1. Calculate the relative relation between $\Delta V/V$, $\Delta P/P$ and $\Delta E/E$. Note $V$ is the velocity of the particle, $P$ is the amplitude of the momentum and $E$ is the energy of the particle.

Answer: From the relation $E = \sqrt{m^2c^4 + P^2c^2}$, we have $dE = c^2PdP/E$. Divide $E^2$ on both side we have $dE/E = (cP/E)^2dP/P$. Note that $\beta = cP/E$, we get $dE/E = \beta^2dP/P$. Then we start from the relation $\beta = cP/E$, the relative change is $d\beta/\beta = dP/P - dE/E = (1 - \beta^2)dP/P = (1/\gamma^2)(dP/P)$

2. In class, we transform the longitudinal map

$$\delta_{n+1} - \delta_n = \frac{eV}{\beta^2 E_0} (\sin \phi_n - \sin \phi_s)$$
$$\phi_{n+1} - \phi_n = 2\pi \eta \delta_{n+1}$$

to longitudinal effective Hamiltonian. Actually we can also establish the one turn matrix for longitudinal motion if assume $\phi_n = \phi_s + \Delta_n$, where $|\Delta_n| \ll 1$. Find this matrix for $(\delta_{n+1}, \Delta_{n+1})$ from $(\delta_n, \Delta_n)$. Find the tune for this map, by assuming the tune is very close to zero, which is true in ring accelerator.

Answer: The matrix can be built as:

$$\begin{pmatrix} \delta_{n+1} \\ \phi_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2\pi \eta \delta_{n+1} & 1 \end{pmatrix} \begin{pmatrix} \delta_n \\ \phi_n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 2\pi \eta \delta_{n+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{eV \cos(\phi_s)}{\beta^2 E_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_n \\ \phi_n \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \frac{eV \cos(\phi_s)}{\beta^2 E_0} \\ 2\pi \eta \delta_{n+1} & 1 + \frac{2\pi eV \eta \cos(\phi_s)}{\beta^2 E_0} \end{pmatrix} \begin{pmatrix} \delta_n \\ \phi_n \end{pmatrix}$$

The trace of the one-turn longitudinal matrix is $Tr = 2 + 2\pi eV \eta \cos(\phi_s)/\beta^2 E_0$.

Compare with the transverse case the one turn map is

$$\begin{pmatrix} \cos \psi + \alpha \sin \psi & \beta \sin \psi \\ -\gamma \sin \psi & \cos \psi - \alpha \sin \psi \end{pmatrix}$$

where $\psi = 2\pi \nu_s$. The longitudinal phase advance can be found from the relation $2 \cos \psi = Tr = 2 + 2\pi eV \eta \cos(\phi_s)/\beta^2 E_0$. Again we know that only if $\eta \cos(\phi_s) < 0$, we can find tune, i.e. the motion is stable. Taylor expend $\cos \psi \sim 1 - \psi^2/2$, we can easily find

$$\nu_s = \frac{-eV \eta \cos(\phi_s)}{2\pi \beta^2 E_0}$$
also we can find 'beta function' for longitudinal motion.

\[ \beta \sin \psi = \frac{eV \cos \phi_s}{\beta^2 E_0} \]

\[ \beta = \sqrt{\frac{-eV \cos \phi_s}{2\pi \hbar \eta \beta^2 E_0}} \]

The ellipse of phase space is near-up-right one \((\alpha \sim \nu_s, \text{small})\).

3. For the example in class, find the synchrotron tune for both 100GeV case and 15GeV proton ring. The relative parameter is the cavity has 5MV voltage, 360 harmonic. Compaction factor \(\alpha_c = 0.002\). The RF phase is zero or \(\pi\). How does the number change if the same ring is for 3GeV electron beam.

Answer: Find the gamma then beta of the the particle with the mass, proton mass is 938 MeV and electron mass is 0.511MeV, then plug in the number to find \(\nu_s\).