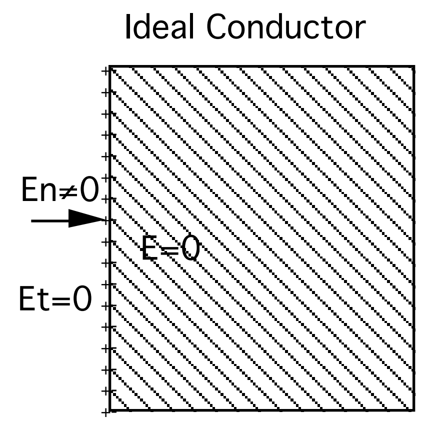
Boundary conditions

- We are considering oscillating EM fields in RF structures
- RF structures are built from highly conducting material, both to contain EM filed inside and to provide low losses
- In first approximation we can consider an ideal boundary conditions and take finite conductivty as a perturbation later
- Q-factor: $Q_{\text{room temp}} \sim 10^4\text{-}10^5$, $Q_{SRF} \sim 10^9\text{-}10^{10}$

$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r})\exp(i\omega t - \alpha t)\right\};$$

$$\alpha = \frac{2\pi\omega}{Q}$$

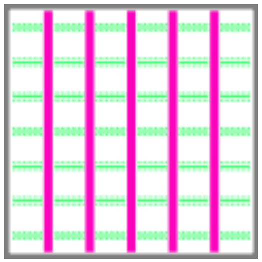


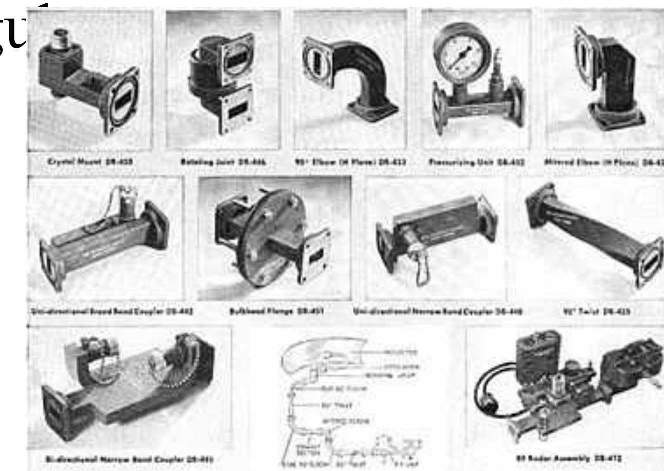


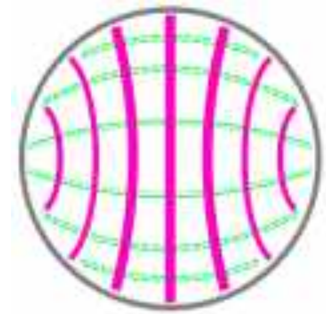
$$\vec{E} = \vec{n}\left(\vec{n}\vec{E}\right) + \vec{E}_{//}; \vec{B} = \vec{n}\left(\vec{n}\vec{B}\right) + \vec{B}_{//};$$

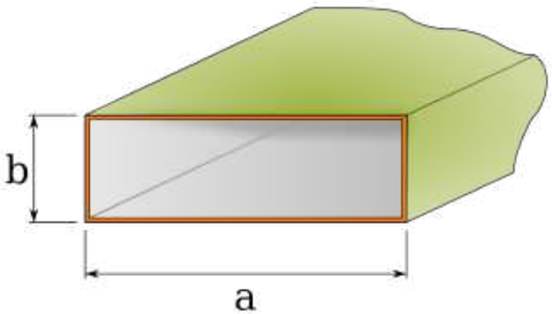
Waveguides

Rectangu ... Circular









$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = 0;\ \ \Delta \equiv \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

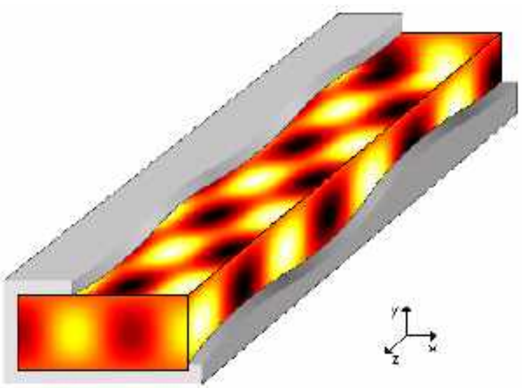
$$\vec{A} = \mathrm{Re}\left\{\vec{A}(\vec{r}_\perp)\exp\left(i\left(\omega t - k_z z\right)\right)\right\};$$

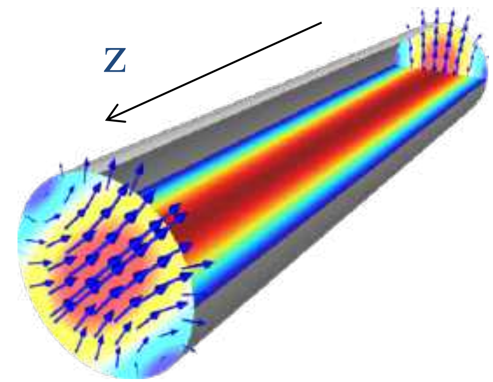
$$\vec{\nabla}_\perp^2\vec{A} + \left(k_o^2 - k_z^2\right)\vec{A} = 0; \quad k_o = \frac{\omega}{c}.$$

$$\vec{\nabla}_\perp^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

At the surfaces

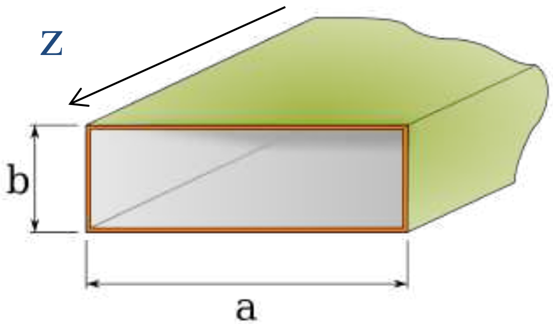
$$\vec{n}\times\vec{E}\Big|_s = 0;\ \vec{n}\cdot\vec{B} = 0 \rightarrow E_z\Big|_s = 0;\ \frac{\partial B_z}{\partial n}\bigg|_s = 0$$



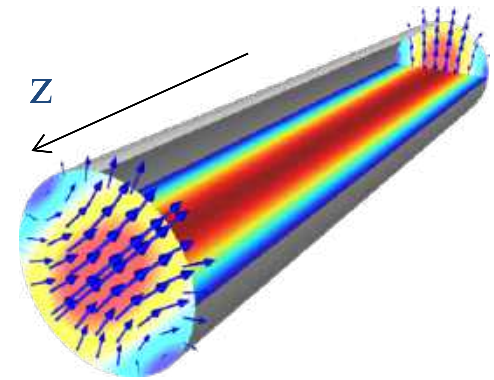


TE and TM waves

Rectangular | **Circular**



- There is simplification
  - The modes are divided into two types: TE (transverse electric) and TM (transverse magnetic)



$$\vec{E} = \vec{E}_z + \vec{E}_\perp; \vec{B} = \vec{B}_z + \vec{B}_\perp; \; \vec{A}_z \equiv \hat{z} A_z;$$

$$\vec{\nabla} \times \vec{E} = i k_o \vec{B}; \; \vec{\nabla} \times \vec{B} = -i k_o \vec{E}; \quad \Rightarrow$$

$$i k_z \vec{E}_\perp + i k_o \left[\hat{z} \times \vec{B}_\perp \right] = \vec{\nabla}_\perp \vec{E}_z;$$

$$i k_z \vec{B}_\perp - i k_o \left[\hat{z} \times \vec{E}_\perp \right] = \vec{\nabla}_\perp \vec{B}_z;$$

At the surfaces

$$\vec{n} \times \vec{E}\Big|_s = 0; \vec{n} \cdot \vec{B} = 0 \rightarrow E_z\Big|_s = 0; \; \frac{\partial B_z}{\partial n}\Bigg|_s = 0$$

$$TM : B_z \equiv 0; \; E_z\Big|_s = 0;$$

$$TE : E_z \equiv 0; \; B_z\Big|_s = 0;$$

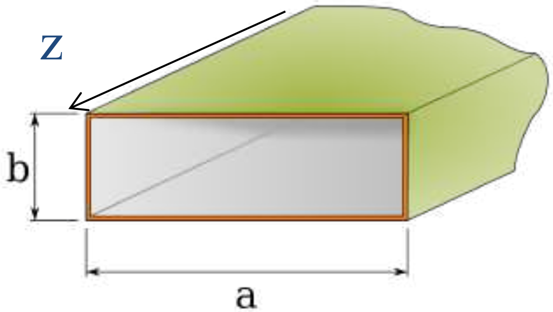
- Last two equations indicated that E_z and B_z fully determine transverse component of the EM field
- It means that we can always consider a linear combination of the fields with *$E_z = 0$ everywhere* (TE) and *$B_z = 0$ everywhere* (TM)
- Naturally, when we interested in accelerating particles, we will need TM mode with $E_z \neq 0$.

$$\vec{B}_\perp = \pm \frac{k_z}{k_o} \left[\hat{z} \times \vec{E}_\perp \right] \quad \text{for both TE and TM modes}$$

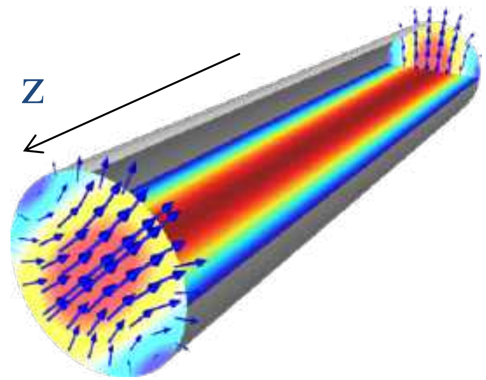
$$\text{TM: } \vec{E}_\perp = \vec{\nabla}_\perp \psi_1 (\vec{r}_\perp); \; \text{TE: } \vec{B}_\perp = \vec{\nabla}_\perp \psi_2 (\vec{r}_\perp);$$

Cut-off frequency

Rectangular



Circular



- EM field is a linear combination of modes with *$E_z = 0$ everywhere* (TE) and *$B_z = 0$ everywhere* (TM)

At the surfaces

$$\vec{n}\times\vec{E}\big|_s = 0;\ \vec{n}\cdot\vec{B} = 0 \rightarrow E_z\big|_s = 0;\ \frac{\partial B_z}{\partial n}\bigg|_s = 0$$

$$\vec{B}_\perp = \pm\frac{k_z}{k_o}\left[\hat{z}\times\vec{E}_\perp\right]$$ for both TE and TM modes

$$TM: B_z \equiv 0;\ E_z\big|_s = 0;$$

$$TE: E_z \equiv 0;\ B_z\big|_s = 0;$$

$$\text{TM: } \vec{E}_\perp = \vec{\nabla}_\perp\psi_1(\vec{r}_\perp);\ \text{TE: } \vec{B}_\perp = \vec{\nabla}_\perp\psi_2(\vec{r}_\perp);$$

$$TM: \psi\big|_s = 0;\ TE: \frac{\partial\psi}{\partial n}\bigg|_s = 0.$$

$$\vec{\nabla}_\perp^2\psi + \left(k_o^2 - k_z^2\right)\psi = 0 + \textit{boundary conditions}$$

Different boundary conditions for TE and TM modes

In general case we need to find eigen function (modes)

$$\vec{\nabla}_\perp^2\psi_\lambda + {\gamma_\lambda}^2\psi_\lambda = 0;\quad \lambda = 1,2,3,\ldots$$

Cut-off frequency

$$k_{z,\lambda}^2 = k_o^2 - \gamma_\lambda^2 > 0$$

$$k_{o\min} = \gamma_\lambda \rightarrow \omega_{cut-off} = c\gamma_\lambda$$

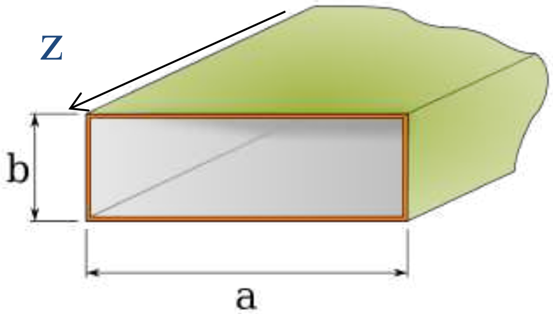
Below cut-off $\omega < \omega_{cut-off}$

evanescent wave: $k_z = \pm i\sqrt{\omega^2{}_{cut-off} - \omega^2} = \pm i\kappa_z$

Exp decay $\psi = \psi_0 e^{\pm\kappa_z z}$

Cut-off frequency

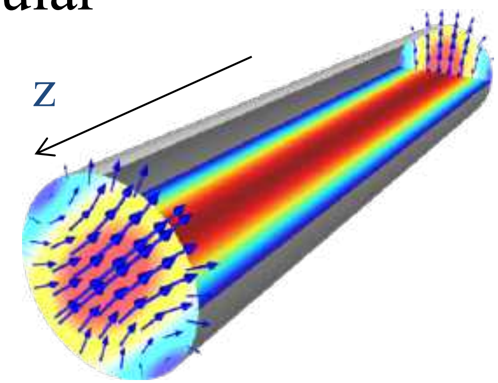
Rectangular



Different boundary conditions for TE and TM modes

$$TM: \psi\big|_s = 0; \quad TE: \frac{\partial \psi}{\partial n}\bigg|_s = 0.$$

Circular



$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + \gamma_{mn}^2 \psi = 0$$

$$TE: \psi^{TE}{}_{mn} = \psi_o \cos k_m x \cos k_n x; \;\; m + n \ge 1;$$

$$TM: \psi^{TM}{}_{mn} = \psi_o \sin k_m x \sin k_n x; \;\; m \ge 1; n \ge 1;$$

$$k_m = \pi \frac{m}{a}; k_n = \pi \frac{n}{b}; \; \gamma_{mn} = k_m{}^2 + k_n{}^2.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \gamma_{mn}^2 \psi = 0$$

$$\psi_{mn} = \varphi_{mn}(r) e^{in\theta} \Rightarrow r \frac{\partial}{\partial r}\left(r \frac{\partial \psi_m}{\partial r}\right) + \left(r^2 \gamma_{mn}^2 - n\right) = 0$$

$$\varphi_{mn} = J_n\left(\gamma_{mn} r\right)$$

Lowes cut-off frequency

Rectangular

$$TE: a > b; \; m = 1; \; n = 0; \;\; \omega_{cut-off} = \frac{\pi c}{a};$$

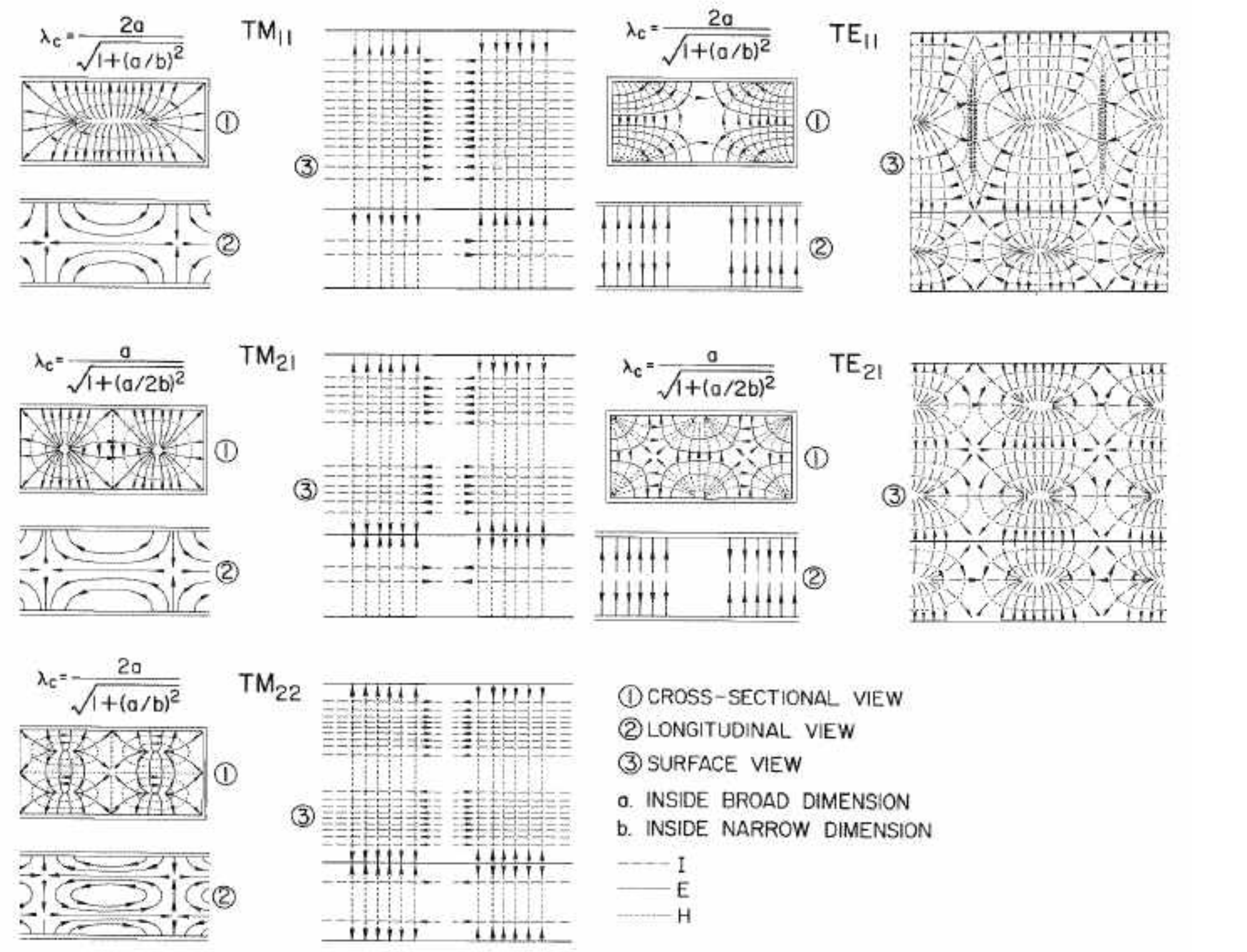
$$TM: \; m = 1; \; n = 1; \;\; \omega_{cut-off} = \frac{\pi c}{a}\sqrt{1 + \frac{a^2}{b^2}}.$$

Circular

$$TM: J_o\left(\gamma_{01} R\right) = 0 \rightarrow \gamma_{01} \cong \frac{2.40483..}{R}; \;\; \omega_{cut-off} \cong \frac{2.40\, c}{R};$$

$$TE: J_1'\left(\gamma_{11} R\right) = 0 \rightarrow \gamma_{11} \cong \frac{1.84118....}{R}; \; \omega_{cut-off} \cong \frac{1.84\, c}{R}.$$

Modes in rectangular waveguide



TM11: $\lambda_c = \frac{2a}{\sqrt{1+(a/b)^2}}$

TE11: $\lambda_c = \frac{2a}{\sqrt{1+(a/b)^2}}$

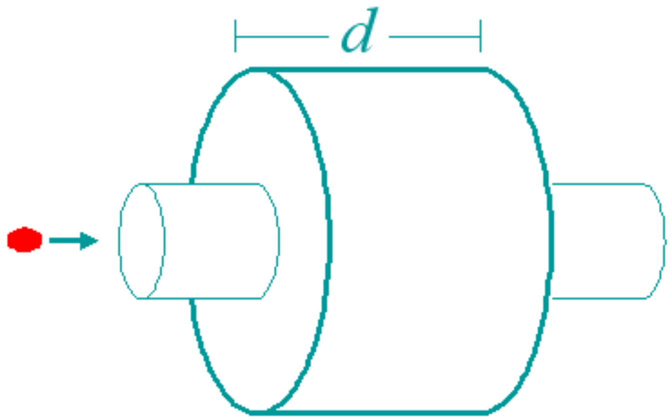
TM21: $\lambda_c = \frac{a}{\sqrt{1+(a/2b)^2}}$

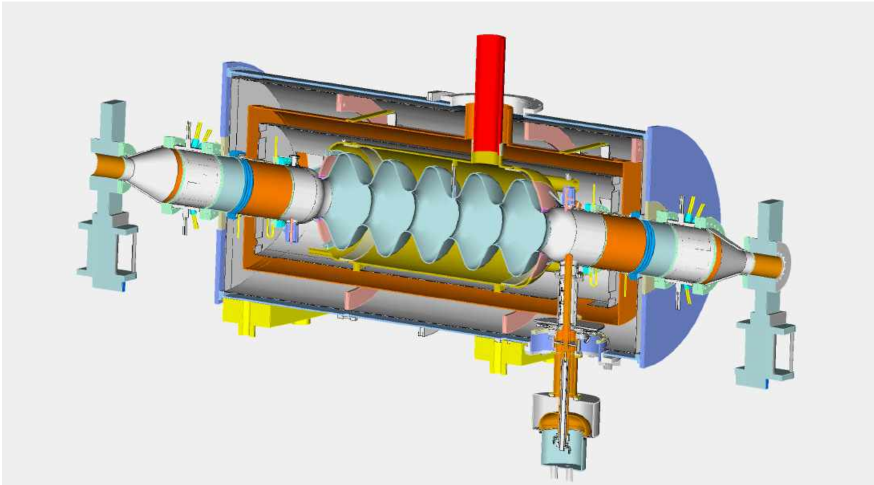
TE21: $\lambda_c = \frac{a}{\sqrt{1+(a/2b)^2}}$

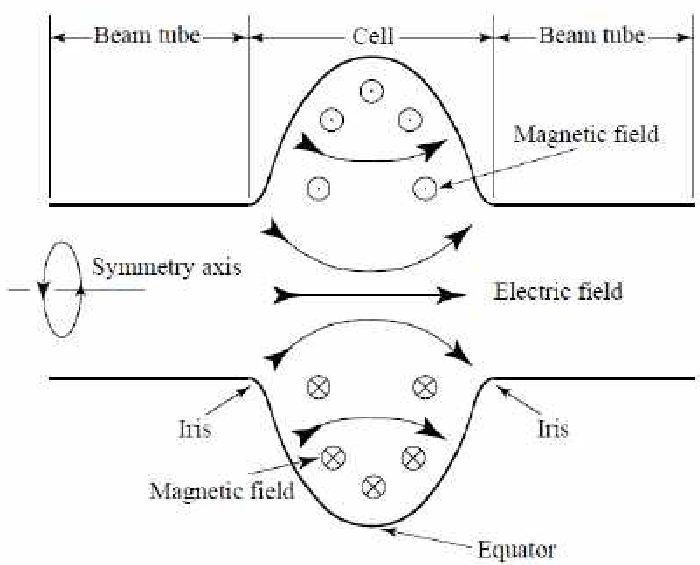
TM22: $\lambda_c = \frac{2a}{\sqrt{1+(a/b)^2}}$

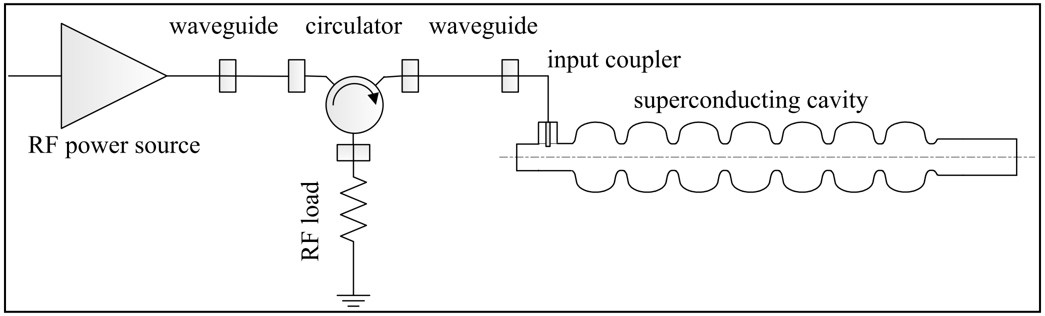
RF cavities

are designed to confine the EM field inside: It means that they operate at frequency below cut-off of the beam-pipes attached to them



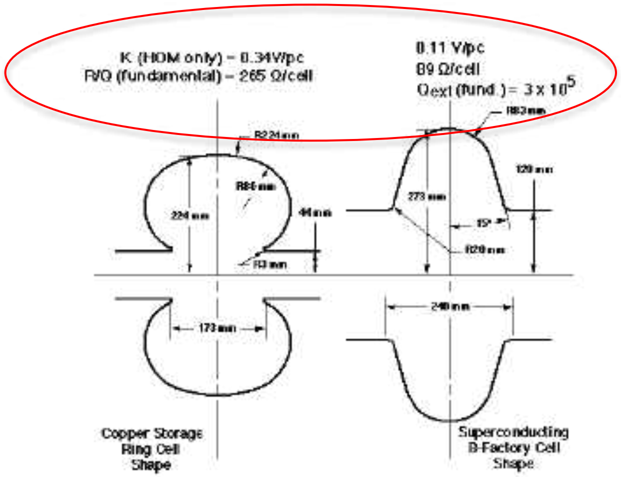


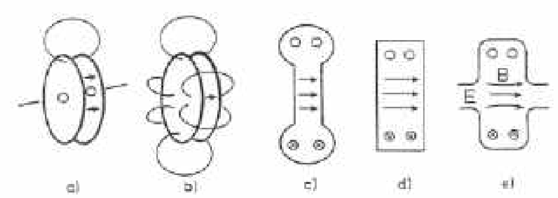


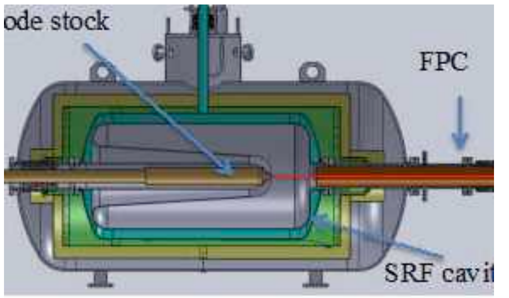


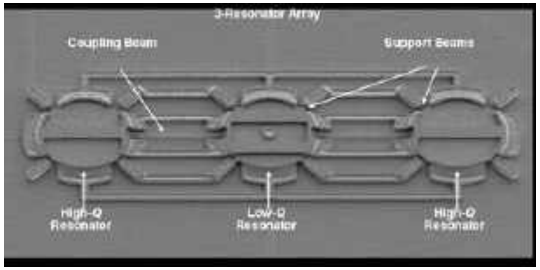
RF cavities come in many shapes, forms and sizes

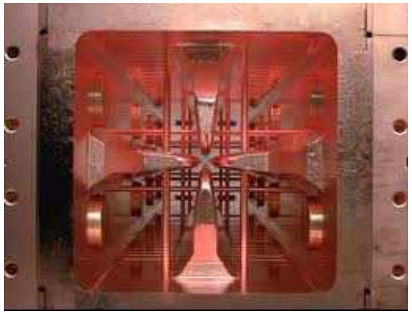
What these mean?

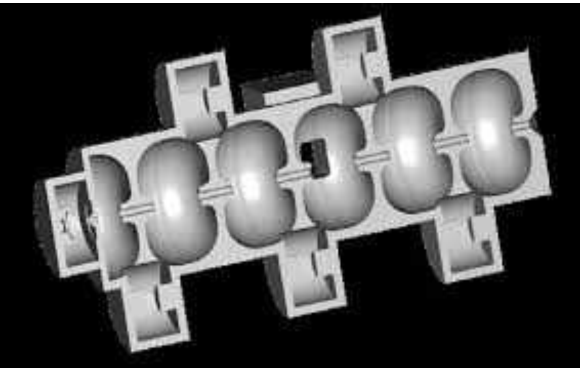


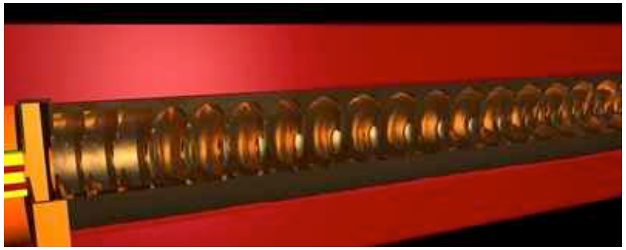












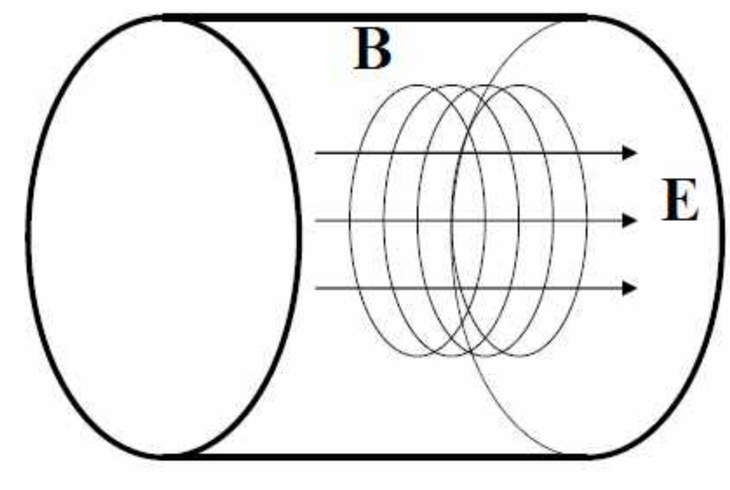
RF Cavity Modes:
the lowest accelerating is TM_{010} mode

- Fields in the cavity are solutions of the equation
$$\left(\nabla^2 - \frac{1}{c}\frac{\partial^2}{\partial t}\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = 0$$
- Subject to the boundary conditions $\hat{n}\times\mathbf{E}=0,\ \hat{n}\cdot\mathbf{H}=0$
- Two extra surfaces (z=0 and z=d): but this is no problem for TM mode
- An infinite number of solutions (eigen modes) belong to two families of modes with different field structure and eigen frequencies: TE modes have only transverse electric fields, TM modes have only transverse magnetic fields.
- One needs longitudinal electric field for acceleration, hence the lowest frequency TM_{010} mode is used.
- For the pillbox cavity w/o beam tubes
- Note that frequency does not depend of the cavity length! But only its radius.

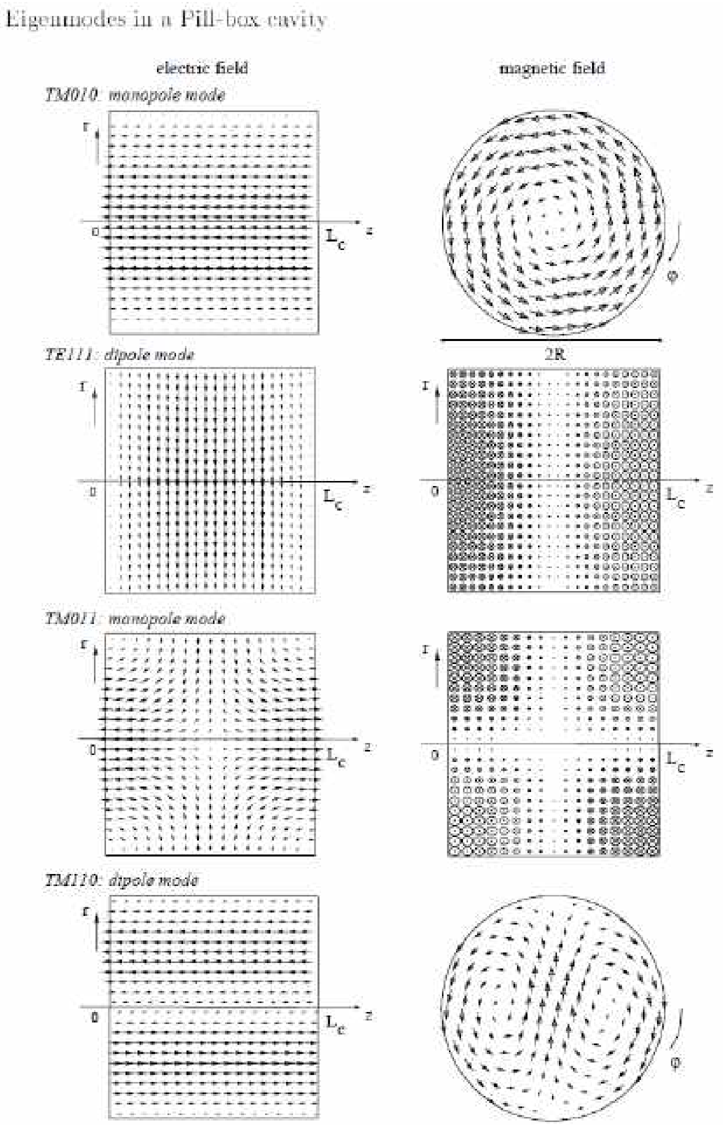
$$E_z = E_0 J_0\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

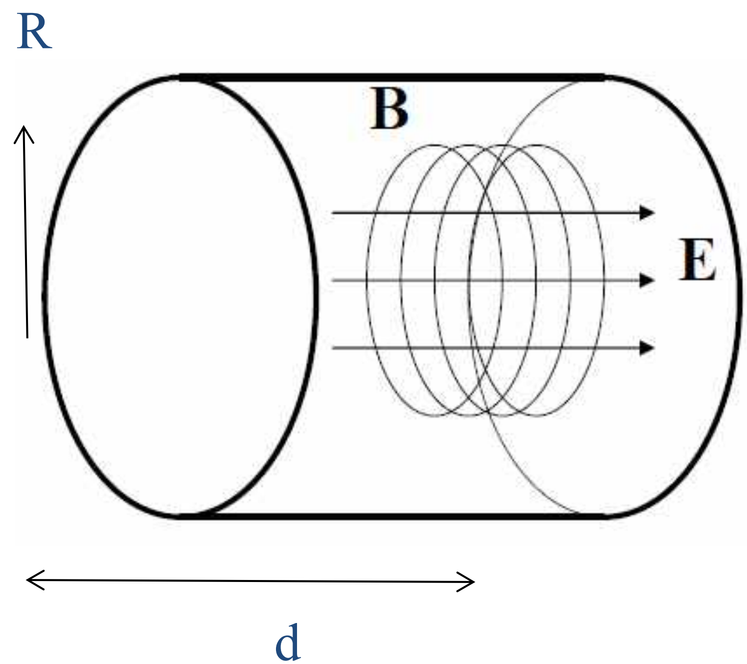
$$H_\phi = -iE_0 J_1\left(\frac{2.405r}{R}\right)e^{i\omega t}$$

$$\omega_{010} = \frac{2.405c}{R},\quad \lambda_{010} = 2.61R$$



Fundamental and high order modes (HOMs)





$$TM:\varphi_{mnl}=J_n(\gamma_{mn}r)\cos k_{z,l}z;\ k_{z,l}=l\frac{\pi}{d};\ J_n(\gamma_{mn}R)=0;$$

$$\omega_{res}=c\sqrt{\gamma_{mn}^2+l^2\frac{\pi^2}{d^2}};l=0,1,2....$$

$$TE:\varphi_{mnl}=J_n(\kappa_{mn}r)\sin k_{z,l}z;\ k_{z,l}=l\frac{\pi}{d};\ J_n'(\kappa_{mn}R)=0;$$

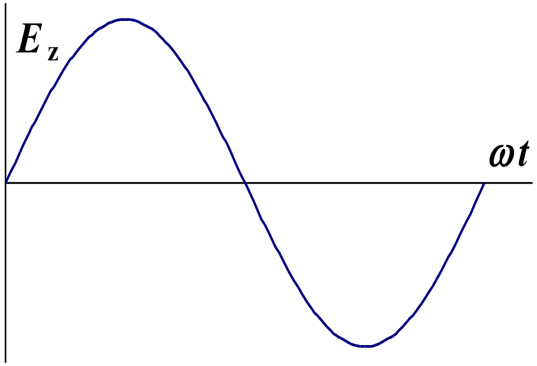
$$\omega_{res}=c\sqrt{\kappa_{mn}^2+l^2\frac{\pi^2}{d^2}};l=1,2....$$

Acceleration inside RF cavity

- Let's consider a cavity terminated by a vacuum pipe for particles to pass
- Let's also consider a charge particle passing on the axis of the cavity the cavity with constant velocity (e.g. either particle is ultra relativistic, or velocity change is very small)
  - Electric field on the axis depending both on z and time

$$\mathbf{E_z}(z,t) = \mathbf{E_o}(z)\cos(\omega_0 t + \varphi)$$

- Specific form of $E_o(z)$ depends on the cavity design
- Energy change of the particle with charge q passing through the cavity is:



$$\Delta E = q\int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos(\omega_0 t + \varphi)\,dz$$

$$t = \frac{z}{\mathrm{v}} \Rightarrow \Delta E = q\int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos\left(\omega_0 \frac{z}{\mathrm{v}} + \varphi\right)dz$$

$$\Delta E = qV_{RF}\cos(\varphi + \varphi_o)$$

$$V_{RF} = \sqrt{V_s^2 + V_c^2};\tan(\varphi_o) = \frac{V_c}{V_s};\ \ V_c = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz;\ V_s = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\sin\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz$$

How it is done

$$t=\frac{z}{\mathrm{v}};\Delta E=q\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos(\omega_0 t+\varphi)dz=q\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}+\varphi\right)dz$$

$$\cos\left(\omega_0\frac{z}{\mathrm{v}}+\varphi\right)=\cos(\varphi)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)-\sin(\varphi)\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)$$

$$\Delta E=q\cos(\varphi)\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)dz-q\sin(\varphi)\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)dz$$

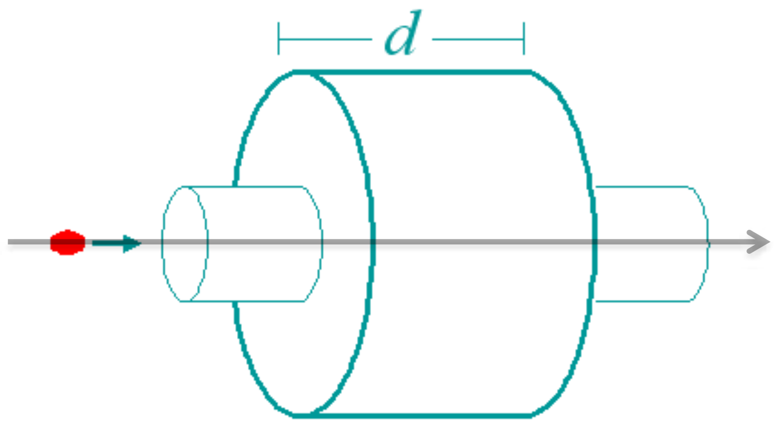
$$V_c=\int_{-\infty}^{\infty}\mathbf{E_o}(z)\cos\left(\omega_0\frac{z}{\mathrm{v}}\right)dz;\ V_s=\int_{-\infty}^{\infty}\mathbf{E_o}(z)\sin\left(\omega_0\frac{z}{\mathrm{v}}\right)dz$$

$$V_{RF}=\sqrt{V_s^{\ 2}+V_c^{\ 2}};\tan(\varphi_o)=\frac{V_c}{V_s};$$

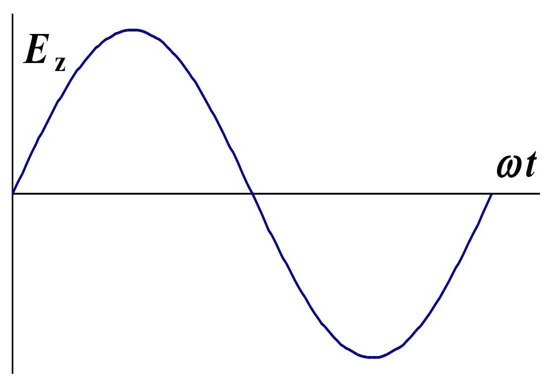
$$\Delta E=qV_{RF}\cos(\varphi+\varphi_o)$$

For particle moving with constant velocity all cavities are described by accelerating voltage and phase! Nothing else

- Now let's consider a pillbox cavity where E_z field is constant and extends from –*d/2* to +*d/2* with a small diameter vacuum pipes attached to it – the later required for particles to get through. With frequency of the RF cavity bellow the cut-off frequency of the vacuum piper, the RF field decays very fast in the pipe.

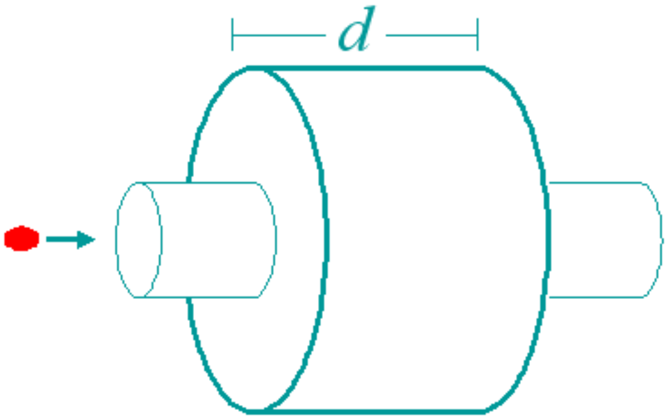


$$\Delta E = qV_{RF} \cdot \cos\varphi_o ; \varphi_o = \omega t$$



$$\mathbf{E_o}(z) = \begin{cases} \mathbf{E_o}, |z| \le d/2 \\ 0, |z| > d/2 \end{cases}$$

Accelerating voltage & transit time



- Assuming charged particles moving along the cavity axis, one can calculate accelerating voltage as

$$V_c = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i\omega_0 z/\beta c} dz \right|$$

For the pillbox cavity one can integrate this analytically:

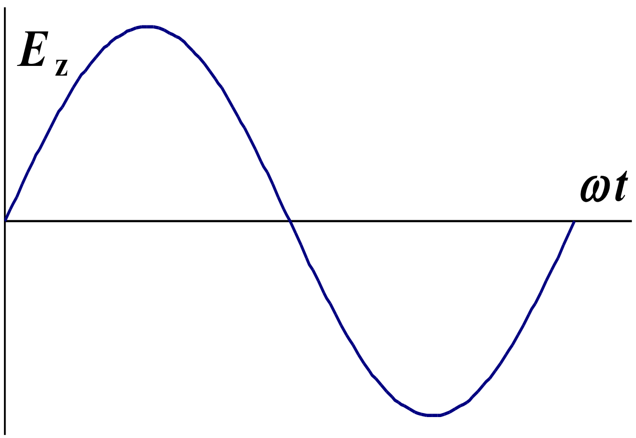
$$V_c = E_0 \left| \int_0^d e^{i\omega_0 z/\beta c} dz \right| = E_0 d \frac{\sin\left(\frac{\omega_0 d}{2\beta c}\right)}{\frac{\omega_0 d}{2\beta c}} = E_0 d \cdot T$$

where T is the transit time factor.

- To get maximum acceleration:

$$T_{transit} = t_{exit} - t_{enter} = \frac{T_0}{2} \Rightarrow d = \beta\lambda/2 \Rightarrow \quad V_c = \frac{2}{\pi} E_0 d$$

Thus for the pillbox cavity $T = 2/p$.

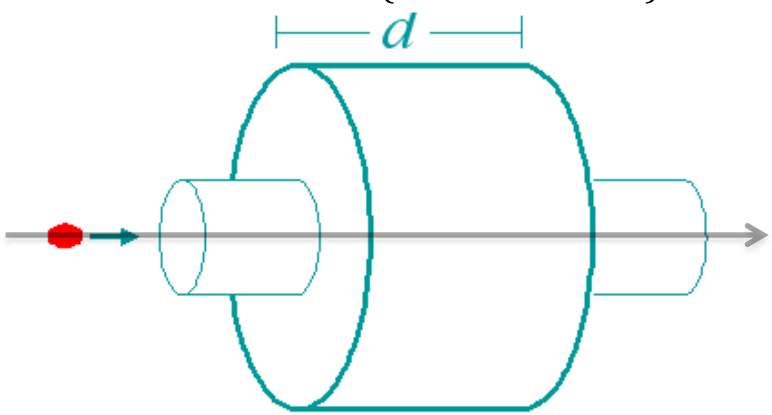


- The accelerating field E_{acc} is defined as $E_{acc} = V_c/d$. Unfortunately the cavity length is not easy to specify for shapes other than pillbox so usually it is assumed to be $d = bl/2$. This works OK for multi-cell cavities, but poorly for single-cell ones.

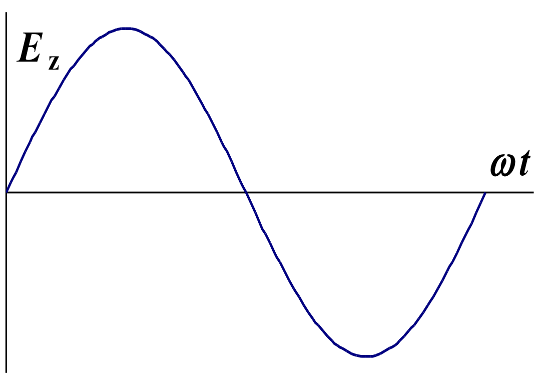
Acceleration inside RF cavity (cont..)

- Now let's consider a pillbox cavity where E_z field is constant and extends from –*d/2* to +*d/2*. Field decays very fact in the pipe

$$\mathbf{E_o}(z) = \begin{cases} \mathbf{E_o}, |z| \le d/2 \\ 0, |z| > d/2 \end{cases}$$



$$\boxed{\Delta E = qV_{RF} \cdot \cos\varphi_o; \varphi_o = \omega t}$$



$$V_s = \mathbf{E_o} \int_{-d/2}^{d/2} \sin\left(\omega_0 \frac{z}{\text{v}}\right) dz = 0$$

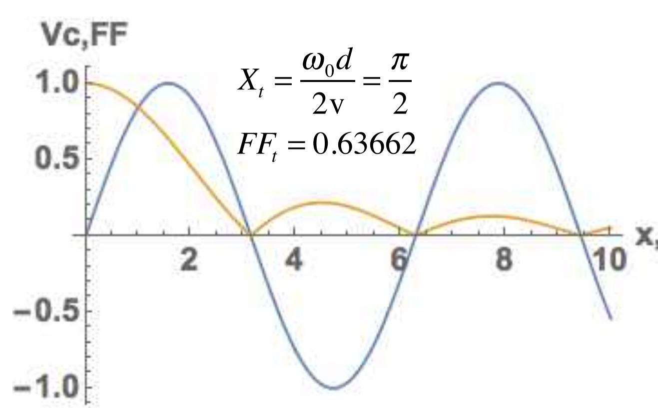
$$V_c = \mathbf{E_o} \int_{-d/2}^{d/2} \cos\left(\omega_0 \frac{z}{\text{v}}\right) dz = \mathbf{E_o} \frac{2\text{v}}{\omega_0} \cdot \sin \frac{\omega_0 d}{2\text{v}} \Rightarrow V_c = \mathbf{E_o} d \cdot \frac{\sin X_t}{X_t}; X_t = \frac{\omega_0 d}{2\text{v}}$$

$$V_{RF} = |V_c|; \tan(\varphi_o) = 0;$$

- Thus, the accelerating voltage differs from the ideal $E_o d$ by the transit time factor

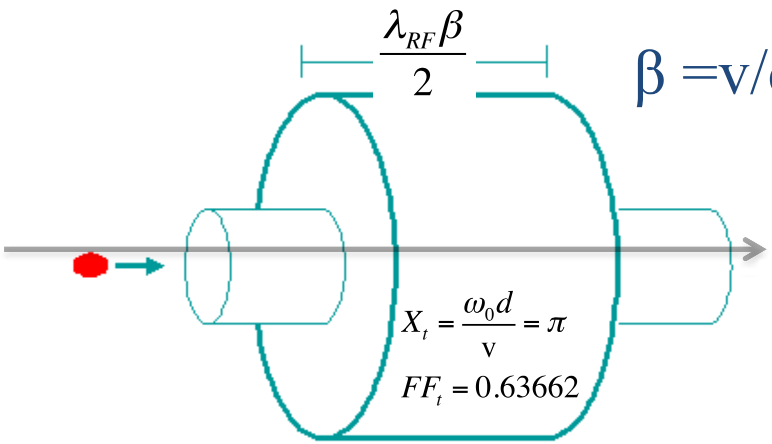
$$\frac{V_{RF}}{\mathbf{E_o} d} = FF_t = \left|\frac{\sin X_t}{X_t}\right|; X_t = \frac{\omega_0 d}{2\text{v}}$$

- Thus making cavity longer than the distance particle passed during ½ of the RF period makes no sense ($X_t = \pi/2$)



What are β=x cavities

- For heavy particles like protons, it takes a lot of RF cavities to accelerate to velocity comparable to speed of the light
- Hence, there are so called low-β cavities designed for slow particles
- You will see in literature β=0.1, β=0.5… cavities – it means that they are designed. For particle traveling nearly speed of light cavities called β=1.



β =v/c

$$\Delta E = qV_{RF} \cdot \cos\varphi_o; \varphi_o = \omega t$$

β =1 Pillbox

$$\frac{\omega_0 d}{c} = \pi$$

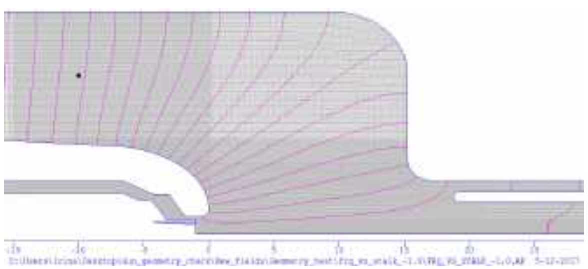
$$FF_t(\beta = 1) = 0.6366$$

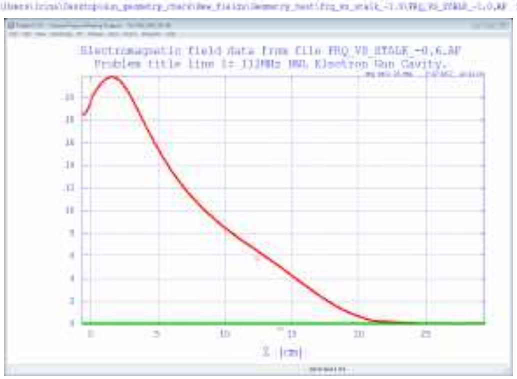
$$FF_t(\beta = 0.8) = 0.4705$$

$$FF_t(\beta = 0.5) = 0$$

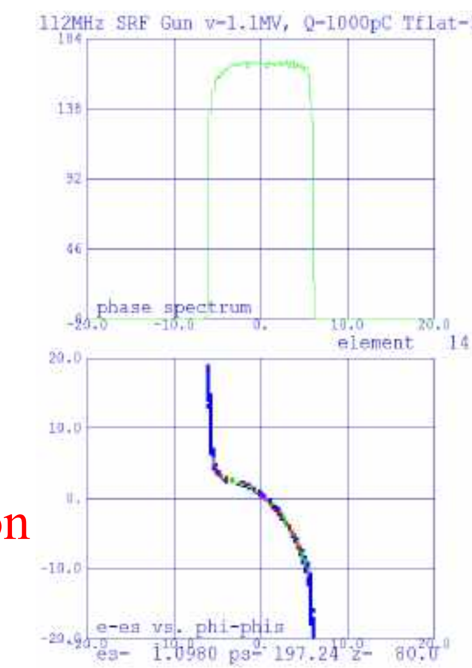
What about $\beta \neq$constant

- Typically we can use approximation that velocity of accelerating beam is nearly constant when it passing the cavity gap.
- This assumption is good for ultra-relativistic electrons/positions which are moving with velocity very close to the speed of light.
- This assumption is also a good approximation for heavy particles (ions or protons) when the energy gain per one accelerator cell is a small portion of the particle's rest mass energy.
- This assumption is violated and can not be used for electron guns, where electrons can accelerate from zero velocity to nearly speed of light. Equation of motion are both time-dependent and non-linear. You can estimate the result, nowadays it is normal to use numerical codes to get all beam dynamics correctly.





113 MHz SRF photo-emission Electron gun for CeC



Multi-cell cavities

- We learned so far that single cell RF cavity has limited accelerating voltage

$$Max\left(V_{RF}\right) = \frac{\mathbf{E_o}\lambda_{RF}}{\pi}$$

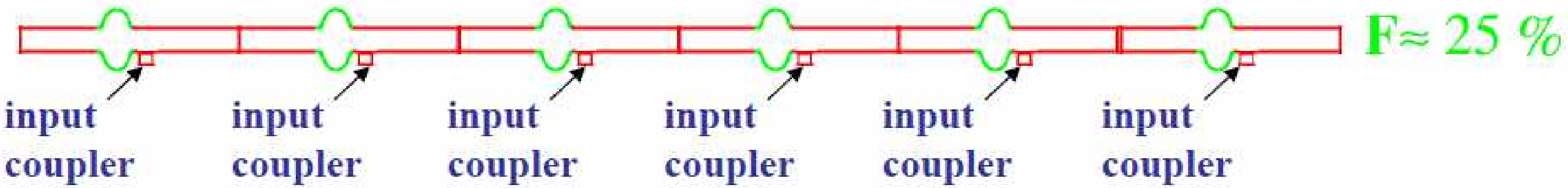
- To gain more energy we can either use more individual cells or use multi-cell cavities
- The first path, while feasible, is expensive (each cavity would need individual transmitter, waveguide, controls, etc.) and less effective – the average accelerating gradient (energy gain per meter of real estate) would be low
- Thus, where the acceleration gradient is important, the accelerator community uses multi-cell cavities

9-cell Tesla design

7-cell

5-cell

Why multi-cell cavities?



higher fill-factor:

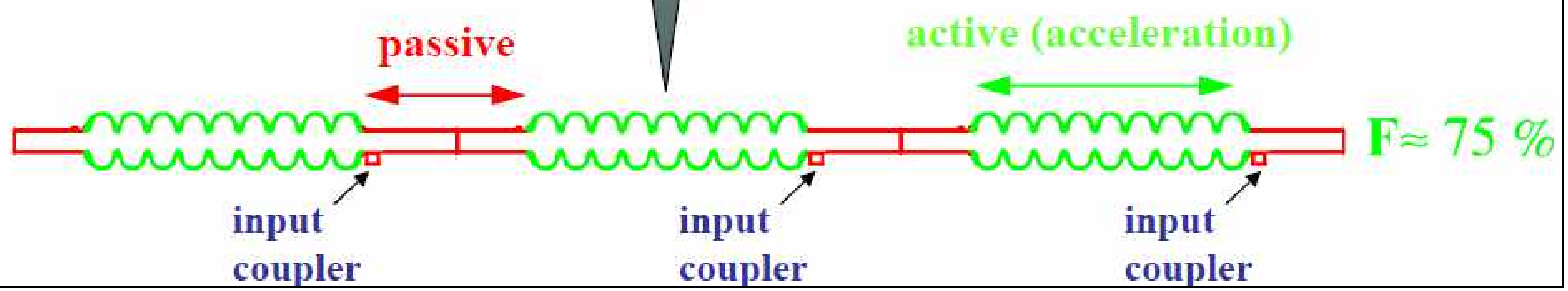
$$F = \frac{\text{active length}}{\text{total length}}$$

⇒ *lower costs*

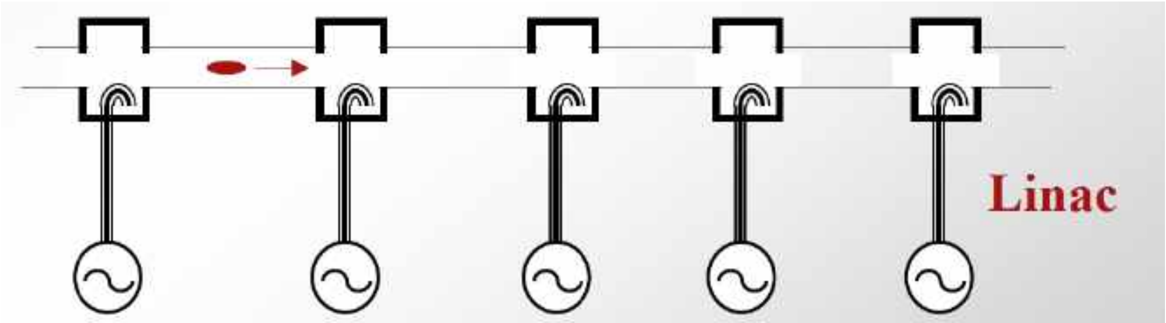
⇒ *better beam*

fewer

- **input couplers**
- **waveguide elements**
- **RF control systems**
- **…**



5-cell linac



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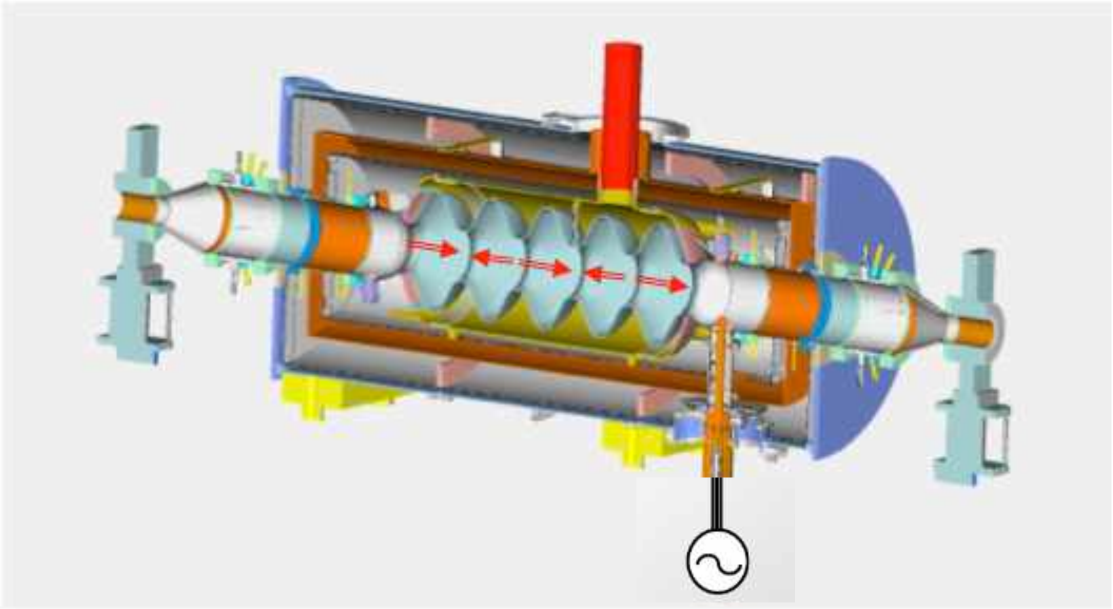
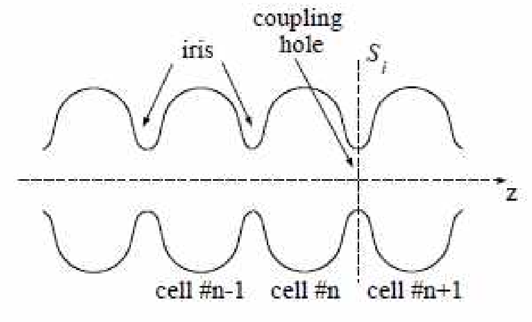
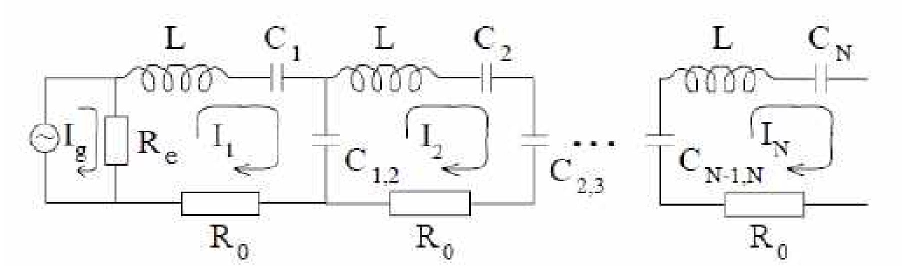


Multi-cell cavities

- Cavity consisting of n-cell is similar to N-coupled linear oscillators or resonant contours
- They all have nearly identical frequencies, but coupling splits then in n modes

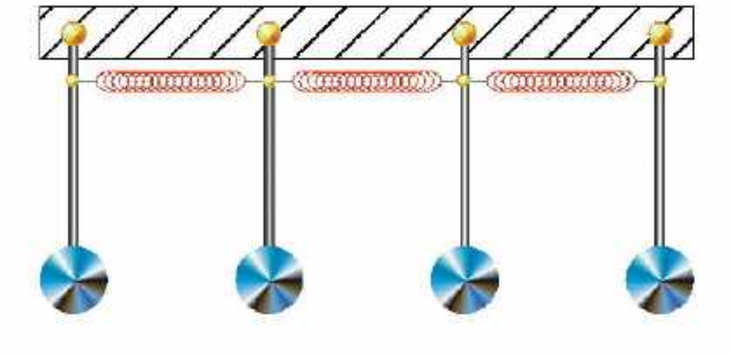




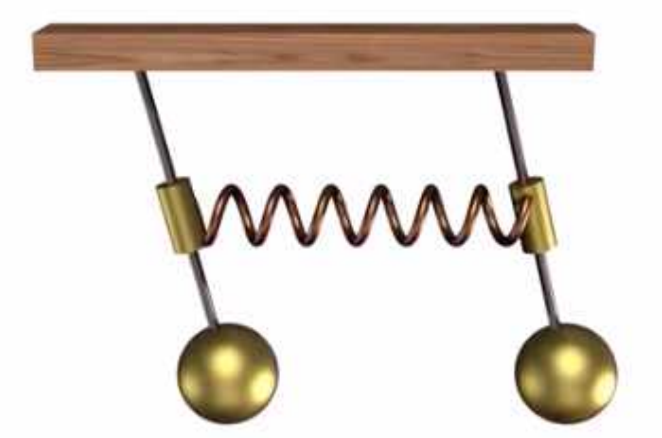
- The width of the pass-band (frequencies of various coupled modes) is determined by the strength of the cell-to-cell coupling *k* and the frequency of the *n*-th mode can be calculated from the dispersion formula

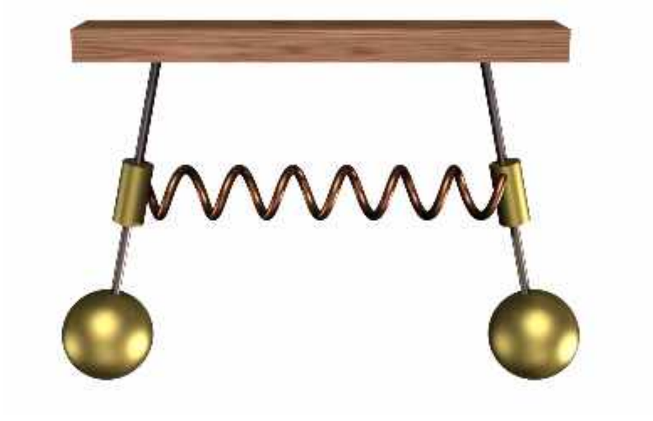
$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

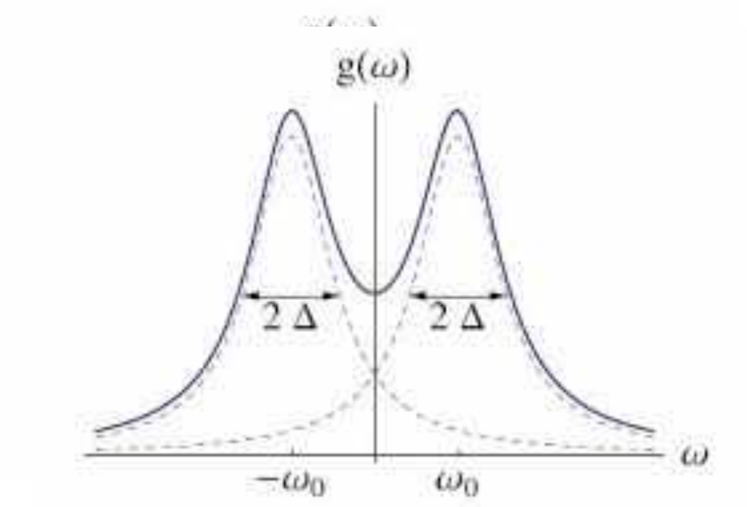
where *N* is the number of cells,
n = 1 … N is the mode number.



Two coupled oscillators: 0-mode and π-mode

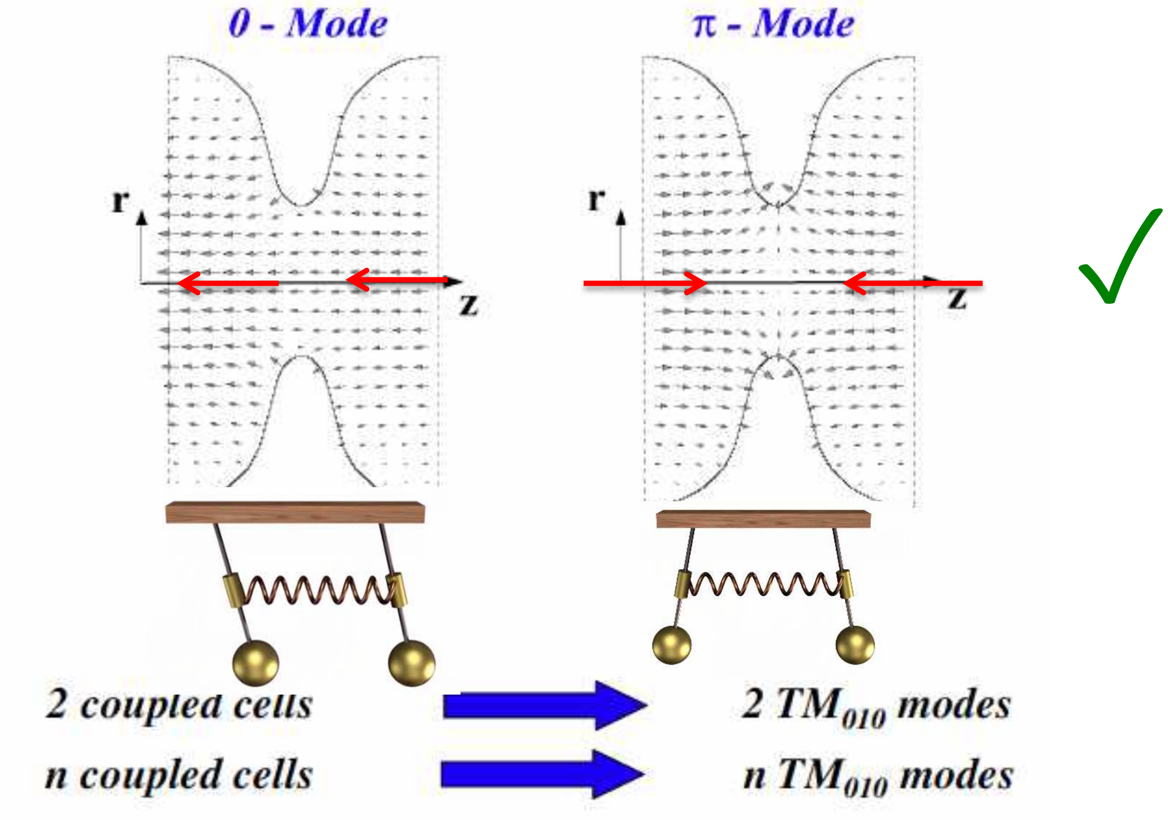






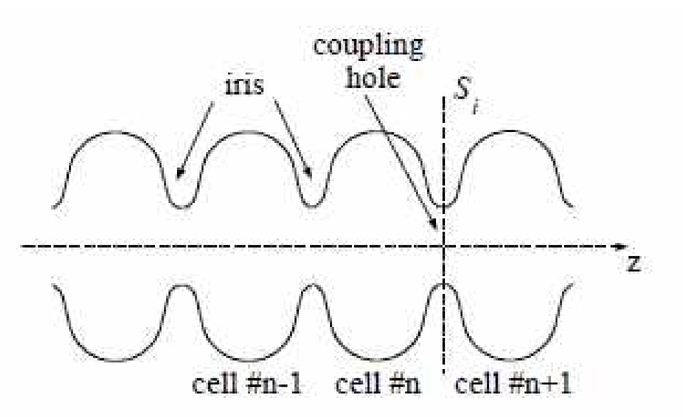
Multi-cell cavities: coupled oscillators

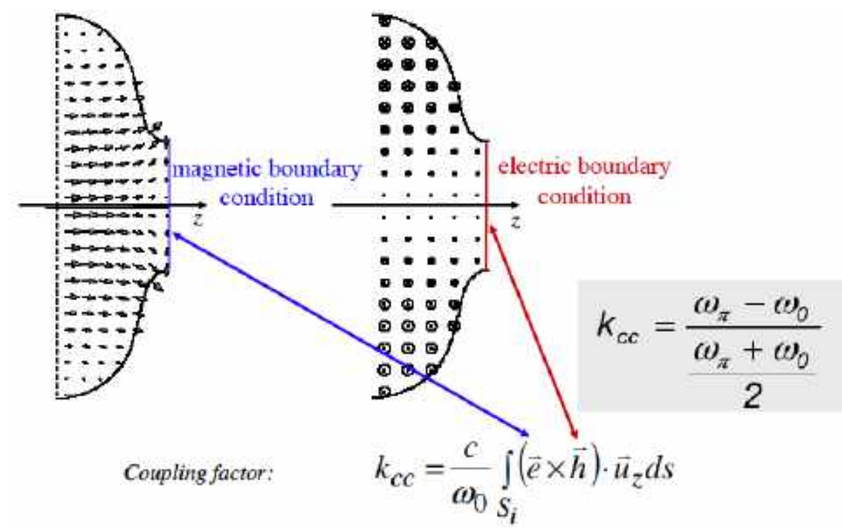
- Several cells can be connected together to form a multi-cell cavity
- Coupling of TM_{010} modes of the individual cells via the iris causes them to split
- 0-mode does not give any advantages – all cavities have the same direction of the field…
- π-mode is of special interest for us:
  - electric field has opposite directions on neighboring cells
  - particle passes through accelerating voltage in a cell in half of RF period
  - when particle crosses to the next cell – it sees again accelerating voltage



Multi-cell cavities - coupling

- Even though calculating coupling between the cavities is straight forwards, in practice is done using EM cavity codes
- For us is important to know that larger iris provides for stronger coupling and better uniformity of the field
- But increasing the iris reduces the electric field on axis (shunt impedance) and reduces accelerating gradient of such accelerator - hence, there is a compromise





$$\frac{d^2 x_1}{dt^2} + \omega_o^2 x_1 = -k x_2$$

......

$$\frac{d^2 x_n}{dt^2} + \omega_o^2 x_n = k x_{n-1} - k x_{n+1}$$

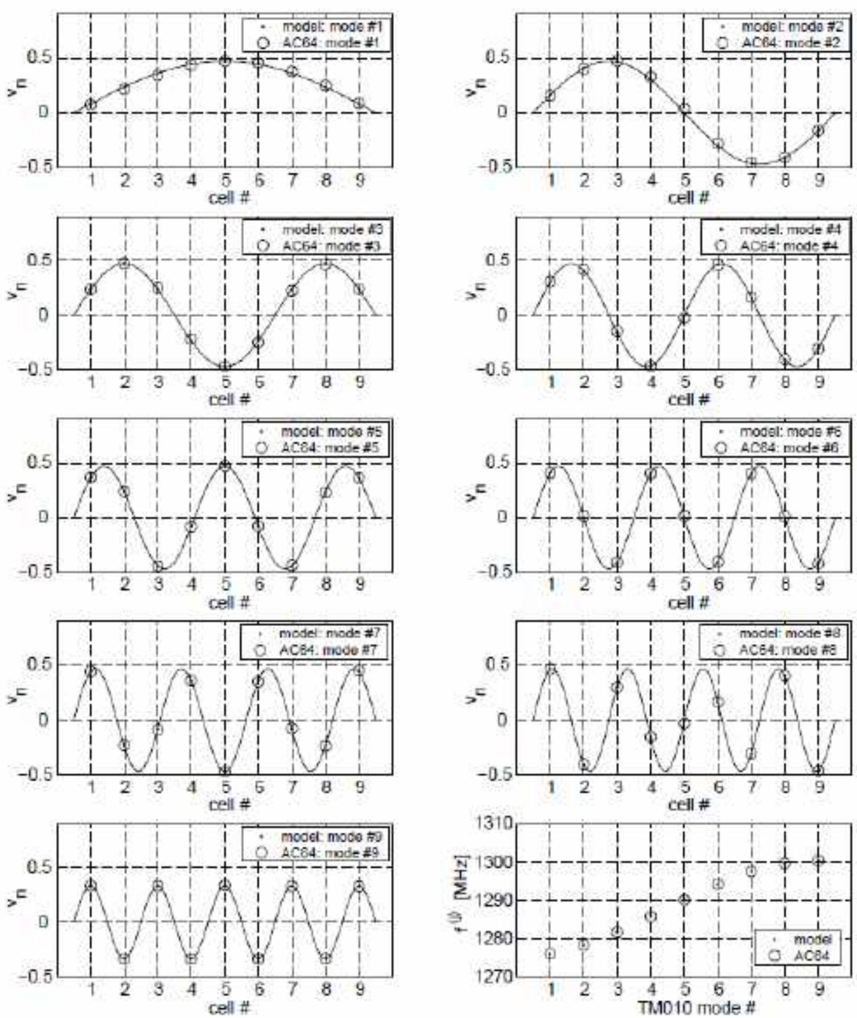
.....

$$\frac{d^2 x_N}{dt^2} + \omega_o^2 x_N = +k x_{N-1}$$

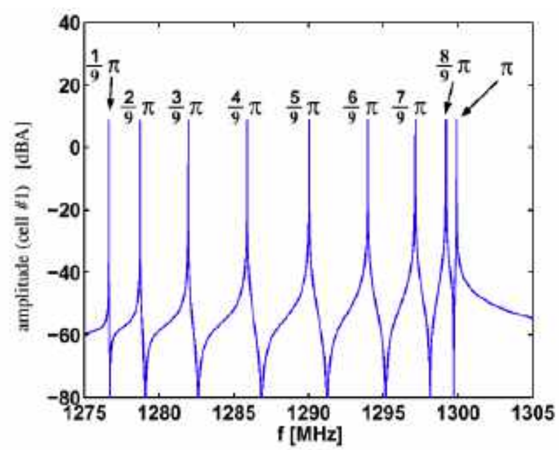
$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

where N is the number of cells,
$n = 1 \dots N$ is the mode number.

Multi-cell cavities



- Figure shows an example of calculated eigenmodes amplitudes in a 9-cell TESLA cavity compared to the measured amplitude profiles. Also shown are the calculated and measured eigenfrequencies.
- A longer cavity with more cells has more modes in the same frequency range, hence the reduction in frequency difference between adjacent modes. The number of cells is usually a result of the accelerating structure optimization.
- The accelerating mode for SC cavities is usually the **p**-mode, which has the highest frequency for electrically coupled structures.
- The same considerations are true for HOMs.



Realistic RF cavity (linac)

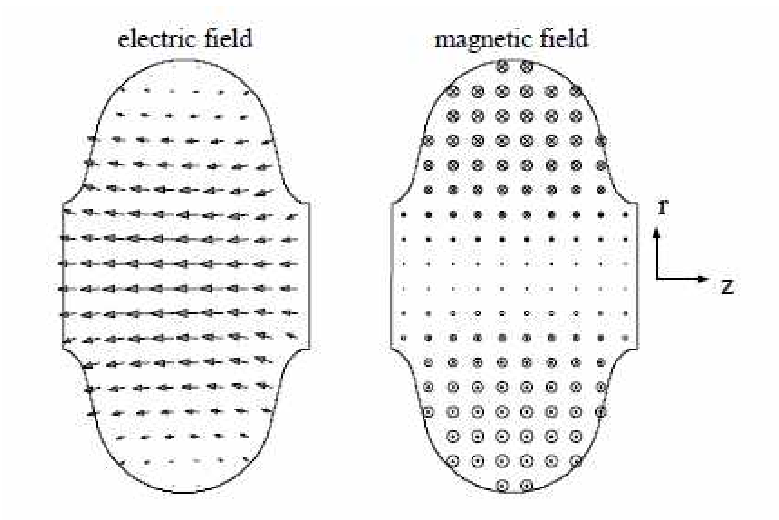
This material is likely for your home reading

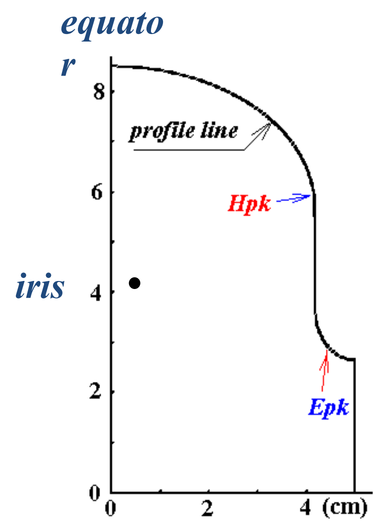
- Final conductivity of the surfaces
  - Approximation of the boundary conditions
  - Surface impedance, losses in the surface
- Main RF cavity characteristics
  - Accelerating voltage, peak electric and magnetic field
    - Q factor: internal, external, total
    - Geometrical factor, G
    - Shunt impedance R_{sh}, R_{sh}/Q
    - Coupling coefficient, *ONE MORE β!*

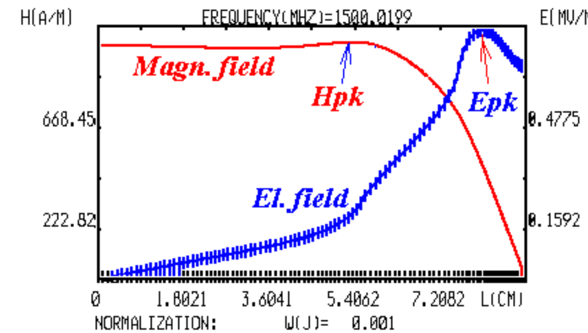
This part is usually related to more "engineering" factors measured in ohms, watts, etc.… – hence, for a change, we are using SI system…

Again, the main idea of this course: you are learning accelerator lingo and basis behind it

Typical SRF Cell fields (simulated using an EM code)





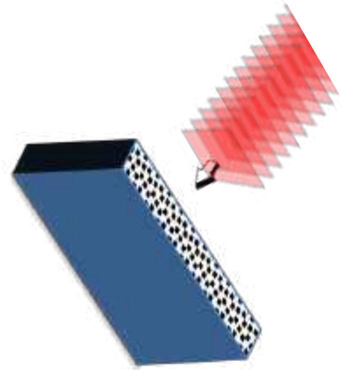


Fields along the profile line

- Important for the cavity performance are the ratios of the peak surface fields to the accelerating field. Peak surface electric field is responsible for field emission; typically for real cavities E_{pk}/E_{acc} = **2…2.6**, as compared to **1.6** for a pillbox cavity.
- Peak surface magnetic field has fundamental limit (critical field for SRF cavities – will discuss at next lecture); surface magnetic field is also responsible for wall current losses; typical values for real cavities H_{pk}/E_{acc} = **40…50 Oe/MV/m**, compare this to **30.5** for the pillbox
- In SGS system 1**Oe -> 1 Gs;** /MV/m is 33.3 Gs, hence ratio H_{pk}/E_{acc} is dimensionless and is close to unity: 0.92 for a pillbox cavity, 1.2 – 1.5 for elliptical cavities.
- Tangential magnetic field on the surface induces Ohmic losses and affect Q-factor

Boundary condition for an ideal conductor

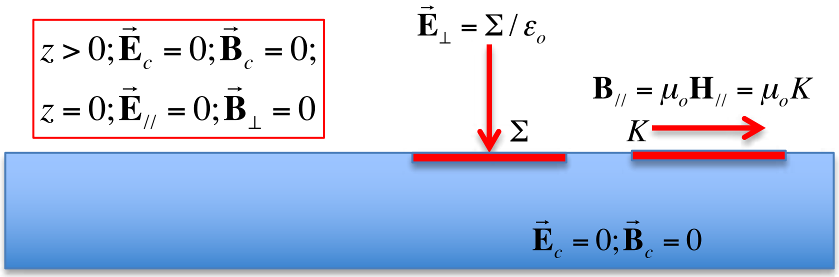
- For an ideal conductor, the condition inside the conductor are simple: both AC electric and magnetic fields are zero



$$\text{Ideal conductor}: \sigma \to \infty$$

$$\left|\vec{\mathbf{E}}_{//}\right| = \frac{\sqrt{2}\left|\vec{\mathbf{B}}_{//}\right|}{\mu\sigma\delta} \propto \frac{1}{\sqrt{\sigma}}\left|\vec{\mathbf{B}}_{//}\right| \to 0$$

$$i\omega\vec{\mathbf{B}}_{\perp} \equiv (\vec{k}\times\vec{\mathbf{E}}_{//})_{\perp} : \ \vec{\mathbf{B}}_{\perp} \to 0$$

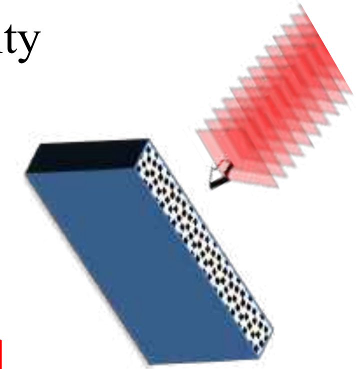


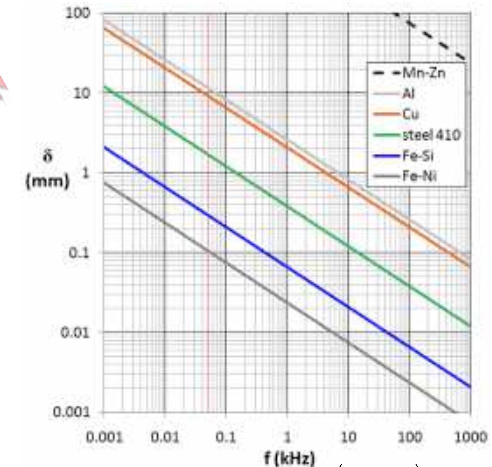
- Good cavities are build using very good conductors (including super-conductors)
- Hence, the electric field component parallel to the surface is very small (nearly zero – "not allowed") while the the magnetic field component parallel to the surface is not limited and in fact is given by the mode structure $\mathbf{B}_{//} = \mu_o \mathbf{H}_{//}$
- This parallel component of the field is compensated by the surface current, which naturally causes dissipation in real conductor

Real: the conducting surface

- As input, we have magnetic field of ideal cavity
- Inside the conductor the EM decays with typical length called skin depth

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$





$$\vec{\mathbf{E}}_{//} \cong \mathrm{Re}\,\mathbf{E}_o\, e^{-\frac{\vec{n}\vec{r}}{\delta}} e^{i\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)}; \mathbf{H}_{//} \cong \sigma\delta\,\mathrm{Re}\frac{\mathbf{E}_o}{1+i} e^{-\frac{\vec{n}\vec{r}}{\delta}} e^{i\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)}; \vec{n}\cdot\vec{\mathbf{E}} = 0;$$

$$\vec{\mathbf{H}}_{//} = e^{-x/\delta}\,\mathrm{Re}\,\mathbf{H_0}\, e^{-\frac{\vec{n}\vec{r}}{\delta}} e^{i\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)}$$

- The current density is

$$J \cong \frac{\sqrt{2}}{\delta}\left|\vec{\mathbf{H}}_{//}\right| e^{-\frac{\vec{n}\vec{r}}{\delta}} \cos\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)$$

$$K = \int J\, d\zeta \cong \vec{\mathbf{H}}_{//}$$

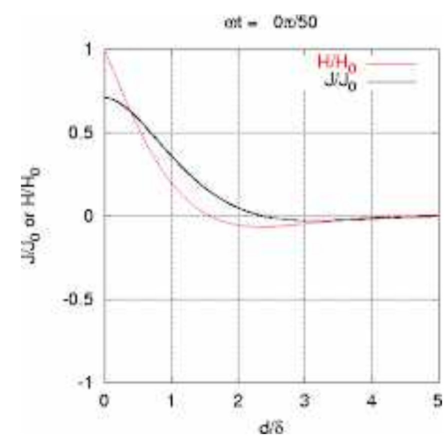
$$\left|\vec{\mathbf{E}}_{//}\right| = \frac{\sqrt{2}}{\sigma\delta}\left|\vec{\mathbf{H}}_{//}\right|$$

- And Ohmic losses per unit area

$$\frac{P_{loss}}{A} = \int \frac{\left\langle J^2(\zeta)\right\rangle_t}{\sigma} d\zeta \cong \frac{1}{2\delta\sigma}\left|\vec{\mathbf{H}}_{//}\right|^2$$

Surface impedance

$$Z_s \equiv \frac{\mathbf{E}_o}{K} = \frac{\mathbf{E}_o}{\mathbf{H_0}} = \frac{1+i}{\sigma\delta} = R_s + iX_s$$



At 10 GHz

Conductor	**Skin depth (µm)**
Aluminum	2.52
Copper	2.06
Gold	2.50
Silver	2.02

$$\frac{P_{loss}}{A} = \frac{1}{2} R_s \left|\vec{\mathbf{H}}_{//}\right|^2$$

$$R_s = \sqrt{\frac{\omega\mu}{\sigma}}\,[\Omega]$$

Beware of factors 2!

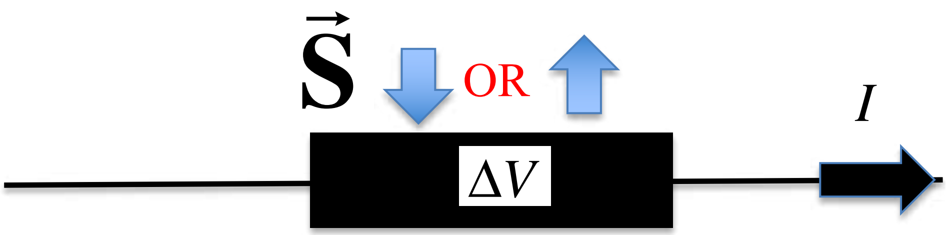
Question

- You should from E&M expression for Pointing vector

$$\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}}$$

indicating the flow of EM energy: direction and power density. Depending on you memory to remember “left hand” or “right hand” screw rule, you may get the direction either right or wrong… I have 50% success.

Based on the energy conservation law, please find direction of the EM energy flow in the case of a simple resistor with a current flowing through it. Is it pointed inside the surface of the resistor or outside? Does the result depends on the direction of the current?



$$\Delta V = -\mathbf{E}_z \cdot L = RI; \quad \oint \vec{\mathbf{H}} d\vec{l} = 2\pi r \vec{\mathbf{H}}_\varphi = I$$

Quality factor (SI)

- Let's consider a stand-alone cavity without any external couplers

- Energy stored in the cavity

$$W = \int \left(\varepsilon_o \frac{\vec{\mathbf{E}}^2}{2} + \mu_o \frac{\vec{\mathbf{H}}^2}{2} \right) dV = \frac{\mu_o}{2} \int \vec{\mathbf{H}}_o^2 \, dV$$

- Losses in the walls

$$P_{loss} = \oiint \frac{1}{2} R_s \left| \vec{\mathbf{H}}_0 \right|^2 dA = \frac{dW}{dt}$$

- Quality factor (definition)

$$Q_0 \equiv \frac{\omega_0 \cdot (\text{stored energy})}{\text{average power loss}} = \frac{\omega_0 U}{P_c} = 2\pi \frac{1}{T_0} \frac{U}{P_c} = \omega_0 \tau_0 = \frac{\omega_0}{\Delta\omega_0}$$

$$Q_0 = \frac{\omega_0 \mu_0 \int_V \left| \vec{\mathbf{H}}_0 \right|^2 dv}{R_s \oiint \left| \vec{\mathbf{H}}_0 \right|^2 dA}$$

- It is number of RF oscillation times 2π required for energy inside the cavity to reduce e-fold.

$$R_s = \sqrt{\frac{\omega \mu}{\sigma}} \, [\Omega]$$

Geometry factor: definition

- The ratio of two integrals determining Q-factor depends only on the cavity geometry: geometry defines eigen mode

$$\vec{u}(\vec{r}); \int |\vec{u}(\vec{r})|^2 dV = 1 \Rightarrow \vec{\mathbf{H}}_0(\vec{r}) = H_o \vec{u}(\vec{r});$$

$$G = \frac{\omega_0 \mu_0 \int_V |\vec{\mathbf{H}}_0|^2 dV}{\oiint |\vec{\mathbf{H}}_o|^2 dA} \equiv \frac{\omega_0 \mu_0 \int_V |\vec{u}|^2 dV}{\oiint |\vec{u}|^2 dA} \equiv$$

The parameter G is the geometry factor (also known as geometry constant)

Obviously

$$Q_0 = \frac{G}{R_s}$$

$$R_s = \sqrt{\frac{\omega\mu}{\sigma}} [\Omega]$$

- The geometry factor depends only on the cavity shape and electromagnetic mode, but not its size: Scaling the cavity size x-fold, increases volume as x^3, reduces frequency as 1/x and increasing surface as x^2. Hence, G does not change.
- **It is very useful for comparing different cavity shapes. TEM_{010} mode in a pillbox cavity had G = 257 Ohm independent the pillbox cavity length (d):**

G_{TEM010}= 257 Ohm for any ratio of the length to the radius.

- At f=1.5 GHz for a normal conducting copper (σ= 5.8 × 10^7 S/m) cavity we get $\delta = 1.7$ µm, R_s = 10 mOhm, and $Q_0 = G/R_s$ = 25,700.

Example: a pillbox cavity

- For a 1.5 GHz RF cavity
- normal conducting copper (σ= 5.8 × 10⁷ S/m)

$$\sigma = 5.8 \cdot 10^7 S/m; \quad \delta = 1.7 \mu m \quad \Rightarrow R_s = 10 m\Omega$$

$$Q_{Cu} = \frac{G}{R_s} = 25{,}700$$

- for superconducting Nb at 1.8 K surface resistance can be as low as few nOhm, but typically is ~20 nOhm.

$$R_s = 20 n\Omega$$

$$Q_{SRF} = \frac{G}{R_s} \propto 1.2 \cdot 10^{10}$$

- Six orders of magnitude in heat losses making SRF cavities very attractive. Even with loss in cooling efficiency 500 to 1,000-fold, there is still three orders of magnitude in cooling.
- Hence, SRF cavity can operate at 30-fold higher accelerating gradient compared with room temperature Cu cavity using the same amount of cooling.

Shunt impedance and R/Q: definitions

- The shunt impedance determines how much acceleration a particle can get for a given power dissipation in a cavity

$$R_{sh} = \frac{V_{RF}^2}{P_{loss}}$$

It characterized the cavity losses.

Often the shunt impedance is defined as in the circuit theory

$$R_{sh} = \frac{V_{RF}^2}{2P_{loss}}$$

and, to add to the confusion, a common definition in linacs is

$$r_{sh} = \frac{E_{acc}^2}{P'_{loss}}$$

where P'_{loss} is the power dissipation per unit length and the shunt impedance is in Ohms per meter.

- A related quantity is the ratio of the shunt impedance to the quality factor, **which is independent of the surface resistivity and the cavity size**:

$$\frac{R_{sh}}{Q_0} = \frac{V_{RF}^2}{\omega_0 W}$$

- This parameter is frequently used as a figure of merit and useful in determining the level of mode excitation by bunches of charged particles passing through the cavity. Sometimes it is called the geometric shunt impedance.
- **Pillbox cavity has *R/Q* = 196 Ohm**.

Dissipated power

- The power loss in the cavity walls is

$$P_{loss} = \frac{V_c^2}{R_{sh}} \equiv \frac{V_c^2}{Q_0 \cdot (R_{sh}/Q_0)} \equiv \frac{V_c^2}{(R_s \cdot Q_0)(R_{sh}/Q_0)/R_s} \equiv \frac{V_c^2 \cdot R_s}{G \cdot (R_{sh}/Q_0)}$$

- To minimize the losses one needs to maximize the denominator.
- The material-independent denominator is *G*R/Q*
- This parameter should be used during cavity shape optimization.

Consider now frequency dependence.

- For normal conductors $R_s \sim \omega^{1/2}$:

$$\frac{P_{loss}}{L} \propto \frac{1}{G \cdot (R_{sh}/Q_0)} \cdot \frac{E_{acc}^2 R_s}{\omega} \propto \omega^{-1/2} \qquad \frac{P}{A} \propto \omega^{1/2}$$

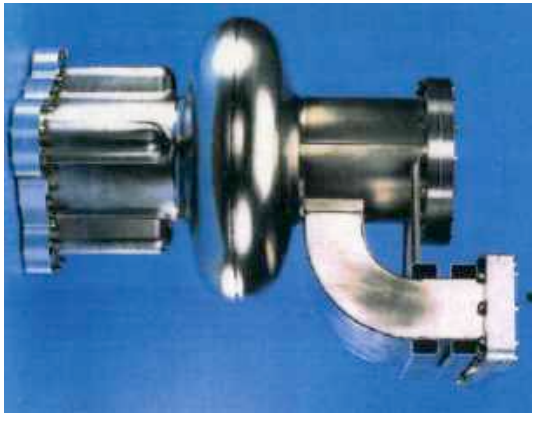
- For superconductors $R_s \sim \omega^2$

$$\frac{P_{loss}}{L} \propto \omega \qquad \frac{P}{A} \propto \omega^2$$

- NC cavities favor high frequencies, SC cavities favor low frequencies.

Pillbox *vs.* "real life" cavity

Quantity	Cornell SC 500 MHz	Pillbox
G	270 Ω	257 Ω
R_a/Q_0	88 Ω /cell	196 Ω/cell
E_{pk}/E_{acc}	2.5	1.6
H_{pk}/E_{acc}	52 Oe/(MV/m)	30.5 Oe/(MV/m)



- **In a high-current storage rings, it is necessary to damp Higher-Order Modes (HOMs) to avoid beam instabilities.**
- The beam pipes are made large to allow HOMs propagation toward microwave absorbers
- This enhances H_{pk} and E_{pk} and reduces R/Q.

Parameters of the 5-cell BNL3 cavity

Parameter	704 MHz BNL3 cavity
V_{acc} [MV]	20
No. of cells	5
Geometry Factor	283
R/Q [Ohm]	506.3
E_{pk}/E_{acc}	2.46
B_{pk}/E_{acc} [mT/MV/m]	4.26
Q_0	$> 2 \times 10^{10}$
Length [cm]	158
Beam pipe radius [mm]	**110**
Operating temperature [K]	1.9

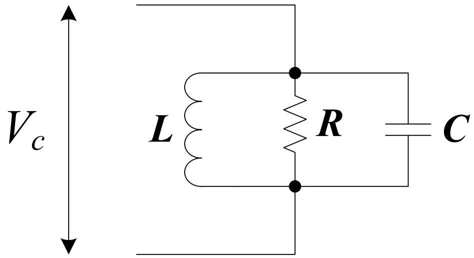
- **It was designed for high current Energy Recovery Linacs. It is necessary to damp dipole Higher-Order Modes (HOMs) to avoid beam instabilities.**
- The beam pipes are made large to allow HOMs propagation toward HOM couplers to damp the modes
- This enhances B_{pk} and E_{pk} and reduces R/Q.

Parallel circuit model

A resonant cavity can be modeled as a series of parallel *RLC* circuits representing the cavity eigen modes. For each mode:

dissipated power $$P_{loss} = \frac{V_c^2}{2R_{sh}}$$

shunt impedance $$R_{sh} = 2R$$

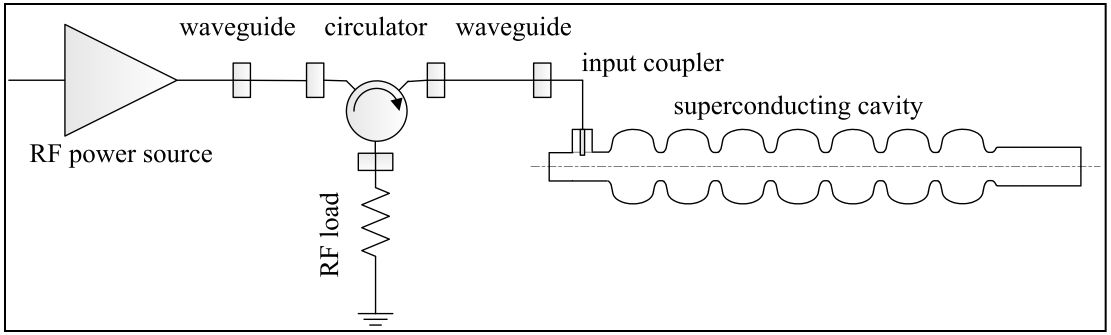


quality factor $$Q_0 = \omega_0 CR = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$$

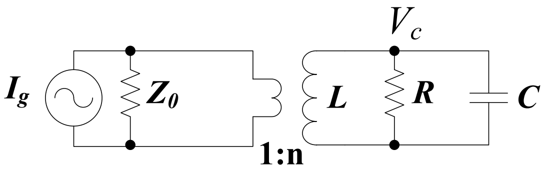
impedance $$Z = \frac{R}{1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \approx \frac{R}{1 + 2iQ\left(\frac{\omega - \omega_0}{\omega_0}\right)}$$

Connecting to a power source

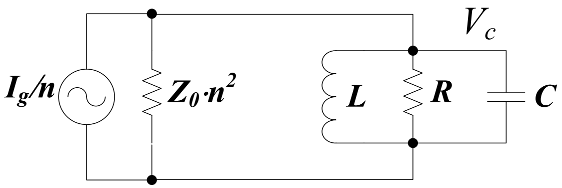
- Consider a cavity connected to an RF power source



- The input coupler can be modeled as an ideal transformer:



or



External & loaded Q factors

- If RF is turned off, stored energy will be dissipated now not only in R, but also in Z_0/n^2, thus

$$P_{tot} = P_o + P_{ext}$$

$$P_o = P_{loss} = \frac{V_c^2}{2R_{sh}} = \frac{V_c^2}{R_{sh}/Q \cdot Q_0} \qquad P_{ext} = \frac{V_c^2}{2Z_0 \cdot n^2} = \frac{V_c^2}{R_{sh}/Q \cdot Q_{ext}}$$

- This is definitions of an external quality factor associated with a coupler.
- Such Q factors can be identified with any external ports on the cavity: input coupler, RF probe, HOM couplers, beam pipes, etc.
- Then the total power loss can be associated with the loaded Q factor of

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext1}} + \frac{1}{Q_{ext2}} + \ldots$$

Coupling parameter β

- Coupling parameter is defined as

$$\beta \equiv \frac{Q_0}{Q_{ext}}$$

e.g.

$$\frac{1}{Q_L} = \frac{1+\beta}{Q_0}$$

- β defines how strongly the couplers interact with the cavity
- Large β implies that the power taken out of the coupler is large compared to the power dissipated in the cavity walls:

$$P_{ext} = \frac{V_c^2}{R/Q \cdot Q_{ext}} = \frac{V_c^2}{R/Q \cdot Q_0} \cdot \beta = \beta P_0$$

- The total power needed from an RF power source is expressed as

$$P_{forward} = (\beta + 1) P_0$$

What we learned about RF accelerators ?

- Resonant modes in a cavity resonator belong to two families: TE and TM.
- There is an infinite number of resonant modes.
- The lowest frequency TM mode is usually used for acceleration.
- All other modes (HOMs) are considered parasitic as they can harm the beam.
- Several figures of merits are used to characterize accelerating cavities:

$$V_{rf}\,,\ E_{peak}\,,\ H_{peak},\ R_s,\ Q_0,\ Q_{ext},\ R/Q,\ G,\ R_{sh}\ldots$$

- Superconducting RF cavities can have quality factor a million times higher than that of best Cu cavities.
- In a multi-cell cavity every eigen mode splits into a pass-band. The number of modes in each pass-band is equal to the number of cavity cells.
- Coaxial lines and rectangular waveguides are commonly used in RF systems for power delivery to cavities.