Homework 5. Due October 2

Problem 1. 2x5 points. Field expansion and field integral.
Accelerator physicists often prefer to express magnetic fields in Cartesian coordinate system. A typical magnetic field (with higher order components) can be conveniently written in a complex form,

$$\vec{B} = B_y(x,y,z) + iB_x(x,y,z) = \sum_{n=0}^{\infty} a_n(z)\,(x+iy)^n$$

where $B_x(x,y,z)$ and $B_y(x,y,z)$ are both real.

a) Show that the fields expressed in this way satisfies relations

$$\left(\frac{\partial B_y}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial x}\right)$$

and

$$\left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right)$$

which effectively shows that the field divergence (transverse) and curl (longitudinal) are zero

$$\nabla_\perp \cdot \vec{B} = \left(\frac{\partial B_x}{\partial x}\right) + \left(\frac{\partial B_y}{\partial y}\right) = 0$$

$$\left(\nabla\times\vec{B}\right)_\parallel = \left(\frac{\partial B_y}{\partial x}\right) - \left(\frac{\partial B_x}{\partial y}\right) = 0$$

i.e., no current in free space.
Solve:

$$B_y(x,y,z) = Re\left(\sum_{n=0}^{\infty} a_n(z)\,(x+iy)^n\right) = \sum_{n=0}^{\infty}(-1)^{(n-k)/2}\sum_{k,even} a_{n-k}(z)\,C_n^k x^k y^{n-k}$$

$$B_x(x,y,z) = Im\left(\sum_{n=0}^{\infty} a_n(z)\,(x+iy)^n\right) = \sum_{n=0}^{\infty}(-1)^{(n-l-1)/2}\sum_{l,odd} a_{n-l}(z)\,C_n^l x^l y^{n-l}$$

to prove $\left(\frac{\partial B_y}{\partial x}\right) = \left(\frac{\partial B_x}{\partial y}\right)$ (assume l = k-1), we just need to prove

$$kC_n^k = (n-k+1)C_n^{k-1}$$

which is obvious, since

$$kC_n^k = \frac{kn!}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-k)!} = \frac{(n-k+1)n!}{(k-1)!\,(n-k+1)!} = (n-k+1)C_n^{k-1}$$

similarly, it is easy to show $\left(\frac{\partial B_y}{\partial y}\right) = -\left(\frac{\partial B_x}{\partial x}\right)$

b) The scalar potential $\Phi(x,y,z)$ of this field ($\vec{B} = -\nabla\Phi$) should satisfy Laplace's equation:

$$\Delta\Phi \stackrel{\text{def}}{=} \nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

using Laplace's equation and by defining a integrated (over z) scalar potential as $\bar{\Phi}(\mathrm{x},\mathrm{y}) \stackrel{\text{def}}{=} \int_{z_1}^{z_2} \Phi(\mathrm{x},\mathrm{y},\mathrm{z})\, dz$

show that we can write the transverse Laplace of the integrated scalar potential as

$$\Delta_\perp\bar{\Phi} \stackrel{\text{def}}{=} \frac{\partial^2\bar{\Phi}}{\partial x^2} + \frac{\partial^2\bar{\Phi}}{\partial y^2} = B_z(x,y,z_2) - B_z(x,y,z_1)$$

discuss under what condition does the integrated field satisfy a two dimensional Laplace's equation? Discuss what are the requirements to use this relation when measuring the integrated field of a magnet, i.e., should the magnet be long or short, should the coil be long or short?

Solve:

Form definition,

$$\Delta_\perp\bar{\Phi} \stackrel{\text{def}}{=} \frac{\partial^2\bar{\Phi}}{\partial x^2} + \frac{\partial^2\bar{\Phi}}{\partial y^2} = -\frac{\partial^2\bar{\Phi}}{\partial z^2} = -\frac{\partial^2 \int_{z_1}^{z_2} \Phi(\mathrm{x},\mathrm{y},\mathrm{z})\, dz}{\partial z^2} = -\left.\frac{\partial\Phi(\mathrm{x},\mathrm{y},\mathrm{z})}{\partial z}\right|_{z_1}^{z_2}$$

use $\vec{B} = -\nabla\Phi$, we can write

$$\Delta_\perp\bar{\Phi} = B_z(x,y,z_2) - B_z(x,y,z_1) = B_2 - B_1$$

thus the transverse Laplace of the integrated scalar potential is zero when B_2=B_1=0, this is typically the case when far out of the magnet or at the center of the magnet (there is no longitudinal magnetic field component for a magnet, what happens for a solenoid?). Thus if magnet is short and the measuring coil is long, this condition is satisfied, this is usually the setup for a rotating coil measurement. On the other hand, when magnet is long and measuring coil is short, there is no longitudinal magnet components, the Laplace's equation is also satisfied, this is usually the setup for a Hall probe measurement.