I. Solving Wave-equation for 4-D potential induced by a moving point charge

In the Lorentz Gauge, the E&M field is described by the 4-D potential

$$A^{\alpha} = \left(\frac{\Phi}{c}, \vec{A}\right), \tag{1}$$

as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} = \mu_0\vec{J}\ , \tag{2}$$

and

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi - \nabla^2\Phi = \frac{1}{\varepsilon_0}\rho \\ &\Rightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\Phi}{c}\right) - \nabla^2\left(\frac{\Phi}{c}\right) = \mu_0\left(c\rho\right) \end{aligned} \ . \tag{3}$$

Eq. (2) and (3) can be written more concisely into

$$\Box A^{\alpha} \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2}A^{\alpha} - \nabla^2 A^{\alpha} = \mu_0 J^{\alpha}\ . \tag{4}$$

Taking 3D Fourier transformation for the spatial coordinates, $\vec{x}$, to eq. (4) yields

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_1 dx_2 dx_3 \left\{\frac{\partial^2}{\partial x_0{}^2}A^{\alpha}(x) - \frac{\partial^2}{\partial x_1{}^2}A^{\alpha}(x) - \frac{\partial^2}{\partial x_2{}^2}A^{\alpha}(x) - \frac{\partial^2}{\partial x_3{}^2}A^{\alpha}(x)\right\}\exp\left(-ik_1x_1 - ik_2x_2 - ik_3x_3\right) \\ &= \mu_0 \int_{-\infty}^{\infty} dx_1 dx_2 dx_3 J^{\alpha}(x)\exp\left(-ik_1x_1 - ik_2x_2 - ik_3x_3\right) \end{aligned} , \tag{5}$$

i.e.

$$
\begin{aligned}
&\left\{\frac{\partial^2}{\partial x_0{}^2}\int_{-\infty}^{\infty}\exp\left(-i\vec{k}\cdot\vec{x}\right)A^\alpha\left(x\right)dx_1\,dx_2\,dx_3-\int_{-\infty}^{\infty}dx_2\,dx_3\exp\left(-ik_2x_2-ik_3x_3\right)\int_{-\infty}^{\infty}\exp\left(-ik_1x_1\right)\frac{\partial^2}{\partial x_1{}^2}A^\alpha\left(x\right)dx_1\right.\\
&\quad-\int_{-\infty}^{\infty}dx_1\,dx_3\exp\left(-ik_1x_1-ik_3x_3\right)\int_{-\infty}^{\infty}\exp\left(-ik_2x_2\right)\frac{\partial^2}{\partial x_2{}^2}A^\alpha\left(x\right)dx_2\\
&\quad\left.-\int_{-\infty}^{\infty}dx_1\,dx_2\exp\left(-ik_1x_1-ik_2x_2\right)\int_{-\infty}^{\infty}\exp\left(-ik_3x_3\right)\frac{\partial^2}{\partial x_3{}^2}A^\alpha\left(x\right)dx_3\right\}=\mu_0\tilde{J}^\alpha\left(\vec{k},x_0\right)
\end{aligned}
$$

.

(6)

with

$$
\tilde{J}^\alpha\left(\vec{k},x_0\right)\equiv\int_{-\infty}^{\infty}\exp\left(-i\vec{k}\cdot\vec{x}\right)J^\alpha\left(x\right)dx_1\,dx_2\,dx_3\ . \tag{7}
$$

Assuming that for $j=1,2,3$

$$
\lim_{x_j\to\pm\infty}A^\alpha\left(x\right)=0\ , \tag{8}
$$

and

$$
\lim_{x_j\to\pm\infty}\frac{\partial}{\partial x_j}A^\alpha\left(x\right)=0, \tag{9}
$$

we obtain the following relation:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\exp\left(-ik_jx_j\right)\frac{\partial^2}{\partial x_j{}^2}A^\alpha\left(x\right)dx_j&=\int_{-\infty}^{\infty}d\left[\exp\left(-ik_jx_j\right)\frac{\partial}{\partial x_j}A^\alpha\left(x\right)\right]-\int_{-\infty}^{\infty}\frac{\partial}{\partial x_j}A^\alpha\left(x\right)d\exp\left(-ik_jx_j\right)\\
&=ik_j\int_{-\infty}^{\infty}\exp\left(-ik_jx_j\right)\frac{\partial}{\partial x_j}A^\alpha\left(x\right)dx_j\\
&=ik_j\left\{\int_{-\infty}^{\infty}d\left[\exp\left(-ik_jx_j\right)A^\alpha\left(x\right)\right]+ik_j\int_{-\infty}^{\infty}A^\alpha\left(x\right)\exp\left(-ik_jx_j\right)dx_j\right\}.\\
&=-k_j^2\int_{-\infty}^{\infty}A^\alpha\left(x\right)\exp\left(-ik_jx_j\right)dx_j
\end{aligned}
$$

(10)

Substituting eq. (10) into eq. (6) produces

$$\left\{\frac{\partial^2}{\partial x_0{}^2}\int_{-\infty}^{\infty}\exp\left(-i\vec{k}\cdot\vec{x}\right)A^{\alpha}\left(x\right)dx_1\,dx_2\,dx_3+k_1^2\int_{-\infty}^{\infty}dx_2\,dx_3\exp\left(-ik_2x_2-ik_3x_3\right)\int_{-\infty}^{\infty}A^{\alpha}\left(x\right)\exp\left(-ik_1x_1\right)dx_1\right.$$
$$+k_2^2\int_{-\infty}^{\infty}dx_1\,dx_3\exp\left(-ik_1x_1-ik_3x_3\right)\int_{-\infty}^{\infty}A^{\alpha}\left(x\right)\exp\left(-ik_2x_2\right)dx_2 \qquad ,$$
$$\left.+k_3^2\int_{-\infty}^{\infty}dx_1\,dx_2\exp\left(-ik_1x_1-ik_2x_2\right)\int_{-\infty}^{\infty}A^{\alpha}\left(x\right)\exp\left(-ik_3x_3\right)dx_3\right\}=\mu_0\tilde{J}^{\alpha}\left(\vec{k},x_0\right)$$

i.e.

$$\frac{\partial^2}{\partial x_0{}^2}\int_{-\infty}^{\infty}\exp\left(-i\vec{k}\cdot\vec{x}\right)A^{\alpha}\left(x\right)dx_1\,dx_2\,dx_3+\kappa^2\int_{-\infty}^{\infty}A^{\alpha}\left(x\right)\exp\left(-i\vec{k}\cdot\vec{x}\right)dx_1\,dx_2\,dx_3=\mu_0\tilde{J}^{\alpha}\left(\vec{k},x_0\right), \qquad (11)$$

with

$$\kappa^2\equiv k_1^2+k_2^2+k_3^2\ . \qquad (12)$$

Defining

$$\tilde{A}^{\alpha}\left(\vec{k},x_0\right)\equiv\int_{-\infty}^{\infty}\exp\left(-i\vec{k}\cdot\vec{x}\right)A^{\alpha}\left(x\right)dx_1\,dx_2\,dx_3\ , \qquad (13)$$

eq. (11) becomes

$$\frac{\partial^2}{\partial x_0{}^2}\tilde{A}^{\alpha}\left(\vec{k},x_0\right)+\kappa^2\tilde{A}^{\alpha}\left(\vec{k},x_0\right)=\mu_0\tilde{J}^{\alpha}\left(\vec{k},x_0\right). \qquad (14)$$

Taking the following integral, $\int_{\tau_0}^{\infty}d\overline{x}_0\exp\left(-s\overline{x}_0\right)\rightarrow$, of eq. (14) yields

$$\int_{\tau_0}^{\infty}\exp\left(-s\overline{x}_0\right)\frac{\partial^2}{\partial\overline{x}_0{}^2}\tilde{A}^{\alpha}\left(\vec{k},\overline{x}_0\right)d\overline{x}_0+\kappa^2\int_{\tau_0}^{\infty}\exp\left(-s\overline{x}_0\right)\tilde{A}^{\alpha}\left(\vec{k},\overline{x}_0\right)d\overline{x}_0=\mu_0\int_{\tau_0}^{\infty}\exp\left(-s\overline{x}_0\right)\tilde{J}^{\alpha}\left(\vec{k},\overline{x}_0\right)d\overline{x}_0\ , \qquad (15)$$

which can be rewritten as

$$\int_{0}^{\infty}\exp\left(-s\left(x_0+\tau_0\right)\right)\frac{\partial^2}{\partial x_0{}^2}\tilde{A}^{\alpha}\left(\vec{k},x_0+\tau_0\right)dx_0+\kappa^2\int_{0}^{\infty}\exp\left(-s\left(x_0+\tau_0\right)\right)\tilde{A}^{\alpha}\left(\vec{k},x_0+\tau_0\right)dx_0$$
$$=\mu_0\int_{0}^{\infty}\exp\left(-s\left(x_0+\tau_0\right)\right)\tilde{J}^{\alpha}\left(\vec{k},x_0+\tau_0\right)dx_0 \qquad , \qquad (16)$$

with $\mathrm{Re}(s)>0$. Multiplying both sides of eq. (16) by $\exp(-s\tau_0)$ leads to

$$\begin{aligned}&\int_0^\infty \exp(-sx_0)\frac{\partial^2}{\partial x_0{}^2}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0+\kappa^2\int_0^\infty \exp(-sx_0)\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0\\&=\mu_0\int_0^\infty \exp(-sx_0)\tilde{J}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0\end{aligned} \quad . \ (17)$$

In deriving the R.H.S. of eq. (15), we used formula found in 'Table of integrals, Series, and products' by I.S. Gradshteyn. Making use of the following relations:

$$\begin{aligned}&\int_0^\infty \exp(-sx_0)\frac{\partial^2}{\partial x_0{}^2}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0\\
&=\int_0^\infty \exp(-sx_0)\,d\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\\
&=\int_0^\infty d\left[\exp(-sx_0)\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\right]-\int_0^\infty \frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)d\exp(-sx_0)\\
&=\lim_{x_0\to\infty}\left[\exp(-sx_0)\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\right]-\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}\\
&+s\int_0^\infty \exp(-sx_0)\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0\\
&=-\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}+s\int_0^\infty \exp(-sx_0)\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0\\
&=-\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}+s\int_0^\infty \frac{\partial}{\partial x_0}\left[\exp(-sx_0)\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\right]dx_0\\
&-s\int_0^\infty \tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\frac{\partial}{\partial x_0}\exp(-sx_0)dx_0\\
&=-\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}+s\lim_{x_0\to\infty}\left[\exp(-sx_0)\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\right]\\
&-s\lim_{x_0\to 0}\left[\exp(-sx_0)\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\right]+s^2\int_0^\infty \tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\exp(-sx_0)dx_0\\
&=-\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{z_0=0}-s\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}+s^2\int_0^\infty \tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\exp(-sx_0)dx_0\end{aligned} \quad , (18)$$

eq. (17)becomes

$$\begin{aligned}&\left(s^2+\kappa^2\right)\int_0^\infty \exp\left(-sx_0\right)\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0\\&=\mu_0\int_0^\infty \exp\left(-sx_0\right)\tilde{J}^\alpha\left(\vec{k},x_0+\tau_0\right)dx_0+\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}+s\,\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}\end{aligned} \quad , (19)$$

where in deriving eq. (18), we assumed that

$$\lim_{x_0\to\infty}\left[\exp\left(-sx_0\right)\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\right]=0 \ , \quad (20)$$

and

$$\lim_{x_0\to\infty}\left[\exp\left(-sz_0\right)\frac{\partial}{\partial z_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\right]=0 \ . \quad (21)$$

With the definition that

$$\tilde{A}^\alpha\left(\vec{k},s\right)\equiv\int_0^\infty U^\alpha\left(\vec{k},x_0\right)\exp\left(-sx_0\right)dx_0 \quad (22)$$

$$U^\alpha\left(\vec{k},x_0\right)\equiv\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right) \ , \quad (23)$$

$$\tilde{J}^\alpha\left(\vec{k},s\right)\equiv\int_0^\infty V^\alpha\left(\vec{k},x_0\right)\exp\left(-sx_0\right)dx_0 \ , \quad (24)$$

and

$$V^\alpha\left(\vec{k},x_0\right)\equiv\tilde{J}^\alpha\left(\vec{k},x_0+\tau_0\right), \quad (25)$$

eq. (19) becomes

$$\left(s^2+\kappa^2\right)\tilde{A}^\alpha\left(\vec{k},s\right)=\mu_0\tilde{J}^\alpha\left(\vec{k},s\right)+\frac{\partial}{\partial x_0}\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0}+s\,\tilde{A}^\alpha\left(\vec{k},x_0+\tau_0\right)\Big|_{x_0=0} \ , \quad (26)$$

i.e.

$$\tilde{A}^\alpha\left(\vec{k},s\right)=\frac{\mu_0\tilde{J}^\alpha\left(\vec{k},s\right)}{s^2+\kappa^2}+\frac{\tilde{A}'_0\left(\vec{k}\right)}{s^2+\kappa^2}+\frac{s\tilde{A}_0\left(\vec{k}\right)}{s^2+\kappa^2} \ , \quad (27)$$

with

$$\tilde{A}'_0\left(\vec{k}\right) \equiv \frac{\partial}{\partial x_0}\tilde{A}^{\alpha}\left(\vec{k}, x_0+\tau_0\right)\Big|_{x_0=0}\ , \tag{28}$$

and

$$\tilde{A}_0\left(\vec{k}\right) \equiv \tilde{A}^{\alpha}\left(\vec{k}, x_0+\tau_0\right)\Big|_{x_0=0}\ . \tag{29}$$

The Fourier components of the 4-D electromagnetic potential in time domain is given by the inverse Laplace transformation of eq. (27):

$$\begin{aligned} U^{\alpha}\left(\vec{k}, x_0\right) &= \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{A}^{\alpha}\left(\vec{k}, s\right)\exp\left(sx_0\right)ds \\ &= \mu_0 \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\tilde{J}^{\alpha}\left(\vec{k}, s\right)}{s^2+\kappa^2}\exp\left(sx_0\right)ds + \tilde{A}'_0\left(\vec{k}\right)\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\exp\left(sx_0\right)}{s^2+\kappa^2}ds + \tilde{A}_0\left(\vec{k}\right)\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{s\exp\left(sx_0\right)}{s^2+\kappa^2}ds \end{aligned}\ , \tag{30}$$

where γ is larger than the real part of all poles of the integrand.

Assuming that there is no electromagnetic field at $x_0 = 0$, it follows that

$$\tilde{A}'_0\left(\vec{k}\right) = 0 \tag{31}$$

$$\tilde{A}_0\left(\vec{k}\right) = 0\ , \tag{32}$$

and hence eq. (30) becomes

$$U^{\alpha}\left(\vec{k}, x_0\right) = \mu_0 \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\tilde{J}^{\alpha}\left(\vec{k}, s\right)}{s^2+\kappa^2}\exp\left(sx_0\right)ds\ . \tag{33}$$

To proceed, we will use the convolution theorem of the Laplace transformation. If the Laplace transformation of function $f\left(x_0\right)$ and $g\left(x_0\right)$ are given by

$$\mathfrak{L}\left[f\left(x_0\right);s\right] = F\left(s\right), \tag{34}$$

and

$$\mathfrak{L}\left[g\left(x_0\right);s\right] = G\left(s\right), \tag{35}$$

the convolution theorem says that the following holds:

$$\mathfrak{L}\left[f*g\left(x_0\right);s\right]=F\left(s\right)G\left(s\right), \tag{36}$$

where

$$f*g\left(x_0\right)=\int_0^{x_0} f\left(x_0-\xi\right)g\left(\xi\right)d\xi . \tag{37}$$

Taking the inverse Laplace transformation of eq. (37) yields

$$\mathfrak{L}\left[f*g\left(x_0\right);s\right]=F\left(s\right)G\left(s\right)\Rightarrow\mathfrak{L}^{-1}\left[F\left(s\right)G\left(s\right)\right]=f*g\left(x_0\right). \tag{38}$$

Making use of eq. (38), the inverse Laplace transformation of eq. (33) leads to

$$U^{\alpha}\left(\vec{k},x_0\right)=\mu_0\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\tilde{J}^{\alpha}\left(\vec{k},s\right)}{s^2+\kappa^2}\exp\left(sx_0\right)ds=\int_0^{x_0}D\left(x_0-\xi\right)V^{\alpha}\left(\vec{k},\xi\right)d\xi\ , \tag{39}$$

with

$$\begin{aligned}D\left(x_0\right)&=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\exp\left(sx_0\right)}{s^2+\kappa^2}ds=\frac{1}{2\pi i}\frac{1}{2i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp\left(sx_0\right)\left[\frac{1}{s-i\kappa}-\frac{1}{s+i\kappa}\right]ds\\&=\frac{1}{2i\kappa}\left[e^{i\kappa x_0}-e^{-i\kappa x_0}\right]=\frac{\sin\left(\kappa x_0\right)}{\kappa}\end{aligned}\ . \tag{40}$$

Inserting eq. (40), (25), and (23) into eq. (39) yields

$$\tilde{A}^{\alpha}\left(\vec{k},x_0+\tau_0\right)=\mu_0\frac{1}{\kappa}\int_0^{x_0}\sin\left(\kappa\left(x_0-\xi\right)\right)\tilde{J}^{\alpha}\left(\vec{k},\xi+\tau_0\right)d\xi\ . \tag{41}$$

The space domain solution of the 4-D potential is given by the 3-D inverse Fourier transformation of eq. (41):

$$\begin{aligned}A^{\alpha}\left(\vec{x},x_0+\tau_0\right)&=\frac{1}{\left(2\pi\right)^3}\int_{-\infty}^{\infty}\tilde{A}^{\alpha}\left(\vec{k},x_0+\tau_0\right)\exp\left(-i\vec{k}\cdot\vec{x}\right)d^3k\\&=\frac{1}{\left(2\pi\right)^3}\int_0^{x_0}d\eta\int_{-\infty}^{\infty}\frac{1}{\kappa}\sin\left(\kappa\left(x_0-\eta\right)\right)\tilde{J}^{\alpha}\left(\vec{k},\eta+\tau_0\right)\exp\left(-i\vec{k}\cdot\vec{x}\right)d^3k\ ,\\&=\frac{1}{\left(2\pi\right)^3}\int_0^{x_0}d\eta\int_{-\infty}^{\infty}F\left(\vec{k}\right)G\left(\vec{k}\right)\exp\left(-i\vec{k}\cdot\vec{x}\right)d^3k\end{aligned} \tag{42}$$

where

$$F \equiv \frac{1}{\kappa}\sin\left(\kappa\left(x_0-\eta\right)\right) , \tag{43}$$

and

$$G \equiv \tilde{J}^{\alpha}\left(\vec{k},\eta+\tau_0\right) . \tag{44}$$

From the convolution theorem, it holds that

$$\mathfrak{F}\left[f*g\left(x\right);k_x\right]=F\left(k_x\right)G\left(k_x\right) , \tag{45}$$

with

$$f*g\left(x\right)=\int_{-\infty}^{\infty} f\left(x-\xi\right)g\left(\xi\right)d\xi , \tag{46}$$

$$f\left(x\right)=\mathfrak{F}^{-1}\left[F\left(k_x\right);x\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty} F\left(k_x\right)\exp\left(-ik_x x\right)dk_x , \tag{47}$$

and

$$g\left(x\right)=\mathfrak{F}^{-1}\left[G\left(k_x\right);x\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty} G\left(k_x\right)\exp\left(-ik_x x\right)dk_x . \tag{48}$$

Taking inverse Fourier transformation of eq. (45) yields

$$\begin{aligned}\mathfrak{F}^{-1}\left[F\left(k_x\right)G\left(k_x\right),x\right]&=\frac{1}{2\pi}\int_{-\infty}^{\infty} F\left(k_x\right)G\left(k_x\right)\exp\left(ik_x x\right)dx\\&=\int_{-\infty}^{\infty} f\left(x-\xi\right)g\left(\xi\right)d\xi\end{aligned} . \tag{49}$$

Inserting eq. (49) into eq. (42) leads to

$$\begin{aligned}A^{\alpha}\left(\vec{x},x_0+\tau_0\right)&=\frac{1}{\left(2\pi\right)^3}\int_0^{x_0} d\eta\int_{-\infty}^{\infty} F\left(\vec{k}\right)G\left(\vec{k}\right)\exp\left(-i\vec{k}\cdot\vec{x}\right)d^3k\\&=\int_0^{x_0} d\eta\int_{-\infty}^{\infty} f\left(\vec{x}-\vec{\xi}\right)g\left(\vec{\xi}\right)d^3\xi\end{aligned} . \tag{50}$$

Making use of eq. (44) and (48) produces

$$g(\vec{x}) = \frac{1}{(2\pi)^3}\mu_0 \int_{-\infty}^{\infty} \tilde{J}^{\alpha}(\vec{k}, \eta + \tau_0)\exp(-i\vec{k}\cdot\vec{x})d^3k \\ = \mu_0 J^{\alpha}(\vec{x}, \eta + \tau_0) \quad , \qquad (51)$$

and similarly, with the help of eq. (43) and (47), it follows that

$$\begin{aligned} f(\vec{x}) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} F(\vec{k})\exp(-i\vec{k}\cdot\vec{x})d^3k \\ &= \frac{1}{(2\pi)^3}\int_0^{\infty} d|k| \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \frac{1}{|k|}\sin(|k|(x_0-\eta))\exp(-i|\vec{k}||\vec{x}|\cos\theta)|k|^2\sin\theta \\ &= \frac{1}{(2\pi)^2}\int_0^{\infty} d|k|\sin(|k|(x_0-\eta))|k|\int_0^{\pi}\exp(-i|\vec{k}||\vec{x}|\cos\theta)\sin\theta\, d\theta \\ &= \frac{1}{(2\pi)^2}\int_0^{\infty} d|k|\sin(|k|(x_0-\eta))|k|\int_{-1}^{1}\exp(-i|\vec{k}||\vec{x}|\tau)\,d\tau \quad . \qquad (52) \\ &= \frac{1}{(2\pi)^2}\int_0^{\infty}\sin(|k|(x_0-\eta))\frac{\exp(i|\vec{k}||\vec{x}|)-\exp(-i|\vec{k}||\vec{x}|)}{i|\vec{x}|}d|k| \\ &= \frac{1}{2\pi^2|\vec{x}|}\int_0^{\infty}\sin(|k|(x_0-\eta))\sin(|\vec{k}||\vec{x}|)d|k| \\ &= -\frac{1}{4\pi^2|\vec{x}|}\int_0^{\infty}\left\{\cos(|k|(x_0-\eta+|\vec{x}|))-\cos(|k|(x_0-\eta-|\vec{x}|))\right\}d|k| \end{aligned}$$

We can further write eq. (52) as

$$\begin{aligned} f(\vec{x}) &= -\frac{1}{8\pi^2|\vec{x}|}\mathrm{Re}\left\{\int_{-\infty}^{\infty}\left\{\exp(i\tau(x_0-\eta+|\vec{x}|))-\exp(i\tau(x_0-\eta-|\vec{x}|))\right\}d\tau\right\} \\ &= -\frac{1}{4\pi|\vec{x}|}\left\{\delta(x_0-\eta+|\vec{x}|)-\delta(x_0-\eta-|\vec{x}|)\right\} \quad . \qquad (53) \\ &= \frac{1}{4\pi|\vec{x}|}\left[\delta(x_0-\eta-|\vec{x}|)-\delta(x_0-\eta+|\vec{x}|)\right] \end{aligned}$$

Inserting eq. (51) and eq. (53) into eq. (50) yields

$$A^{\alpha}\left(\vec{x},x_0+\tau_0\right)=\mu_0\int_0^{x_0}d\eta\int_{-\infty}^{\infty}\frac{1}{4\pi\left|\vec{x}-\vec{\xi}\right|}\left[\delta\left(x_0-\eta-\left|\vec{x}-\vec{\xi}\right|\right)-\delta\left(x_0-\eta+\left|\vec{x}-\vec{\xi}\right|\right)\right] \qquad (54)$$
$$J^{\alpha}\left(\vec{\xi},\eta+\tau_0\right)d^3\xi \, .$$

Assuming there is no current sources at the location of the observation point $\vec{x}$, i.e.

$$J^{\alpha}\left(\vec{x},\eta+\tau_0\right)=0 \, , \qquad (55)$$

the second term in the square bracket of eq. (54) does not contribute and hence

$$A^{\alpha}\left(\vec{x},x_0+\tau_0\right)=\mu_0\int_0^{x_0}d\eta\int_{-\infty}^{\infty}\frac{1}{4\pi\left|\vec{x}-\vec{\xi}\right|}\delta\left(x_0-\eta-\left|\vec{x}-\vec{\xi}\right|\right)J^{\alpha}\left(\vec{\xi},\eta+\tau_0\right)d^3\xi \, . \qquad (56)$$

Making the following changes of the integration variables:

$$\eta\rightarrow x'_0 \, , \qquad (57)$$

and

$$\vec{\xi}\rightarrow\vec{x}' \, , \qquad (58)$$

eq. (54) becomes

$$A^{\alpha}\left(\vec{x},x_0+\tau_0\right)=\mu_0\int_0^{x_0}dx'_0\int_{-\infty}^{\infty}\frac{\delta\left(x_0-x'_0-\left|\vec{x}-\vec{x}'\right|\right)}{4\pi\left|\vec{x}-\vec{x}'\right|}J^{\alpha}\left(\vec{x}',x'_0+\tau_0\right)d^3x' \, . \qquad (59)$$

Now, we will try to write eq. (59) into Lorentz invariant form. Using the following property of the Dirac delta function:

$$\delta\left(g(x)\right)=\sum_i\frac{\delta\left(x-x_i\right)}{\left|g'\left(x_i\right)\right|} \, , \qquad (60)$$

it follows that

$$\begin{aligned}\delta\left(\left(x-x'\right)^2\right)&=\delta\left(\left(x_0-x'_0\right)^2-\left|\vec{x}-\vec{x}'\right|^2\right)\\&=\delta\left(\left(x_0-x'_0+\left|\vec{x}-\vec{x}'\right|\right)\left(x_0-x'_0-\left|\vec{x}-\vec{x}'\right|\right)\right)\\&=\frac{\delta\left(x_0-x'_0+\left|\vec{x}-\vec{x}'\right|\right)}{\left|2\left(x_0'-\left|\vec{x}-\vec{x}'\right|-x_0'\right)\right|}+\frac{\delta\left(x_0-x'_0-\left|\vec{x}-\vec{x}'\right|\right)}{\left|2\left(x_0'+\left|\vec{x}-\vec{x}'\right|-x_0'\right)\right|}\ ,\qquad(61)\\&=\frac{\delta\left(x_0-x'_0+\left|\vec{x}-\vec{x}'\right|\right)+\delta\left(x_0-x'_0-\left|\vec{x}-\vec{x}'\right|\right)}{2\left|\vec{x}-\vec{x}'\right|}\end{aligned}$$

i.e.

$$\frac{\delta\left(x_0-x'_0-\left|\vec{x}-\vec{x}'\right|\right)}{\left|\vec{x}-\vec{x}'\right|}=2\delta\left(\left(x-x'\right)^2\right)-\frac{\delta\left(x_0-x'_0+\left|\vec{x}-\vec{x}'\right|\right)}{\left|\vec{x}-\vec{x}'\right|}\ .\qquad(62)$$

Inserting eq. (62) into eq. (59) yields

$$\begin{aligned}A^\alpha\left(\vec{x},x_0+\tau_0\right)&=\mu_0\int_0^{x_0}dx'_0\int_{-\infty}^{\infty}\frac{2\delta\left(\left(x-x'\right)^2\right)}{4\pi}J^\alpha\left(\vec{x}',x'_0\right)d^3x'\\&\quad+\mu_0\int_0^{x_0}dx'_0\int_{-\infty}^{\infty}\frac{\delta\left(x_0-x'_0+\left|\vec{x}-\vec{x}'\right|\right)}{4\pi\left|\vec{x}-\vec{x}'\right|}J^\alpha\left(\vec{x}',x'_0+\tau_0\right)d^3x'\end{aligned}\ .\qquad(63)$$

With the condition of eq. (55), the second terms in the R.H.S. of eq. (63) vanishes and consequently we obtain

$$A^\alpha\left(\vec{x},x_0+\tau_0\right)=\frac{1}{2\pi}\mu_0\int_0^{x_0}dx'_0\int_{-\infty}^{\infty}\delta\left(\left(x-x'\right)^2\right)J^\alpha\left(\vec{x}',x'_0+\tau_0\right)d^3x'\ .\qquad(64)$$

Eq. (64) can also be written as

$$A^\alpha\left(\vec{x},x_0+\tau_0\right)=\frac{1}{2\pi}\mu_0\int_0^{\infty}dx'_0\int_{-\infty}^{\infty}H\left(x_0-x'_0\right)\delta\left(\left(x-x'\right)^2\right)J^\alpha\left(\vec{x}',x'_0+\tau_0\right)d^3x'\ ,\qquad(65)$$

or

$$
\begin{aligned}
A^{\alpha}\left(\vec{\bar{x}},\bar{x}_0\right) &= \frac{\mu_0}{2\pi}\int_0^{\infty} dx'_0 \int_{-\infty}^{\infty} H\left(\bar{x}_0 - x'_0 - \tau_0\right) \\
&\delta\left(\left(\bar{x}_0 - x'_0 - \tau_0\right)^2 - \left(\vec{\bar{x}} - \vec{x}'\right)^2\right) J^{\alpha}\left(\vec{x}', x'_0 + \tau_0\right) d^3x' \\
&= \frac{\mu_0}{2\pi}\int_0^{\infty} d\left(\bar{x}'_0 - \tau_0\right) \int_{-\infty}^{\infty} H\left(\bar{x}_0 - \left(\bar{x}_0' - \tau_0\right) - \tau_0\right) \\
&\delta\left(\left(\bar{x}_0 - \left(\bar{x}_0' - \tau_0\right) - \tau_0\right)^2 - \left(\vec{\bar{x}} - \vec{\bar{x}}'\right)^2\right) J^{\alpha}\left(\vec{\bar{x}}', \left(\bar{x}_0' - \tau_0\right) + \tau_0\right) d^3\bar{x}' \\
&= \frac{\mu_0}{2\pi}\int_{\tau_0}^{\infty} d\bar{x}_0' \int_{-\infty}^{\infty} H\left(\bar{x}_0 - \bar{x}_0'\right)\delta\left(\left(\bar{x}_0 - \bar{x}_0'\right)^2 - \left(\vec{\bar{x}} - \vec{\bar{x}}'\right)^2\right) J^{\alpha}\left(\vec{\bar{x}}', \bar{x}_0'\right) d^3\bar{x}' \\
&= \frac{\mu_0}{2\pi}\int_{\tau_0}^{\infty} d\bar{x}_0' \int_{-\infty}^{\infty} H\left(\bar{x}_0 - \bar{x}_0'\right)\delta\left(\left(\bar{x} - \bar{x}'\right)^2\right) J^{\alpha}\left(\vec{\bar{x}}', \bar{x}_0'\right) d^3\bar{x}'
\end{aligned}
\quad , (66)
$$

where

$$\left(\vec{x}, x_0\right) \equiv \left(\vec{\bar{x}}, \bar{x}_0 - \tau_0\right) \ , \qquad (67)$$

$$\left(\vec{x}', x_0'\right) \equiv \left(\vec{\bar{x}}', \bar{x}_0' - \tau_0\right) \ , \qquad (68)$$

and $H\left(x\right)$ is the Heaviside step function. For simplicity, we drop the bar from eq. (66) and get

$$A^{\alpha}\left(\vec{x}, x_0\right) = \frac{\mu_0}{2\pi}\int_{\tau_0}^{\infty} dx_0' \int_{-\infty}^{\infty} H\left(x_0 - x_0'\right)\delta\left(\left(x - x'\right)^2\right) J^{\alpha}\left(\vec{x}', x_0'\right) d^3x' \ . \qquad (69)$$

Now we assume $\tau_0 \to -\infty$ (but τ_0 is still much larger than all the negative infinity integral limits that we took in the derivation), and eq. (69) becomes

$$
\begin{aligned}
A^{\alpha}\left(\vec{x}, x_0\right) &= \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} dx_0' \int_{-\infty}^{\infty} d^3x' \delta\left(\left(x - x'\right)^2\right) H\left(x_0 - x_0'\right) J^{\alpha}\left(\vec{x}', x_0'\right) \\
&= \mu_0 \int_{-\infty}^{\infty} dx_0' \int_{-\infty}^{\infty} d^3x' D_r\left(x - x'\right) J^{\alpha}\left(\vec{x}', x_0'\right)
\end{aligned}
\quad , \qquad (70)
$$

with

$$D_r\left(x - x'\right) \equiv \frac{1}{2\pi} H\left(x_0 - x_0'\right)\delta\left(\left(x - x'\right)^2\right) . \qquad (71)$$

Eq. (70) and (71) reproduce the result of eq. (14.1) and eq. (12.133) of 'Classical Electrodynamics' by Jackson. The 4-D current density is given by

$$J^{\alpha}(x)=\left(c\rho(x),\vec{J}(x)\right) \quad . \tag{72}$$

For a particle moving along a known 3-D trajectory $\vec{r}(t)$, the current density is given by

$$J^{\alpha}(\vec{x},t)=\left(ec\delta^{(3)}\left(\vec{x}-\vec{r}(t)\right),e\vec{v}(t)\delta^{(3)}\left(\vec{x}-\vec{r}(t)\right)\right), \tag{73}$$

which can also be written into the following integral:

$$J^{\alpha}(x)=ec\int_{-\infty}^{\infty}d\tau U^{\alpha}(\tau)\delta^{(4)}\left[x-r(\tau)\right], \tag{74}$$

where the proper time is defined as

$$d\tau=\gamma^{-1}dt, \tag{75}$$

the 4-D velocity is defined as

$$U^{\alpha}(\tau)=\left(\gamma c,\gamma\vec{v}(t)\right)=\left(\gamma c,\gamma\vec{v}(\tau\gamma)\right), \tag{76}$$

and the trajectory of the particle in the 4-D space-time coordinate system is defined as

$$r(\tau)=\left(ct,\vec{r}(t)\right)=\left(r_0(\tau),\vec{r}(\tau\gamma)\right) \quad . \tag{77}$$

The validity of eq. (74) can be checked by inserting eq. (75)-(77) back into eq. (74):

$$\begin{aligned}
J^{0}(x)&=ec\int_{-\infty}^{\infty}d\tau U^{0}(\tau)\delta^{(4)}\left[x-r(\tau)\right]\\
&=ec^{2}\int_{-\infty}^{\infty}dt_1\delta\left[x_0-r_0(t_1)\right]\delta^{(3)}\left[\vec{x}-\vec{r}(t_1)\right]\\
&=ec^{2}\int_{-\infty}^{\infty}dt_1\delta\left[x_0-ct_1\right]\delta^{(3)}\left[\vec{x}-\vec{r}(t_1)\right] \quad ,\\
&=ec^{2}\frac{\delta^{(3)}\left[\vec{x}-\vec{r}(x_0/c)\right]}{c}\\
&=ec\delta^{(3)}\left[\vec{x}-\vec{r}(x_0/c)\right]
\end{aligned} \tag{78}$$

$$\begin{aligned}
J^1(x) &= ec\int_{-\infty}^{\infty} d\tau U^1(\tau)\delta^{(4)}\left[x-r(\tau)\right] \\
&= ec\int_{-\infty}^{\infty} dt_1 v_1(t_1)\delta\left[x_0-r_0(t_1)\right]\delta^{(3)}\left[\vec{x}-\vec{r}(t_1)\right] \\
&= ec\int_{-\infty}^{\infty} dt_1 v_1(t_1)\delta\left[x_0-ct_1\right]\delta^{(3)}\left[\vec{x}-\vec{r}(t_1)\right] \quad , \\
&= ecv_1(x_0/c)\frac{\delta^{(3)}\left[\vec{x}-\vec{r}(x_0/c)\right]}{c} \\
&= ev_1(x_0/c)\delta^{(3)}\left[\vec{x}-\vec{r}(x_0/c)\right]
\end{aligned} \tag{79}$$

and similarly, we can also derive

$$J^2(x) = ev_2(x_0/c)\delta^{(3)}\left[\vec{x}-\vec{r}(x_0/c)\right] , \tag{80}$$

and

$$J^3(x) = ev_3(x_0/c)\delta^{(3)}\left[\vec{x}-\vec{r}(x_0/c)\right] . \tag{81}$$

After recognizing that

$$t = \frac{x_0}{c} ,$$

and inserting it into eq. (78)-(81), we obtain

$$J^\alpha(\vec{x},t) = \left(ec\delta^{(3)}\left(\vec{x}-\vec{r}(t)\right), e\vec{v}(t)\delta^{(3)}\left(\vec{x}-\vec{r}(t)\right)\right) , \tag{82}$$

which is identical to eq. (73).

II. Solving radiation the 4-D potential induced by a moving charge (Lienard-Wiechert potential)

In the previous section we derived the 4-D potential, i.e. eq. (70),

$$A^\alpha(\vec{x},x_0) = \mu_0\int_{-\infty}^{\infty} dx_0'\int_{-\infty}^{\infty} d^3x' D_r(x-x')J^\alpha(\vec{x}',x_0'), \tag{83}$$

with

$$D_r(x-x') \equiv \frac{1}{2\pi} H(x_0 - x_0')\,\delta\left((x-x')^2\right), \tag{84}$$

and the current density due to a charge moving with prescribed trajectories, i.e. eq. (74),

$$J^\alpha(x) = ec\int_{-\infty}^{\infty} d\tau U^\alpha(\tau)\,\delta^{(4)}\left[x - r(\tau)\right], \tag{85}$$

with

$$U^\alpha(\tau) = \left(\gamma c, \gamma\vec{v}(t)\right) = \left(\gamma c, \gamma\vec{v}(\tau\gamma)\right), \tag{86}$$

$$d\tau = \gamma(\tau)^{-1}\, dt\ , \tag{87}$$

and

$$r(\tau) = \left(ct, \vec{r}(t)\right) = \left(r_0(\tau), \vec{r}(\tau)\right). \tag{88}$$

Inserting eq. (85) into eq. (83) yields

$$\begin{aligned} A^\alpha(\vec{x}, x_0) &= \mu_0 c\int_{-\infty}^{\infty} dx_0'\int_{-\infty}^{\infty} d^3x' D_r(x-x')\, e\int_{-\infty}^{\infty} d\tau U^\alpha(\tau)\,\delta^{(4)}\left[x' - r(\tau)\right] \\ &= e\mu_0 c\int_{-\infty}^{\infty} d\tau U^\alpha(\tau)\int_{-\infty}^{\infty} dx_0'\,\delta\left[x'_0 - r_0(\tau)\right]\int_{-\infty}^{\infty} d^3x'\,\delta^{(3)}\left[x' - \vec{r}(\tau\gamma)\right] D_r(x-x') \end{aligned}, \tag{89}$$

Carrying out the integrals over x' in eq. (89) yields

$$\begin{aligned} A^\alpha(\vec{x}, x_0) &= \mu_0 c\int_{-\infty}^{\infty} dx_0'\int_{-\infty}^{\infty} d^3x' D_r(x-x')\, e\int_{-\infty}^{\infty} d\tau U^\alpha(\tau)\,\delta^{(4)}\left[x' - r(\tau)\right] \\ &= e\mu_0 c\int_{-\infty}^{\infty} d\tau U^\alpha(\tau)\int_{-\infty}^{\infty} dx_0'\,\delta\left[x'_0 - r_0(\tau)\right]\int_{-\infty}^{\infty} d^3x'\,\delta^{(3)}\left[\vec{x}' - \vec{r}(\tau)\right] D_r(x-x')\ , \\ &= e\mu_0 c\int_{-\infty}^{\infty} U^\alpha(\tau) D_r\left(x - r(\tau)\right) d\tau \end{aligned} \tag{90}$$

where

$$r(\tau) = \left(r_0(\tau), \vec{r}(\tau)\right). \tag{91}$$

Inserting eq. (84) into eq. (90) yields

$$A^{\alpha}\left(\vec{x},x_0\right)=\frac{e\mu_0 c}{2\pi}\int_{-\infty}^{\infty}U^{\alpha}\left(\tau\right)H\left(x_0-r_0\left(\tau\right)\right)\delta\left(\left(x-r\left(\tau\right)\right)^2\right)d\tau \ . \tag{92}$$

The condition for any values of $\tau=\tau_0$ to have non-vanishing contribution to the R.H.S. of eq. (92) is (from the delta function)

$$\left(x-r\left(\tau_0\right)\right)^2=\left(x_0-r_0\left(\tau_0\right)\right)^2-\left|\vec{x}-\vec{r}\left(\tau_0\right)\right|^2=0 \ , \tag{93}$$

and (from the Heaviside function)

$$x_0>cr_0\left(\tau_0\right) . \tag{94}$$

Combining eq. (93) and (94) gives

$$\left[x_0-r_0\left(\tau_0\right)\right]-\left|\vec{x}-\vec{r}\left(\tau_0\right)\right|=0 \tag{95}$$

i.e. at a specific location ($\tau=\tau_0$) along its 4-D trajectory, the particle must sit on the light-cone opened up from the observation point as shown in fig. 1 (The particle can not intersect with the light-cone twice as shown in APPENDIX A):

$$\left(x_0-X_0\right)-\sqrt{\left(X_1-x_1\right)^2+\left(X_2-x_3\right)^2+\left(X_3-x_3\right)^2}=0 \ . \tag{96}$$

Using eq. (60) and (93), eq. (92) can be written as

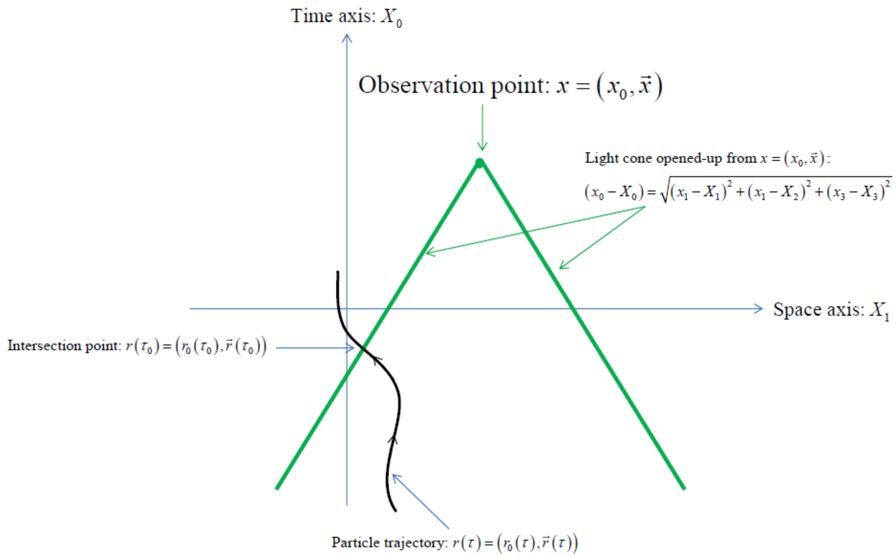


Figure 1: illustration of retarded potential.

$$
\begin{aligned}
A^\alpha(\vec{x},x_0) &= \frac{e\mu_0 c}{2\pi}\int_{-\infty}^{\infty} U^\alpha(\tau)\frac{\delta(\tau-\tau_0)}{\left[\frac{d}{d\tau}(x-r(\tau))^2\Big|_{\tau=\tau_0}\right]}H(x_0-r_0(\tau))\,d\tau \\
&= \frac{e\mu_0 c}{2\pi}U^\alpha(\tau_0)H(x_0-r_0(\tau_0))\left[\frac{d}{d\tau}(x-r(\tau))^2\Big|_{\tau=\tau_0}\right]^{-1} \\
&= \frac{e\mu_0 c}{2\pi}U^\alpha(\tau_0)H(x_0-r_0(\tau_0))\left[2(x-r(\tau))_\beta\frac{d}{d\tau}r^\beta(\tau)\Big|_{\tau=\tau_0}\right]^{-1} \\
&= \frac{e\mu_0 c}{2\pi}U^\alpha(\tau_0)H(x_0-r_0(\tau_0))\left[2(x-r(\tau))_\beta U^\beta(\tau)\Big|_{\tau=\tau_0}\right]^{-1} \\
&= \frac{e\mu_0 c}{4\pi}\frac{U^\alpha(\tau_0)H(x_0-r_0(\tau_0))}{(x-r(\tau_0))_\beta U^\beta(\tau_0)}
\end{aligned}
\quad . \qquad (97)
$$

From eq. (95) and eq. (88), it follows that

$$x_0 = r_0(\tau_0)+|\vec{x}-\vec{r}(\tau_0)| \geq r_0(\tau_0)\,, \qquad (98)$$

and therefore eq. (97) becomes

$$A^{\alpha}\left(\vec{x},x_0\right)=\frac{e\mu_0 c}{4\pi}\frac{U^{\alpha}\left(\tau_0\right)}{\left(x-r\left(\tau_0\right)\right)_{\beta}U^{\beta}\left(\tau_0\right)}. \tag{99}$$

The potential described by eq. (99) is called the Lienard-Wiechert potential and the retarded proper time, τ_0, is determined by eq. (98):

$$x_0=r_0\left(\tau_0\right)+\left|\vec{x}-\vec{r}\left(\tau_0\right)\right| . \tag{100}$$

Eq. (99) can also be explicitly written into non-covariant form (3D space component+1D time component). Making use of eq. (86) and eq. (91), the denominator of eq. (99) can be written as

$$\begin{aligned}\left(x-r\left(\tau_0\right)\right)_{\beta}U^{\beta}\left(\tau_0\right)&=\left(x_0-r_0\left(\tau_0\right)\right)\gamma c-\left(\vec{x}-\vec{r}\left(\tau_0\right)\right)\cdot\vec{U}\left(\tau_0\right)\\&=\left(x_0-r_0\left(\tau_0\right)\right)\gamma c-\left(\vec{x}-\vec{r}\left(\tau_0\right)\right)\cdot\gamma\vec{v}\left(\tau_0\gamma\right)\\&=\left|\vec{x}-\vec{r}\left(\tau_0\right)\right|\gamma c-\vec{R}\left(\tau_0\right)\cdot\gamma\vec{v}\left(\tau_0\gamma\right)\qquad ,\\&=\gamma cR\left(\tau_0\right)\left[1-\vec{n}\left(\tau_0\right)\cdot\frac{\vec{v}\left(\tau_0\gamma\right)}{c}\right]\\&=\gamma cR\left(\tau_0\right)\left[1-\vec{n}\left(\tau_0\right)\cdot\vec{\beta}\left(\tau_0\gamma\right)\right]\end{aligned} \tag{101}$$

Where

$$\vec{R}\left(\tau_0\right)\equiv\left(\vec{x}-\vec{r}\left(\tau_0\right)\right)=R\left(\tau_0\right)\vec{n}\left(\tau_0\right), \tag{102}$$

$$R\left(\tau_0\right)=\left|\vec{x}-\vec{r}\left(\tau_0\right)\right| , \tag{103}$$

and

$$\vec{n}\equiv\frac{\vec{R}\left(\tau_0\right)}{R\left(\tau_0\right)} . \tag{104}$$

Inserting eq. (101) into eq. (99) and writing it into its components yield

$$A^{0}\left(\vec{x},x_0\right)=\frac{e\mu_0 c}{4\pi}\frac{1}{R\left(\tau_0\right)\left[1-\vec{n}\left(\tau_0\right)\cdot\vec{\beta}\left(\tau_0\gamma\right)\right]} , \tag{105}$$

and

$$\vec{A}(\vec{x},x_0)=\frac{e\mu_0 c}{4\pi}\frac{\vec{v}(\tau_0\gamma)}{cR(\tau_0)\left[1-\vec{n}(\tau_0)\cdot\vec{\beta}(\tau_0\gamma)\right]}$$
$$=\frac{e\mu_0 c}{4\pi}\frac{\vec{\beta}(\tau_0\gamma)}{R(\tau_0)\left[1-\vec{n}(\tau_0)\cdot\vec{\beta}(\tau_0\gamma)\right]} . \tag{106}$$

Inserting eq. (105) into eq. (1) leads to

$$\Phi(\vec{x},x_0)=A^0(\vec{x},x_0)c$$
$$=\frac{e\mu_0 c^2}{4\pi}\frac{1}{R(\tau_0)\left[1-\vec{n}(\tau_0)\cdot\vec{\beta}(\tau_0\gamma)\right]} . \tag{107}$$
$$=\frac{e}{4\pi\varepsilon_0}\frac{1}{R(\tau_0)\left[1-\vec{n}(\tau_0)\cdot\vec{\beta}(\tau_0\gamma)\right]}$$

Since $r_0(\tau_0)$ is uniquely determined by τ_0, we can also write eq. (106) and (107) as a function of the retarded time to be solved from

$$t_r=\frac{r_0(\tau_0)}{c}=\frac{1}{c}\left[x_0-R(t_r)\right] , \tag{108}$$

i.e.

$$\vec{A}(\vec{x},x_0)=\frac{e\mu_0 c}{4\pi}\frac{\vec{\beta}(t_r)}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]} , \tag{109}$$

and

$$\Phi(\vec{x},x_0)=\frac{e}{4\pi\varepsilon_0}\frac{1}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]} . \tag{110}$$

III. Radiation field E and B from the 4-D Lienard-Wiechert potential

The electric field $\vec{E}$ and magnetic field $\vec{B}$ can be determined from the 4-D potential as

$$\vec{E}(\vec{x},t)=-\vec{\nabla}_x\Phi(\vec{x},t)-\frac{\partial}{\partial t}\vec{A}(\vec{x},t) , \tag{111}$$

and

$$\vec{B}(\vec{x},t)=\vec{\nabla}_x\times\vec{A}(\vec{x},t). \qquad (112)$$

Since the 4-D potential is a function of retarded time, we will then derive a few relations that are useful in carrying out the derivatives in eq. (111) and (112). Taking the first derivation of eq. (108) with respect to the observation time gives

$$\begin{aligned}
\frac{d}{dt}t_r &= \frac{1}{c}\left[c-\frac{d}{dt}R(t_r)\right] \\
&= 1-\frac{1}{c}\frac{dt_r}{dt}\cdot\frac{d}{dt_r}R(t_r) \\
&= 1-\frac{1}{c}\frac{dt_r}{dt}\cdot\frac{d}{dt_r}\sqrt{\left(x_1-r_1(t_r)\right)^2+\left(x_2-r_2(t_r)\right)^2+\left(x_3-r_3(t_r)\right)^2}\ , \qquad (113)\\
&= 1+\frac{1}{cR(t_r)}\left[\left(\vec{x}-\vec{r}(t_r)\right)\cdot\vec{v}(t_r)\right]\frac{dt_r}{dt} \\
&= 1+\left[\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]\frac{dt_r}{dt}
\end{aligned}$$

and it follows that

$$\frac{dt_r}{dt}=\frac{1}{1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)}\ . \qquad (114)$$

Taking the 3D gradient of eq. (108) yields

$$
\begin{aligned}
\vec{\nabla}_x t_r &= -\frac{1}{c}\vec{\nabla}_x R(t_r) \\
&= -\frac{1}{c}\vec{\nabla}_x \sqrt{\left(x_1 - r_1(t_r)\right)^2 + \left(x_2 - r_2(t_r)\right)^2 + \left(x_3 - r_3(t_r)\right)^2} \\
&= -\frac{1}{2cR(t_r)}\vec{\nabla}_x \left[\left(x_1 - r_1(t_r)\right)^2 + \left(x_2 - r_2(t_r)\right)^2 + \left(x_3 - r_3(t_r)\right)^2\right] \\
&= -\frac{1}{2cR(t_r)}\sum_{i=1}^{3} 2\left(x_i - r_i(t_r)\right)\left(\vec{\nabla}_x x_i - \vec{\nabla}_x r_i(t_r)\right) \\
&= -\frac{1}{cR(t_r)}\sum_{i=1}^{3} \left(x_i - r_i(t_r)\right)\left(\hat{i} - \frac{d}{dt_r} r_i(t_r)\vec{\nabla}_x t_r\right) \qquad , \qquad (115) \\
&= -\frac{1}{cR(t_r)}\sum_{i=1}^{3} \left(x_i - r_i(t_r)\right)\left(\hat{i} - v_i(t_r)\vec{\nabla}_x t_r\right) \\
&= -\frac{1}{cR(t_r)}\left[\vec{R}(t_r) - \left(\vec{R}(t_r)\cdot\vec{v}(t_r)\right)\vec{\nabla}_x t_r\right] \\
&= \left(\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right)\vec{\nabla}_x t_r - \frac{\vec{n}(t_r)}{c}
\end{aligned}
$$

which leads to

$$
\vec{\nabla}_x t_r = -\frac{\vec{n}(t_r)}{c\left(1 - \vec{n}(t_r)\cdot\vec{\beta}(t_r)\right)} \ . \qquad (116)
$$

Another two useful relations are

$$
\begin{aligned}
\frac{d}{dt}R(t_r) &= \frac{dt_r}{dt}\frac{d}{dt_r}R(t_r) \\
&= \frac{dt_r}{dt}\frac{d}{dt_r}\sqrt{\left(x_1 - r_1(t_r)\right)^2 + \left(x_2 - r_2(t_r)\right)^2 + \left(x_3 - r_3(t_r)\right)^2} \\
&= -\frac{dt_r}{dt}\frac{1}{2R}\left[2\left(\vec{x} - \vec{r}(t_r)\right)\cdot\frac{d}{dt_r}\vec{r}(t_r)\right] \qquad , \qquad (117) \\
&= -\frac{dt_r}{dt}\left[\vec{n}\cdot\vec{v}(t_r)\right] \\
&= \frac{-\vec{n}\cdot\vec{\beta}(t_r)c}{1 - \vec{n}(t_r)\cdot\vec{\beta}(t_r)}
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{\nabla}_x R(t_r) &= \vec{\nabla}_x \sqrt{\left(x_1 - r_1(t_r)\right)^2 + \left(x_2 - r_2(t_r)\right)^2 + \left(x_3 - r_3(t_r)\right)^2}\Bigg|_{t_r = const} + \left(\vec{\nabla}_x t_r\right) \cdot \frac{d}{dt_r} R(t_r) \\
&= \left(\vec{\nabla}_x t_r\right) \frac{d}{dt_r} \sqrt{\left(x_1 - r_1(t_r)\right)^2 + \left(x_2 - r_2(t_r)\right)^2 + \left(x_3 - r_3(t_r)\right)^2} + \vec{n}(t_r) \\
&= -\left(\vec{\nabla}_x t_r\right) \frac{1}{2R} \left[2\left(\vec{x} - \vec{r}(t_r)\right) \cdot \frac{d}{dt_r} \vec{r}(t_r) \right] + \vec{n}(t_r) \\
&= -\left(\vec{\nabla}_x t_r\right) \left[\vec{n} \cdot \vec{v}(t_r) \right] + \vec{n}(t_r) \\
&= \frac{\vec{n}(t_r)}{1 - \vec{n}(t_r) \cdot \vec{\beta}(t_r)}
\end{aligned}
\quad . \quad (118)
$$

Now we can calculate the gradient of the scalar potential as

$$
\begin{aligned}
\vec{\nabla}_x \Phi(\vec{x}, x_0) &= \frac{e}{4\pi\varepsilon_0} \vec{\nabla}_x \frac{1}{R(t_r)\left[1 - \vec{n}(t_r) \cdot \vec{\beta}(t_r)\right]} \\
&= -\frac{e}{4\pi\varepsilon_0} \frac{\vec{\nabla}_x R(t_r) - \vec{\nabla}_x \left[\vec{R}(t_r) \cdot \vec{\beta}(t_r)\right]}{R(t_r)^2 \left[1 - \vec{n}(t_r) \cdot \vec{\beta}(t_r)\right]^2}
\end{aligned}
\quad . \quad (119)
$$

Making use of the following relation,

$$
\begin{aligned}
\left\{\vec{\nabla}_x \left[\vec{R}(t_r) \cdot \vec{\beta}(t_r)\right]\right\}_j &= \frac{\partial}{\partial x_j} \left[\sum_{i=1}^{3} \left(\vec{x} - \vec{r}(t_r)\right)_i \vec{\beta}_i(t_r) \right] \\
&= \left[\sum_{i=1}^{3} \left(\vec{x} - \vec{r}(t_r)\right)_i \frac{\partial}{\partial x_j} \vec{\beta}_i(t_r) \right] + \left[\sum_{i=1}^{3} \vec{\beta}_i(t_r) \frac{\partial}{\partial x_j} \left(\vec{x} - \vec{r}(t_r)\right)_i \right] \\
&= \left[\sum_{i=1}^{3} \left(\vec{x} - \vec{r}(t_r)\right)_i \frac{dt_r}{dx_j} \frac{d}{dt_r} \vec{\beta}_i(t_r) \right] + \left[\sum_{i=1}^{3} \vec{\beta}_i(t_r) \frac{\partial}{\partial x_j} \left(x_i - r_i(t_r)\right) \right] \\
&= \left[-\frac{n_j(t_r)}{c\left(1 - \vec{n}(t_r) \cdot \vec{\beta}(t_r)\right)} \sum_{i=1}^{3} \left(\vec{x} - \vec{r}(t_r)\right)_i \dot{\vec{\beta}}_i(t_r) \right] + \left[\sum_{i=1}^{3} \vec{\beta}_i(t_r) \left(\delta_{i,j} - \frac{\partial}{\partial x_j} t_r \frac{d}{dt_r} r_i(t_r) \right) \right] , \quad (120) \\
&= \left[\frac{-\vec{R} \cdot \dot{\vec{\beta}}(t_r)}{c\left(1 - \vec{n}(t_r) \cdot \vec{\beta}(t_r)\right)} \vec{n}(t_r) \right]_j + \vec{\beta}_j(t_r) + \left[\frac{\vec{v}(t_r) \cdot \vec{\beta}(t_r)}{c\left(1 - \vec{n}(t_r) \cdot \vec{\beta}(t_r)\right)} \vec{n}(t_r) \right]_j \\
&= \left[\frac{\vec{v}(t_r) \cdot \vec{\beta}(t_r) - \vec{R} \cdot \dot{\vec{\beta}}(t_r)}{c\left(1 - \vec{n}(t_r) \cdot \vec{\beta}(t_r)\right)} \vec{n}(t_r) \right]_j + \vec{\beta}_j(t_r)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\vec{\nabla}_x \Phi(\vec{x},x_0) &= \frac{e}{4\pi\varepsilon_0}\vec{\nabla}_x \frac{1}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]} \\
&= -\frac{e}{4\pi\varepsilon_0}\frac{\vec{\nabla}_x R(t_r)-\vec{\nabla}_x\left[\vec{R}(t_r)\cdot\vec{\beta}(t_r)\right]}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2} \\
&= -\frac{e}{4\pi\varepsilon_0}\frac{\dfrac{\vec{n}(t_r)}{1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)}-\left[\dfrac{\vec{v}(t_r)\cdot\vec{\beta}(t_r)-\vec{R}\cdot\dot{\vec{\beta}}(t_r)}{c\left(1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right)}\vec{n}(t_r)\right]-\vec{\beta}(t_r)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2} \\
&= -\frac{e}{4\pi\varepsilon_0}\frac{\left[1-\beta^2(t_r)+\vec{R}\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}\right]\vec{n}(t_r)-\vec{\beta}(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}
\end{aligned}
\quad . \qquad (121)
$$

$$
\frac{d}{dt}\vec{A}(\vec{x},x_0)=\frac{e\mu_0 c}{4\pi}\frac{d}{dt}\frac{\vec{\beta}(t_r)}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]}
$$

$$
\begin{aligned}
&= \frac{e\mu_0 c}{4\pi}\frac{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]\frac{d}{dt}\vec{\beta}(t_r)-\vec{\beta}(t_r)\frac{d}{dt}\left\{\left[R(t_r)-\vec{R}(t_r)\cdot\vec{\beta}(t_r)\right]\right\}}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2} \\
&= \frac{e\mu_0 c}{4\pi}\frac{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]\frac{dt_r}{dt}\dot{\vec{\beta}}(t_r)-\vec{\beta}(t_r)\left\{\frac{d}{dt}R(t_r)-\frac{d}{dt}\left[\vec{R}(t_r)\cdot\vec{\beta}(t_r)\right]\right\}}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2} \\
&= \frac{e\mu_0 c}{4\pi}\frac{R(t_r)\dot{\vec{\beta}}(t_r)-\vec{\beta}(t_r)\frac{d}{dt}R(t_r)+\vec{\beta}(t_r)\vec{R}(t_r)\cdot\frac{d}{dt}\left[\vec{\beta}(t_r)\right]+\vec{\beta}(t_r)\left[\frac{d}{dt}\vec{R}(t_r)\right]\cdot\vec{\beta}(t_r)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2} \quad . \qquad (122) \\
&= \frac{e\mu_0 c}{4\pi}\frac{R(t_r)\dot{\vec{\beta}}(t_r)-\vec{\beta}(t_r)\frac{d}{dt}R(t_r)+\frac{dt_r}{dt}\left\{\vec{R}(t_r)\cdot\dot{\vec{\beta}}(t_r)-c\beta^2(t_r)\right\}\vec{\beta}(t_r)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2} \\
&= \frac{e\mu_0 c}{4\pi}\frac{R(t_r)\dot{\vec{\beta}}(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]+\vec{\beta}(t_r)\left[\vec{n}\cdot\vec{\beta}(t_r)c\right]+\left\{\vec{R}(t_r)\cdot\dot{\vec{\beta}}(t_r)-c\beta^2(t_r)\right\}\vec{\beta}(t_r)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3} \\
&= \frac{e\mu_0 c}{4\pi}\frac{R(t_r)\dot{\vec{\beta}}(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]+\left\{\vec{R}(t_r)\cdot\dot{\vec{\beta}}(t_r)-c\beta^2(t_r)+\vec{n}\cdot\vec{\beta}(t_r)c\right\}\vec{\beta}(t_r)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}
\end{aligned}
$$

Inserting eq. (121) and eq. (122) into eq. (111) yields

$$\vec{E}(\vec{x},t) = -\vec{\nabla}_x \Phi(\vec{x},t) - \frac{\partial}{\partial t}\vec{A}(\vec{x},t)$$
$$= \frac{e}{4\pi\varepsilon_0}\frac{1}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}\left\{\left[1-\beta^2(t_r)+\vec{R}\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}\right]\vec{n}(t_r)-\vec{\beta}(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]\right.$$
$$\left.-R(t_r)\dot{\vec{\beta}}(t_r)\frac{1}{c}\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]-\left\{\vec{R}(t_r)\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}-\beta^2(t_r)+\vec{n}\cdot\vec{\beta}(t_r)\right\}\vec{\beta}(t_r)\right\}$$
$$= \frac{e}{4\pi\varepsilon_0}\frac{1}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}\left\{\left[1-\beta^2(t_r)+\vec{R}\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}\right]\vec{n}(t_r)\right.$$
$$\left.-R(t_r)\dot{\vec{\beta}}(t_r)\frac{1}{c}\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]-\left\{\vec{R}(t_r)\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}-\beta^2(t_r)+1\right\}\vec{\beta}(t_r)\right\} .$$
$$= \frac{e}{4\pi\varepsilon_0}\frac{1}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}$$
$$\left\{\left[1-\beta^2(t_r)\right]\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)+\left[\vec{R}\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}\right]\left[\vec{n}(t_r)-\vec{\beta}(t_r)\right]-R(t_r)\dot{\vec{\beta}}(t_r)\frac{1}{c}\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]\right\}$$
$$= \frac{e}{4\pi\varepsilon_0}\frac{1}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}$$
$$\left\{\left[1-\beta^2(t_r)\right]\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)+\frac{1}{c}\left[\vec{R}\cdot\dot{\vec{\beta}}(t_r)\right]\left[\vec{n}(t_r)-\vec{\beta}(t_r)\right]-\frac{1}{c}\left[\vec{R}(t_r)\cdot\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\right]\dot{\vec{\beta}}(t_r)\right\}$$

(123)

Using the relation

$$\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w} \ , \qquad (124)$$

The last two terms in the curly bracket of eq. (123) reduces to

$$\vec{E}(\vec{x},t) = \frac{e}{4\pi\varepsilon_0\gamma^2(t_r)}\frac{\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}+\frac{e}{4\pi\varepsilon_0 c}\frac{\vec{n}\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3} \ . \qquad (125)$$

The magnetic field is given by eq. (112)

$$
\begin{aligned}
\vec{B}(\vec{x},t) &= \vec{\nabla}_x \times \vec{A}(\vec{x},t) \\
&= \frac{e\mu_0 c}{4\pi}\vec{\nabla}_x \times \left\{ \frac{\vec{\beta}(t_r)}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]} \right\} \\
&= \frac{e\mu_0 c}{4\pi}\frac{\vec{\nabla}_x \times \vec{\beta}(t_r)}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]} + \frac{e}{4\pi\varepsilon_0 c}\left[\vec{\nabla}_x \frac{1}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]}\right]\times\vec{\beta}(t_r) \\
&= \frac{e\mu_0 c}{4\pi}\frac{\vec{\nabla}_x \times \vec{\beta}(t_r)}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]} + \frac{1}{c}\left[\vec{\nabla}_x\Phi\right]\times\vec{\beta}(t_r)
\end{aligned}
\quad . \qquad (126)
$$

Making use of eq. (121) and the following relation

$$
\begin{aligned}
\left[\vec{\nabla}_x \times \vec{\beta}(t_r)\right]_k &= \sum_{i,j}\varepsilon_{ijk}\frac{\partial}{\partial x_i}\beta_j(t_r) \\
&= \sum_{i,j}\varepsilon_{ijk}\frac{\partial}{\partial x_i}t_r\frac{d}{dt_r}\beta_j(t_r) \\
&= -\frac{1}{c\left(1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right)}\sum_{i,j}\varepsilon_{ijk}n_i(t_r)\dot{\beta}_j(t_r) \\
&= \left\{-\frac{\vec{n}(t_r)\times\dot{\vec{\beta}}(t_r)}{c\left(1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right)}\right\}_k
\end{aligned}
\quad , \qquad (127)
$$

eq. (126) becomes

$$
\begin{aligned}
\vec{B}(\vec{x},t) &= -\frac{e}{4\pi\varepsilon_0 c^2}\frac{\vec{n}(t_r)\times\dot{\vec{\beta}}(t_r)}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2} - \frac{e}{4\pi\varepsilon_0 c}\frac{\left[1-\beta^2(t_r)+\vec{R}\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}\right]\left[\vec{n}(t_r)\times\vec{\beta}(t_r)\right]}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3} \\
&= -\frac{e}{4\pi\varepsilon_0 c}\frac{1}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}\left\{\frac{1}{c}R(t_r)\vec{n}(t_r)\cdot\left[\vec{n}(t_r)-\vec{\beta}(t_r)\right]\left[\vec{n}(t_r)\times\dot{\vec{\beta}}(t_r)\right]\right. \\
&\qquad \left. +\left[1-\beta^2(t_r)+\vec{R}\cdot\dot{\vec{\beta}}(t_r)\frac{1}{c}\right]\left[\vec{n}(t_r)\times\vec{\beta}(t_r)\right]\right\}
\end{aligned}
\quad .
$$

(128)

If we take the electric field of eq. (123) and calculate the following quantity:

$$\frac{1}{c}\vec{n}\times\vec{E}(\vec{x},t) = -\frac{e}{4\pi\varepsilon_0 c}\frac{1}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}$$
$$\left\{\left[1-\beta^2(t_r)+\frac{1}{c}\vec{R}\cdot\dot{\vec{\beta}}(t_r)\right]\left(\vec{n}(t_r)\times\vec{\beta}(t_r)\right)+\frac{1}{c}\left[\vec{R}(t_r)\cdot\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\right]\left[\vec{n}(t_r)\times\dot{\vec{\beta}}(t_r)\right]\right\}, \quad (129)$$

we arrived at the same expression as that of the R.H.S. of eq. (128). Therefore, from eq. (128) and (129), we obtain

$$\vec{B}(\vec{x},t) = \frac{1}{c}\vec{n}\times\vec{E}(\vec{x},t). \quad (130)$$

IV. Power radiated from a moving particle

The electric field as derived in eq. (125) has two pieces:

$$\begin{aligned}\vec{E}(\vec{x},t) &= \frac{e}{4\pi\varepsilon_0\gamma^2(t_r)}\frac{\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}+\frac{e}{4\pi\varepsilon_0 c}\frac{\vec{n}\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}, \\ &= \vec{E}_{static}(\vec{x},t)+\vec{E}_{rad}(\vec{x},t)\end{aligned}$$

where the first term, falling off fast with R^{-2}, (one can verify it equals to the field emitted from a charge with constant velocity)

$$\vec{E}_{static}(\vec{x},t) = \frac{e}{4\pi\varepsilon_0\gamma^2(t_r)}\frac{\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)}{R(t_r)^2\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}, \quad (131)$$

is the static field moving with the charged particles and the second term, falling off with R^{-1}

$$\vec{E}_{rad} = \frac{e}{4\pi\varepsilon_0 c}\frac{\vec{n}\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{R(t_r)\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}, \quad (132)$$

is the radiation field that we are interested in.

The energy of an electro-magnetic field passing through a unit area per unit time at the observer's location, $(\vec{x},t)$, is given by the Poynting vector:

$$\begin{aligned}\vec{S}(\vec{x},t) &= \frac{1}{\mu_0}\vec{E}_{rad}(\vec{x},t)\times\vec{B}_{rad}(\vec{x},t) \\ &= \frac{1}{c\mu_0}\vec{E}_{rad}(\vec{x},t)\times\left[\vec{n}\times\vec{E}_{rad}(\vec{x},t)\right] \\ &= \frac{1}{c\mu_0}\left(\vec{E}_{rad}(\vec{x},t)\cdot\vec{E}_{rad}(\vec{x},t)\right)\vec{n}-\left(\vec{E}_{rad}(\vec{x},t)\cdot\vec{n}\right)\vec{E}_{rad}(\vec{x},t) \\ &= \frac{1}{c\mu_0}E_{rad}^2(\vec{x},t)\vec{n}\end{aligned} \quad . \tag{133}$$

The power crossing unit area along $\vec{n}$ at the observation point $\vec{x}$ and at the observation time t is

$$\vec{S}(\vec{x},t)\cdot\vec{n} \tag{134}$$

and so the total energy crossing this unit area is

$$W(\vec{x},t)=\int_{-\infty}^{\infty}dt\vec{S}(\vec{x},t)\cdot\vec{n}=\int_{-\infty}^{\infty}dt_r\left[\frac{dt}{dt_r}\vec{S}(\vec{x},t)\cdot\vec{n}\right], \tag{135}$$

where the integrand in the second integral is the rate at which the particle radiates what eventually passes through the unit area at the observation point. From eq. (135), we obtain the instantaneous radiated power per solid angle along the direction $\vec{n}$:

$$\frac{dP(t_r)}{d\Omega}=\left(\vec{n}\cdot\vec{S}\right)R(t_r)^2\frac{dt}{dt_r}=\frac{E_{rad}^2(\vec{x},t)}{c\mu_0}R(t_r)^2\frac{dt}{dt_r}=c\varepsilon_0E_{rad}^2(\vec{x},t)R(t_r)^2\frac{dt}{dt_r}\,. \tag{136}$$

The derivative, dt/dt_r, can be calculated from eq. (108) as

$$\begin{aligned}& t_r=\frac{1}{c}\left(x_0-R(t_r)\right) \\ &\Rightarrow t_r=t-\frac{1}{c}R(t_r) \\ &\Rightarrow 1=\frac{dt}{dt_r}-\frac{1}{c}\frac{d}{dt_r}\sqrt{\left(x_1-r_1\right)^2+\left(x_2-r_2\right)^2+\left(x_3-r_3\right)^2} \\ &\Rightarrow 1=\frac{dt}{dt_r}+\frac{1}{c}\frac{2\vec{R}\cdot\vec{v}}{2R} \\ &\Rightarrow \frac{dt}{dt_r}=1-\vec{n}\cdot\vec{\beta}\end{aligned} \quad . \tag{137}$$

Inserting eq. (132) and (137) into eq. (136) yields

$$\frac{dP(t_r)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\left|\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]\right|^2}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^5} \ . \qquad (138)$$

IV.1 Charge moving with non-relativistic velocity

For $\left|\vec{\beta}(t_r)\right| << 1$, eq. (138) is reduced to

$$\frac{dP(t_r)}{d\Omega} \approx \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\left|\vec{n}(t_r)\times\left[\vec{n}(t_r)\times\dot{\vec{\beta}}(t_r)\right]\right|^2 \ . \qquad (139)$$

Let θ be the angle between $\vec{n}(t_r)$ and $\dot{\vec{\beta}}(t_r)$, eq. (139) becomes

$$\frac{dP(t_r)}{d\Omega} \approx \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\sin^2\theta\left|\dot{\vec{\beta}}(t_r)\right|^2\left|\vec{n}(t_r)\times\vec{k}\right|^2 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c^3}\left|\dot{\vec{v}}(t_r)\right|^2\sin^2\theta \ , \qquad (140)$$

where $\vec{k}$ is the unit vector along $\vec{n}(t_r)\times\dot{\vec{\beta}}(t_r)$ and hence is perpendicular to $\vec{n}(t_r)$ with

$$\left|\vec{n}(t_r)\times\vec{k}\right| = 1 \, . \qquad (141)$$

Choosing the z axis along the direction of $\dot{\vec{\beta}}(t_r)$ and integrating eq. (140) over

$$d\Omega = \sin\theta d\theta d\phi \qquad (142)$$

lead to

$$
\begin{aligned}
P(t_r) &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{d\vec{P}(\vec{x},t)}{d\Omega} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c^3}\left|\dot{\vec{v}}(t_r)\right|^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \sin^2\theta \\
&= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2c^3}\left|\dot{\vec{v}}(t_r)\right|^2 \int_{-1}^{1} \sin^2\theta \, d\cos\theta \\
&= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2c^3}\left|\dot{\vec{v}}(t_r)\right|^2 \int_{-1}^{1} \left(1-\xi^2\right) d\xi \\
&= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{2c^3}\left|\dot{\vec{v}}(t_r)\right|^2 \left(\xi - \frac{\xi^3}{3}\right)\Bigg|_{\xi=-1}^{1} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3c^3}\left|\dot{\vec{v}}(t_r)\right|^2
\end{aligned} \quad . \tag{143}
$$

Eq. (143) is called the Larmor equation for a nonrelativistic accelerated charge.

IV.2 Charge moving with arbitrary velocity

For arbitrary velocity of the moving particle, we can integrate eq. (138) over the solid angle to calculate the total power:

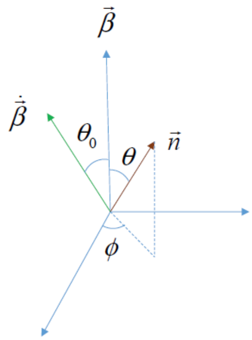
$$
P(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\int \frac{d\Omega}{\kappa^5}\left|\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]\right|^2 , \tag{144}
$$

where we define

$$
\kappa \equiv 1 - \vec{n}(t_r)\cdot\vec{\beta}(t_r) . \tag{145}
$$

Using eq. (124), the expression in the integrand of eq. (144) can be written as

$$
\begin{aligned}
&\left|\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right|^2 \\
&= \left|\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\left(\vec{n}-\vec{\beta}\right) - \left(1-\vec{n}\cdot\vec{\beta}\right)\dot{\vec{\beta}}\right|^2 \\
&= \left(\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\left(\vec{n}-\vec{\beta}\right) - \left(1-\vec{n}\cdot\vec{\beta}\right)\dot{\vec{\beta}}\right)\cdot\left(\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\left(\vec{n}-\vec{\beta}\right) - \left(1-\vec{n}\cdot\vec{\beta}\right)\dot{\vec{\beta}}\right) \\
&= \left(\vec{n}\cdot\dot{\vec{\beta}}\right)^2\left(1-2\vec{n}\cdot\vec{\beta}+\beta^2\right) - 2\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\left(1-\vec{n}\cdot\vec{\beta}\right)\left(\vec{n}\cdot\dot{\vec{\beta}}-\vec{\beta}\cdot\dot{\vec{\beta}}\right) + \left(1-\vec{n}\cdot\vec{\beta}\right)^2\dot{\beta}^2 \\
&= \left(\vec{n}\cdot\dot{\vec{\beta}}\right)^2\left[\beta^2-1\right] + 2\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\left(1-\vec{n}\cdot\vec{\beta}\right)\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right) + \left(1-\vec{n}\cdot\vec{\beta}\right)^2\dot{\beta}^2
\end{aligned} \quad . \tag{146}
$$



$$\dot{\vec{\beta}} = \dot{\beta}\left(\sin\theta_0, 0, \cos\theta_0\right)$$

$$\vec{\beta} = \beta\left(0,0,1\right)$$

$$\vec{n} = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

$$\vec{n}\cdot\dot{\vec{\beta}} = \dot{\beta}\left(\sin\theta\sin\theta_0\cos\phi + \cos\theta\cos\theta_0\right)$$

Figure 2: illustration of the reference frame used for integrating solid angle in eq. (144).

Using the reference frame as illustrated in fig. 2, eq. (146) becomes

$$\begin{aligned}
&\left|\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right|^2 = \dot{\beta}^2\left(\sin\theta\sin\theta_0\cos\phi+\cos\theta\cos\theta_0\right)^2\left(\beta^2-1\right)\\
&+2\dot{\beta}^2\left(\sin\theta\sin\theta_0\cos\phi+\cos\theta\cos\theta_0\right)\left(1-\beta\cos\theta\right)\beta\cos\theta_0+\left(1-\beta\cos\theta\right)^2\dot{\beta}^2\\
&=\dot{\beta}^2\left(\sin^2\theta\sin^2\theta_0\cos^2\phi+\cos^2\theta\cos^2\theta_0+\frac{1}{2}\sin\left(2\theta\right)\sin\left(2\theta_0\right)\cos\phi\right)\left(\beta^2-1\right)\\
&+2\dot{\beta}^2\left(\sin\theta\sin\theta_0\cos\phi+\cos\theta\cos\theta_0\right)\left(1-\beta\cos\theta\right)\beta\cos\theta_0+\left(1-\beta\cos\theta\right)^2\dot{\beta}^2
\end{aligned} \quad . \qquad (147)$$

Integrating eq. (147) over ϕ yields

$$\begin{aligned}
&\int_0^{2\pi}\left|\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right|^2 d\phi = 2\pi\left\{\dot{\beta}^2\left(\frac{1}{2}\sin^2\theta\sin^2\theta_0+\cos^2\theta\cos^2\theta_0\right)\left(\beta^2-1\right)\right.\\
&\qquad\left.+2\dot{\beta}^2\left(\cos\theta\cos\theta_0\right)\left(1-\beta\cos\theta\right)\beta\cos\theta_0+\left(1-\beta\cos\theta\right)^2\dot{\beta}^2\right\}\\
&=2\pi\dot{\beta}^2\left[\left(\frac{1}{2}\left(1-u^2\right)\sin^2\theta_0+u^2\cos^2\theta_0\right)\left(\beta^2-1\right)+2u\beta\cos^2\theta_0\left(1-\beta u\right)+\left(1-2\beta u+\beta^2u^2\right)\right]\\
&\qquad=2\pi\dot{\beta}^2\left\{\left[\left(\frac{3}{2}+\frac{1}{2}\beta^2\right)u^2-2u\beta-\frac{1}{2}\left(1-\beta^2\right)\right]\left(\sin^2\theta_0\right)-u^2+1\right\}
\end{aligned} \quad , \qquad (148)$$

where we define

$$u \equiv \cos\theta \ . \qquad (149)$$

Inserting eq. (148) into eq. (144) yields

$$P(t_r)=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\int_0^{\pi}\frac{d\theta\sin\theta}{\kappa^5}\int_0^{2\pi}\left|\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]\right|^2 d\phi$$
$$=\frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{2c}\int_{-1}^{1}\frac{du}{\kappa^5}\left\{\left[\left(\frac{3}{2}+\frac{1}{2}\beta^2\right)u^2-2u\beta-\frac{1}{2}\left(1-\beta^2\right)\right]\left(\sin^2\theta_0\right)-u^2+1\right\} . \quad (150)$$

Noticing that

$$\kappa=1-\vec{n}\cdot\vec{\beta}=1-\beta u \ , \quad (151)$$

and changing the integration variable from u to

$$x=1-\beta u \ , \quad (152)$$

with $u=\frac{1}{\beta}(1-x)$ yield

$$P(t_r)=\frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{2c\beta^3}\int_{1-\beta}^{1+\beta}\frac{dx}{x^5}$$
$$\left\{\left[\left(\frac{3}{2}+\frac{1}{2}\beta^2\right)(1-x)^2-2\beta^2(1-x)-\frac{1}{2}\left(\beta^2-\beta^4\right)\right]\sin^2\theta_0-(1-x)^2+\beta^2\right\}$$
$$=\frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{2c\beta^3}\int_{1-\beta}^{1+\beta}\frac{dx}{x^5}$$
$$\left\{\left[\frac{3}{2}-2\beta^2+\frac{\beta^4}{2}+\left(-3+\beta^2\right)x+\left(\frac{3}{2}+\frac{1}{2}\beta^2\right)x^2\right]\sin^2\theta_0+\beta^2-1+2x-x^2\right\}$$
$$=\frac{1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{2c\beta^3}\int_{1-\beta}^{1+\beta}dx \quad . \quad (153)$$
$$\frac{d}{dx}\left\{\left[-\left(3-4\beta^2+\beta^4\right)\frac{1}{8x^4}+\frac{3-\beta^2}{3x^3}-\frac{3+\beta^2}{4x^2}\right]\sin^2\theta_0-\frac{\beta^2-1}{4x^4}-\frac{2}{3x^3}+\frac{1}{2x^2}\right\}$$
$$=\frac{-1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{2c\beta^3}\int_{1-\beta}^{1+\beta}dx$$
$$\frac{d}{dx}\left\{\left[\frac{\left(1-\beta^2\right)\left(3-\beta^2\right)}{8x^4}+\frac{\beta^2-3}{3x^3}+\frac{3+\beta^2}{4x^2}\right]\sin^2\theta_0+\frac{\beta^2-1}{4x^4}+\frac{2}{3x^3}-\frac{1}{2x^2}\right\}$$

The integral involves the following two algebraic relation:

$$
\begin{aligned}
&\frac{\beta^2-1}{4\left(1+\beta\right)^4}+\frac{2}{3\left(1+\beta\right)^3}-\frac{1}{2\left(1+\beta\right)^2}-\frac{\beta^2-1}{4\left(1-\beta\right)^4}-\frac{2}{3\left(1-\beta\right)^3}+\frac{1}{2\left(1-\beta\right)^2}\\
&=\frac{3\left(\beta^2-1\right)+8\left(1+\beta\right)-6\left(1+\beta\right)^2}{12\left(1+\beta\right)^4}-\frac{3\left(\beta^2-1\right)+8\left(1-\beta\right)-6\left(1-\beta\right)^2}{12\left(1-\beta\right)^4}\\
&=\frac{3\left(\beta-1\right)+8-6\left(1+\beta\right)}{12\left(1+\beta\right)^3}-\frac{-3\left(\beta+1\right)+8-6\left(1-\beta\right)}{12\left(1-\beta\right)^3}\\
&=\frac{1-3\beta}{12\left(1-\beta\right)^3}-\frac{1+3\beta}{12\left(1+\beta\right)^3}\\
&=\frac{\left(1-3\beta\right)\left(1+\beta\right)^3-\left(1+3\beta\right)\left(1-\beta\right)^3}{12\left(1-\beta^2\right)^3}\\
&=\frac{\left(1+\beta\right)^3-\left(1-\beta\right)^3-3\beta\left[\left(1+\beta\right)^3+\left(1-\beta\right)^3\right]}{12\left(1-\beta^2\right)^3}\\
&=\frac{\left(1+\beta-1+\beta\right)\left[\left(1+\beta\right)^2+1-\beta^2+\left(1-\beta\right)^2\right]}{12\left(1-\beta^2\right)^3}\\
&-\frac{3\beta\left(1+\beta+1-\beta\right)\left[\left(1+\beta\right)^2-1+\beta^2+\left(1-\beta\right)^2\right]}{12\left(1-\beta^2\right)^3}\\
&=\frac{6\beta+2\beta^3-6\beta-18\beta^3}{12\left(1-\beta^2\right)^3}\\
&=\frac{-4\beta^3}{3\left(1-\beta^2\right)^3}\\
&=-\frac{4}{3}\gamma^6\beta^3
\end{aligned}
\qquad , (154)
$$

and

$$
\begin{aligned}
&\frac{(1-\beta^2)(3-\beta^2)}{8(1+\beta)^4}+\frac{\beta^2-3}{3(1+\beta)^3}+\frac{3+\beta^2}{4(1+\beta)^2}-\left\{\frac{(1-\beta^2)(3-\beta^2)}{8(1-\beta)^4}+\frac{\beta^2-3}{3(1-\beta)^3}+\frac{3+\beta^2}{4(1-\beta)^2}\right\} \\
&=\frac{3(1-\beta^2)(3-\beta^2)+8(\beta^2-3)(1+\beta)+6(3+\beta^2)(1+\beta)^2}{24(1+\beta)^4} \\
&-\frac{3(1-\beta^2)(3-\beta^2)+8(\beta^2-3)(1-\beta)+6(3+\beta^2)(1-\beta)^2}{24(1-\beta)^4} \\
&=\frac{3(1-\beta)(3-\beta^2)+8(\beta^2-3)+6(3+\beta^2)(1+\beta)}{24(1+\beta)^3} \\
&-\frac{3(1+\beta)(3-\beta^2)+8(\beta^2-3)+6(3+\beta^2)(1-\beta)}{24(1-\beta)^3} \\
&=\frac{3+11\beta^2+9\beta+9\beta^3}{24(1+\beta)^3}-\frac{3+11\beta^2-9\beta-9\beta^3}{24(1-\beta)^3} \\
&=\frac{(3+11\beta^2)\left[(1-\beta)^3-(1+\beta)^3\right]+(9\beta+9\beta^3)\left[(1-\beta)^3+(1+\beta)^3\right]}{24(1-\beta^2)^3} \\
&=\frac{(3+11\beta^2)\left[(1-\beta-1-\beta)\left((1-\beta)^2+1-\beta^2+(1+\beta)^2\right)\right]}{24(1-\beta^2)^3} \\
&+\frac{(9\beta+9\beta^3)\left[(1-\beta+1+\beta)\left((1-\beta)^2-1+\beta^2+(1+\beta)^2\right)\right]}{24(1-\beta^2)^3} \\
&=\frac{(9\beta+9\beta^3)(1+3\beta^2)-\beta(3+11\beta^2)(3+\beta^2)}{12(1-\beta^2)^3} \\
&=\frac{(9\beta+27\beta^3+9\beta^3+27\beta^5)-(9\beta+36\beta^3+11\beta^5)}{12(1-\beta^2)^3} \\
&=\frac{4\beta^5}{3(1-\beta^2)^3} \\
&=\frac{4\beta^5\gamma^6}{3}
\end{aligned}
\qquad .\ (155)
$$

Inserting eq. (154) and eq. (155) into eq. (153) yields

$$
P\left(t_r\right)=\frac{-1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{2c\beta^3}\int_{1-\beta}^{1+\beta}dx
$$

$$
\frac{d}{dx}\left\{\left[\frac{\left(1-\beta^2\right)\left(3-\beta^2\right)}{8x^4}+\frac{\beta^2-3}{3x^3}+\frac{3+\beta^2}{4x^2}\right]\sin^2\theta_0+\frac{\beta^2-1}{4x^4}+\frac{2}{3x^3}-\frac{1}{2x^2}\right\}. \qquad (156)
$$

$$
=\frac{-1}{4\pi\varepsilon_0}\frac{e^2\dot{\beta}^2}{2c\beta^3}\left\{\frac{4\beta^5\gamma^6}{3}\sin^2\theta_0-\frac{4}{3}\gamma^6\beta^3\right\}
$$

$$
=\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2\dot{\beta}^2}{c}\gamma^6\left(1-\beta^2\sin^2\theta_0\right)
$$

Noticing that

$$
\sin\theta_0=\frac{\left|\vec{\beta}\times\dot{\vec{\beta}}\right|}{\beta\dot{\beta}}, \qquad (157)
$$

inserting eq. (157) into eq. (156) leads to

$$
P\left(t_r\right)=\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2\dot{\beta}^2}{c}\gamma^6\left(1-\frac{\left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2}{\dot{\beta}^2}\right)=\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\beta}^2-\left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right]. \qquad (158)
$$

IV.3 Charge moving with arbitrary velocity

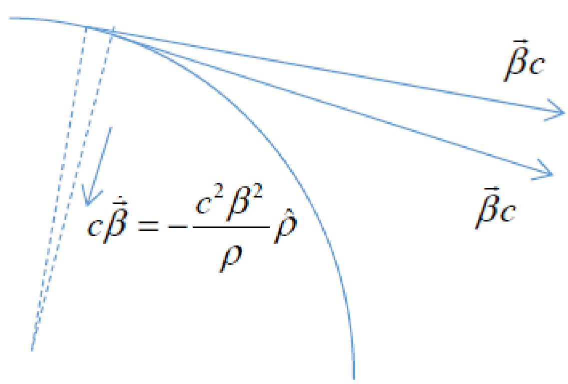


Figure 3: Illustration of charged particle following circular orbit.

For electron traveling along a circular orbit, the acceleration is

$$\dot{a} = -\frac{v^2}{\rho}\hat{\rho} \Rightarrow \dot{\vec{\beta}} = -\frac{\beta^2 c}{\rho}\hat{\rho} \quad . \tag{159}$$

Inserting eq. (159) into eq. (158) yields

$$P\left(t_r\right) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\left(\dot{\beta}^2 - \beta^2\dot{\beta}^2\right) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\dot{\beta}^2\left(1-\beta^2\right) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2c\beta^4\gamma^4}{3\rho^2} \quad . \tag{160}$$

IV.4 Charge moving in straight line with acceleration parallel to its velocity

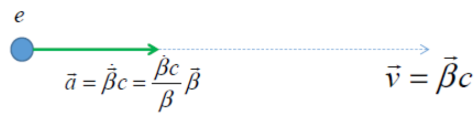


Figure 4: illustration of charged particle following a straight orbit with acceleration parallel to its velocity.

In the case that the acceleration of the charged particle is along the direction of its velocity, it follows that

$$\vec{a} = \dot{\vec{\beta}}c = \frac{\dot{\beta}}{\beta}\vec{\beta}c \Rightarrow \dot{\vec{\beta}} = \frac{\dot{\beta}}{\beta}\vec{\beta} \quad . \tag{161}$$

Inserting eq. (161) into eq. (158) yields

$$P\left(t_r\right) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\dot{\beta}^2 \quad . \tag{162}$$

Using the following relation:

$$\beta^2 = 1-\gamma^{-2} \Rightarrow 2\beta\dot{\beta} = 2\gamma^{-3}\dot{\gamma} \Rightarrow \dot{\beta} = \frac{1}{\beta\gamma^3}\dot{\gamma} = \frac{1}{\beta\gamma^3 mc^2}\frac{dE}{dt} = \frac{1}{\gamma^3 mc}\frac{dE}{dx} \ ,$$

eq. (162) can also be written into the following form:

$$P\left(t_r\right) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3m^2c^3}\left(\frac{dE}{dx}\right)^2 \ , \tag{163}$$

or

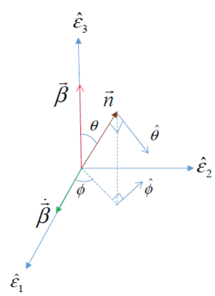
$$\frac{P(t_r)}{dE/dt}=\frac{1}{4\pi\varepsilon_0}\frac{2e^2}{3m^2c^4\beta}\frac{dE}{dx}=\left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2}\right)\frac{2}{3mc^2\beta}\frac{dE}{dx}=\frac{2}{3\beta}\frac{r_e}{mc^2}\frac{dE}{dx}=\frac{2}{3\beta}\frac{dE/dx}{mc^2/r_e}. \quad (164)$$

Eq. (164) is the ratio of the energy gain rate due to acceleration and the energy loss due to radiation, which become significant only when the energy gain rate is comparable to

$$\frac{mc^2}{r_e}=\frac{0.55MeV}{2.8\cdot 10^{-15}m}=1.9\times 10^{14}\frac{MeV}{m}. \quad (165)$$

The state of art accelerating rate at the moment is below 100 MeV/m and hence synchrotron radiation is negligible in linear accelerators.

IV.3 Angular distribution of radiation for charge moving in circular orbit



$$\dot{\vec{\beta}}=\dot{\beta}(1,0,0)$$

$$\vec{\beta}=\beta(0,0,1)$$

$$\vec{n}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$$

$$\hat{\theta}=(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta)$$

$$\vec{n}\cos\theta-\hat{\theta}\sin\theta=(0,0,1)$$

$$\hat{\phi}=(-\sin\phi,\cos\phi,0)$$

$$\vec{n}\sin\theta+\hat{\theta}\cos\theta=(\cos\phi,\sin\phi,0)$$

$$\left(\vec{n}\sin\theta+\hat{\theta}\cos\theta\right)\cos\phi-\hat{\phi}\sin\phi=(1,0,0)$$

Figure 4: coordinate system set up and some relations between vectors.

We start from eq. (138)

$$\frac{dP(t_r)}{d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\left|\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]\right|^2}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^5}, \quad (166)$$

and choose the coordinate system as shown in fig. 4. Using the relations derived in fig. 4,

$$\hat{\varepsilon}_1=\left(\vec{n}\sin\theta-\hat{\theta}\cos\theta\right)\cos\phi-\hat{\phi}\sin\phi, \quad (167)$$

$$\hat{\varepsilon}_3=\vec{n}\cos\theta-\hat{\theta}\sin\theta, \quad (168)$$

we obtain

$$\dot{\vec{\beta}} = \dot{\beta}\hat{\varepsilon}_1 = \dot{\beta}\left(\vec{n}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi\right) \,, \tag{169}$$

and

$$\vec{\beta} = \beta\hat{\varepsilon}_3 = \beta\left(\vec{n}\cos\theta - \hat{\theta}\sin\theta\right) \,. \tag{170}$$

Noting that $\vec{n}$ is along the radial direction of the observation point for the spherical coordinate system $\left(\hat{r},\hat{\theta},\hat{\phi}\right)$, it follows that

$$\begin{aligned}
\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}} &= \left[\vec{n}-\beta\left(\vec{n}\cos\theta-\hat{\theta}\sin\theta\right)\right]\times\dot{\beta}\left(\vec{n}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi\right)\\
&= \vec{n}\times\dot{\beta}\left(\vec{n}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi\right)\\
&\quad -\beta\dot{\beta}\cos\theta\vec{n}\times\left(\vec{n}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi\right)\\
&\quad +\beta\dot{\beta}\sin\theta\hat{\theta}\times\left(\vec{n}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi\right)\\
&= \hat{\phi}\dot{\beta}\cos\theta\cos\phi+\hat{\theta}\dot{\beta}\sin\phi-\beta\dot{\beta}\cos^2\theta\cos\phi\hat{\phi}-\beta\dot{\beta}\cos\theta\sin\phi\hat{\theta}\\
&\quad -\beta\dot{\beta}\sin\theta\sin\theta\cos\phi\hat{\phi}-\beta\dot{\beta}\sin\theta\sin\phi\hat{n}\\
&= \dot{\beta}\left(\cos\theta-\beta\right)\cos\phi\hat{\phi}+\hat{\theta}\dot{\beta}\sin\phi-\beta\dot{\beta}\cos\theta\sin\phi\hat{\theta}-\beta\dot{\beta}\sin\theta\sin\phi\hat{n}
\end{aligned} \,, \tag{171}$$

$$\begin{aligned}
\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right] &= \vec{n}\times\left[\dot{\beta}\left(\cos\theta-\beta\right)\cos\phi\hat{\phi}+\left(\dot{\beta}\sin\phi-\beta\dot{\beta}\cos\theta\sin\phi\right)\hat{\theta}\right]\\
&= -\dot{\beta}\left(\cos\theta-\beta\right)\cos\phi\hat{\theta}+\dot{\beta}\sin\phi\left(1-\beta\cos\theta\right)\hat{\phi}
\end{aligned} \,, \tag{172}$$

and

$$\begin{aligned}
\left|\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right|^2 &= \dot{\beta}^2\left[\left(\cos\theta-\beta\right)^2\cos^2\phi+\sin^2\phi\left(1-\beta\cos\theta\right)^2\right]\\
&= \dot{\beta}^2\left[\left(\cos^2\theta-2\beta\cos\theta+\beta^2\right)\cos^2\phi+\left(1-\cos^2\phi\right)\left(1-2\beta\cos\theta+\beta^2\cos^2\theta\right)\right]\\
&= \dot{\beta}^2\left[\left(\cos^2\theta-1\right)\left(1-\beta^2\right)\cos^2\phi+\left(1-2\beta\cos\theta+\beta^2\cos^2\theta\right)\right]\\
&= \dot{\beta}^2\left[\left(1-\beta\cos\theta\right)^2-\sin^2\theta\left(1-\beta^2\right)\cos^2\phi\right]
\end{aligned} \,. \tag{173}$$

Inserting eq. (173) into eq. (166) yields

$$
\begin{aligned}
\frac{dP(t_r)}{d\Omega} &= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\dot{\beta}^2\left[(1-\beta\cos\theta)^2-\sin^2\theta\left(1-\beta^2\right)\cos^2\phi\right]}{(1-\beta\cos\theta)^5} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\left(1-\beta^2\right)\cos^2\phi}{(1-\beta\cos\theta)^2}\right] \quad . \qquad (174) \\
&= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]
\end{aligned}
$$

For $\frac{1}{\gamma^4} << \theta << 1$ and $\gamma >> 1$, we can use the following approximation

$$
\begin{aligned}
1-\beta\cos\theta &\approx 1-\beta\left(1-\frac{1}{2}\theta^2\right) \\
&= 1-\beta+\frac{1}{2}\beta\theta^2 \\
&= \frac{1}{\gamma^2(1+\beta)}+\frac{1}{2}\theta^2 \\
&= \frac{1}{\gamma^2}\left[\frac{1}{2-(1-\beta)}\right]+\frac{1}{2}\theta^2 \quad , \\
&\approx \frac{1}{2\gamma^2}\left[1+\frac{1-\beta}{2}\right]+\frac{1}{2}\theta^2 \\
&\approx \frac{1}{2\gamma^2}\left[1+\frac{1}{2\gamma^2}+\ldots\right]+\frac{1}{2}\theta^2 \\
&\approx \frac{1}{2\gamma^2}+\frac{1}{2}\theta^2
\end{aligned}
$$

and eq. (174) becomes

$$
\frac{dP(t_r)}{d\Omega} \approx \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{\pi c}\frac{\gamma^6\dot{\beta}^2}{\left(1+\gamma^2\theta^2\right)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\phi}{\left(1+\gamma^2\theta^2\right)^2}\right]. \qquad (175)
$$

Since the factor inside the square bracket is between 0 and 1, the angular width of eq. (175) is determined by the factor $\left(1+\gamma^2\theta^2\right)^{-3}$, i.e. the radiation power drops substantially when $\theta \geq \frac{1}{\gamma}$.

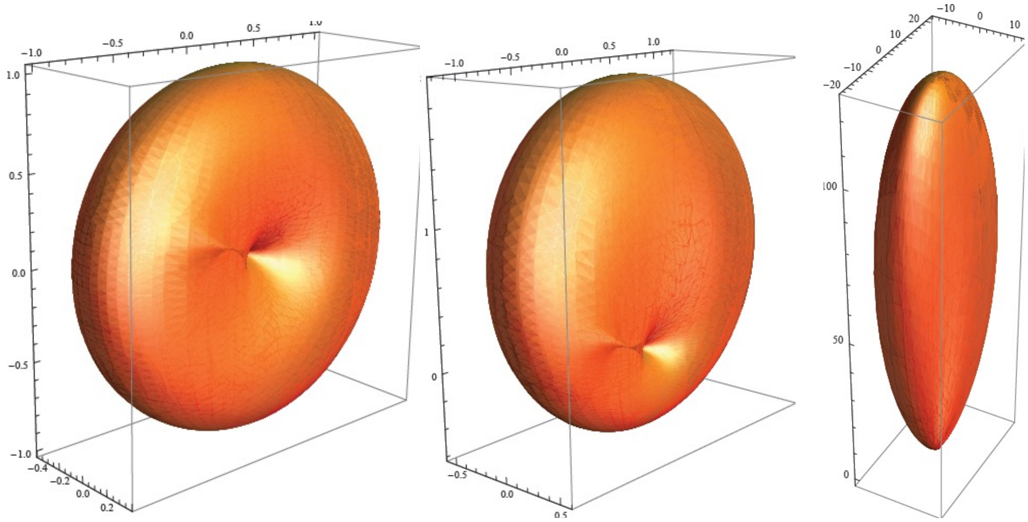


Figure 5: angular distribution of synchrotron radiation as calculated from eq. (175). The plots are generated with Mathematica spherical 3-D plot of function $r = f(\theta,\phi)$ with the radius representing the value of the radiation. Left: $\beta = 0.002$; middle: $\beta = 0.2$; right: $\beta = 0.8$.

IV.3 Spectrum of radiation

Since we are now looking at the frequency spectrum at the observation point and observation time, the power per solid angle at the observation time can be derived from eq. (135)

$$\frac{dP(t)}{d\Omega} = \left(\vec{n}\cdot\vec{S}\right) R\left(t_r\right)^2 = c\varepsilon_0 E_{rad}^2\left(\vec{x},t\right) R\left(t_r\right)^2 \; . \qquad (176)$$

Compared with eq. (135), eq. (176) does not have the dt/dt_r term and the total energy radiated is obtained from

$$dW = \int \frac{dP(t)}{d\Omega} dt \; , \qquad (177)$$

with t being the time lapsed at the observation point. We can also write eq. (176) into the following form

$$\frac{dP(t)}{d\Omega} = c\varepsilon_0 R\left(t_r\right)^2 \vec{E}_{rad}\left(\vec{x},t\right)\cdot\vec{E}_{rad}\left(\vec{x},t\right) = \vec{a}\left(\vec{x},t\right)\cdot\vec{a}\left(\vec{x},t\right) \; , \qquad (178)$$

with

$$\vec{a}(\vec{x},t)=\sqrt{c\varepsilon_0}R(t_r)\vec{E}_{rad}(\vec{x},t)\ . \qquad (179)$$

Inserting eq. (132) into eq. (179) yields

$$\vec{a}(\vec{x},t)=\frac{e}{4\pi\sqrt{\varepsilon_0 c}}\frac{\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}\ . \qquad (180)$$

The Fourier transformation of $\vec{a}(\vec{x},t)$ is defined as

$$\tilde{\vec{a}}(\vec{x},\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{a}(\vec{x},t)e^{i\omega t}\,dt\ , \qquad (181)$$

and the inverse Fourier transformation of eq. (181) reads

$$\vec{a}(\vec{x},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{a}}(\vec{x},\omega)e^{-i\omega t}\,d\omega\ . \qquad (182)$$

Since $\vec{a}(\vec{x},t)$ is real, the following relation hold:

$$\vec{a}(\vec{x},t)=\vec{a}^*(\vec{x},t). \qquad (183)$$

Inserting eq. (182) to eq. (183) leads to

$$\int_{-\infty}^{\infty}\tilde{\vec{a}}(\vec{x},\omega)e^{-i\omega t}\,d\omega=\int_{-\infty}^{\infty}\tilde{\vec{a}}^*(\vec{x},\omega)e^{i\omega t}\,d\omega=-\int_{\infty}^{-\infty}\tilde{\vec{a}}^*(\vec{x},-\omega')e^{-i\omega' t}\,d\omega'=\int_{-\infty}^{\infty}\tilde{\vec{a}}^*(\vec{x},-\omega)e^{-i\omega t}\,d\omega\ , \qquad (184)$$

and therefore, it follows

$$\tilde{\vec{a}}(\vec{x},\omega)=\tilde{\vec{a}}^*(\vec{x},-\omega)\ . \qquad (185)$$

Inserting eq. (182) into eq. (178) yields

$$\begin{aligned}\frac{dP(t)}{d\Omega}&=\vec{a}(\vec{x},t)\cdot\vec{a}(\vec{x},t)\\&=\vec{a}(\vec{x},t)\cdot\vec{a}^*(\vec{x},t)\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{a}}(\vec{x},\omega)\cdot e^{-i\omega t}\,d\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{a}}^*(\vec{x},\omega')e^{i\omega' t}\,d\omega'\ .\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,d\omega'\,\tilde{\vec{a}}(\vec{x},\omega)\cdot\tilde{\vec{a}}^*(\vec{x},\omega')e^{i\omega' t}e^{-i\omega t}\end{aligned} \qquad (186)$$

Integrating eq. (186) over t gives the total radiated energy per solid angle:

$$\begin{aligned}\frac{dW}{d\Omega} &= \int_{-\infty}^{\infty} \frac{dP(t)}{d\Omega} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, d\omega' \tilde{\vec{a}}(\vec{x},\omega) \cdot \tilde{\vec{a}}^{*}(\vec{x},\omega') \int_{-\infty}^{\infty} e^{i\omega' t} e^{-i\omega t}\, dt \\ &= \int_{-\infty}^{\infty} d\omega\, d\omega' \tilde{\vec{a}}(\vec{x},\omega) \cdot \tilde{\vec{a}}^{*}(\vec{x},\omega') \delta(\omega' - \omega) \\ &= \int_{-\infty}^{\infty} \left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2 d\omega \\ &= \int_{0}^{\infty} \left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2 d\omega + \int_{-\infty}^{0} \left|\tilde{\vec{a}}(\vec{x},-\omega')\right|^2 d(-\omega') \\ &= \int_{0}^{\infty} \left[\left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2 + \left|\tilde{\vec{a}}(\vec{x},-\omega)\right|^2 \right] d\omega \end{aligned} \quad . \qquad (187)$$

From eq. (185), it follows that

$$\left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2 = \left|\tilde{\vec{a}}^{*}(\vec{x},-\omega)\right|^2 = \left|\tilde{\vec{a}}(\vec{x},-\omega)\right|^2 . \qquad (188)$$

Inserting eq. (188) into eq. (187) produces

$$\frac{dW}{d\Omega} = 2\int_{0}^{\infty} \left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2 d\omega \equiv \int_{0}^{\infty} \frac{dI(\omega)}{d\Omega} d\omega \ , \qquad (189)$$

where we introduced the total radiation received per unit frequency per unit solid angle during the entire pulse of radiation with

$$\frac{dI(\omega)}{d\Omega} = 2\left|\tilde{\vec{a}}(\vec{x},\omega)\right|^2 . \qquad (190)$$

Inserting eq. (180) into eq. (181) yields

$$
\begin{aligned}
\tilde{\vec{a}}(\vec{x},\omega) &= \frac{e}{4\pi\sqrt{2\pi\varepsilon_0 c}}\int_{-\infty}^{\infty}\frac{\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}e^{i\omega t}\,dt \\
&= \frac{e}{4\pi\sqrt{2\pi\varepsilon_0 c}}\int_{-\infty}^{\infty}\frac{\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^3}e^{i\omega t}\frac{dt}{dt_r}dt_r \qquad , \quad (191) \\
&= \frac{e}{4\pi\sqrt{2\pi\varepsilon_0 c}}\int_{-\infty}^{\infty}\frac{\vec{n}(t_r)\times\left[\left(\vec{n}(t_r)-\vec{\beta}(t_r)\right)\times\dot{\vec{\beta}}(t_r)\right]}{\left[1-\vec{n}(t_r)\cdot\vec{\beta}(t_r)\right]^2}e^{i\omega\left(t_r+R(t_r)/c\right)}dt_r
\end{aligned}
$$

where in the last step, we used eq. (137) and eq. (108). In the limit of far field approximation, $|\vec{x}| >> |\vec{r}(\tau_0)|$ in eq. (103), it follows

$$
\begin{aligned}
R(\tau_0)^2 &= \left(\vec{x}-\vec{r}(\tau_0)\right)\cdot\left(\vec{x}-\vec{r}(\tau_0)\right) \\
&= |\vec{x}|^2 - 2\vec{x}\cdot\vec{r}(\tau_0) + |\vec{r}(\tau_0)|^2 \\
&= |\vec{x}|^2\left(1-2\frac{\vec{x}\cdot\vec{r}(\tau_0)}{|\vec{x}|^2}+\frac{|\vec{r}(\tau_0)|^2}{|\vec{x}|^2}\right), \qquad (192) \\
&\approx |\vec{x}|^2\left(1-2\frac{\vec{x}\cdot\vec{r}(\tau_0)}{|\vec{x}|^2}\right) \\
&\approx |\vec{x}|^2\left(1-2\frac{\vec{n}\cdot\vec{r}(\tau_0)}{|\vec{x}|}\right)
\end{aligned}
$$

where in the last step, we used

$$
\vec{n} = \frac{\vec{R}}{R} = \frac{\vec{x}-\vec{r}(\tau_0)}{|\vec{x}-\vec{r}(\tau_0)|} \approx \frac{\vec{x}}{|\vec{x}|} \ . \qquad (193)
$$

Taking square root of eq. (192) leads to

$$
R(\tau_0) \approx |\vec{x}|\sqrt{\left(1-2\frac{\vec{n}\cdot\vec{r}(t_r)}{|\vec{x}|}\right)} \approx |\vec{x}|\left(1-\frac{\vec{n}\cdot\vec{r}(t_r)}{|\vec{x}|}\right) = |\vec{x}| - \vec{n}\cdot\vec{r}(t_r) \ . \qquad (194)
$$

Inserting eq. (194) into eq. (191) produces

$$\tilde{\vec{a}}\left(\vec{x},\omega\right)=\frac{e}{4\pi\sqrt{2\pi\varepsilon_0 c}}e^{i|\vec{x}|\omega/c}\int_{-\infty}^{\infty}\frac{\vec{n}\left(t_r\right)\times\left[\left(\vec{n}\left(t_r\right)-\vec{\beta}\left(t_r\right)\right)\times\dot{\vec{\beta}}\left(t_r\right)\right]}{\left[1-\vec{n}\left(t_r\right)\cdot\vec{\beta}\left(t_r\right)\right]^2}e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)}dt_r \quad . \tag{195}$$

Since

$$\begin{aligned}\frac{d}{dt_r}\vec{n}\left(t_r\right)&=\frac{d}{dt_r}\frac{\vec{x}-\vec{r}\left(t_r\right)}{\left|\vec{x}-\vec{r}\left(t_r\right)\right|}\\&=-\frac{\vec{\beta}\left(t_r\right)c}{\left|\vec{x}-\vec{r}\left(t_r\right)\right|}+\left(\vec{x}-\vec{r}\left(t_r\right)\right)\frac{d}{dt_r}\frac{1}{\left|\vec{x}-\vec{r}\left(t_r\right)\right|}\ ,\\&=-\frac{\vec{\beta}\left(t_r\right)c}{\left|\vec{x}-\vec{r}\left(t_r\right)\right|}+\frac{\vec{n}\cdot\vec{\beta}\left(t_r\right)c}{\left|\vec{x}-\vec{r}\left(t_r\right)\right|}\vec{n}\end{aligned} \tag{196}$$

if we assume

$$\Delta\vec{n}\left(t_r\right)=-\frac{\vec{\beta}\left(t_r\right)c\Delta t_r}{\left|\vec{x}-\vec{r}\left(t_r\right)\right|}+\frac{\vec{n}\cdot\vec{\beta}\left(t_r\right)c\Delta t_r}{\left|\vec{x}-\vec{r}\left(t_r\right)\right|}\vec{n}\sim\frac{r}{R}<<\Delta\beta\ , \tag{197}$$

we can ignore the derivative with respect to $\vec{n}$ in the following derivative

$$\frac{d}{dt_r}\left(\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)\ ,$$

and obtain

$$\begin{aligned}\frac{d}{dt_r}\left(\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)&\approx\frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}+\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\\&=\frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}\left(1-\vec{n}\cdot\vec{\beta}\right)+\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\\&=\frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}+\frac{\left[\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)\right]\left(\vec{n}\cdot\dot{\vec{\beta}}\right)-\left[\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)\right]\left(\vec{n}\cdot\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}\ .\\&=\frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}+\frac{\vec{n}\times\left\{\vec{n}\times\left[\left(\vec{n}\cdot\dot{\vec{\beta}}\right)\vec{\beta}-\left(\vec{n}\cdot\vec{\beta}\right)\dot{\vec{\beta}}\right]\right\}}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}\end{aligned} \tag{198}$$

Making use of eq. (124)

$$\begin{aligned}
\frac{d}{dt_r}\left(\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}\right) &\approx \frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}+\frac{\vec{n}\times\left\{\vec{n}\times\left[\vec{n}\times\left(\vec{\beta}\times\dot{\vec{\beta}}\right)\right]\right\}}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2} \\
&= \frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}+\frac{\left\{\vec{n}\cdot\left[\vec{n}\times\left(\vec{\beta}\times\dot{\vec{\beta}}\right)\right]\right\}\vec{n}-\left\{\vec{n}\cdot\vec{n}\right\}\left[\vec{n}\times\left(\vec{\beta}\times\dot{\vec{\beta}}\right)\right]}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2} \\
&= \frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}+\frac{-\vec{n}\times\left(\vec{\beta}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2} \\
&= \frac{\vec{n}\times\left(\vec{n}\times\dot{\vec{\beta}}-\vec{\beta}\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2} \\
&= \frac{\vec{n}\times\left(\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)^2}
\end{aligned} \quad , \qquad (199)$$

where in the second step, we also used eq. (124). Inserting eq. (199) into eq. (195) yields

$$\begin{aligned}
\tilde{\vec{a}}\left(\vec{x},\omega\right) &= \frac{e}{4\pi\sqrt{2\pi\varepsilon_0 c}}e^{i|\vec{x}|\omega/c}\int_{-\infty}^{\infty}\frac{d}{dt_r}\left(\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)}\,dt_r \\
&= -\frac{i\omega e}{4\pi\sqrt{2\pi\varepsilon_0 c}}e^{i|\vec{x}|\omega/c}\int_{-\infty}^{\infty}\left(\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)}\frac{d}{dt_r}\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)dt_r \\
&= -\frac{i\omega e}{4\pi\sqrt{2\pi\varepsilon_0 c}}e^{i|\vec{x}|\omega/c}\int_{-\infty}^{\infty}\left(\frac{\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)}{\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)}\left(1-\vec{n}\cdot\vec{\beta}\left(t_r\right)\right)dt_r \\
&= -\frac{i\omega e}{4\pi\sqrt{2\pi\varepsilon_0 c}}e^{i|\vec{x}|\omega/c}\int_{-\infty}^{\infty}\left(\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)}\,dt_r
\end{aligned} \quad . \qquad (200)$$

Inserting eq. (200) into eq. (190) leads to

$$\frac{dI\left(\omega\right)}{d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2e^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\left(\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)}\,dt_r\right|^2 \ . \qquad (201)$$

IV.4 Synchrotron radiation spectrum for charge moving in circular orbit

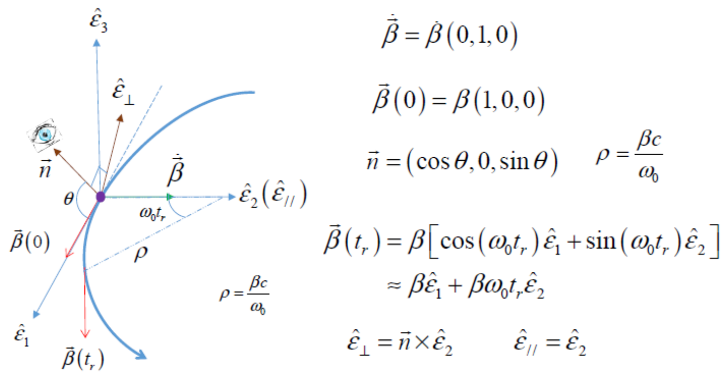


Figure 6: illustration of instantaneous circular motion of charged particles and coordinate system set up.

In the configuration of fig. 6, it is assumed that the overall circular orbit is much shorter than the bending radius, i.e. $\omega_0 t_r << 1$, and the observation point is limited to the x-z plane. From fig. 6, the direction of the observation point is

$$\vec{n} = \hat{\varepsilon}_1 \cos\theta + \hat{\varepsilon}_3 \sin\theta \ , \qquad (202)$$

and the velocity vector for $\omega_0 t_r << 1$ is

$$\vec{\beta}(t_r) = \beta\left[\cos(\omega_0 t_r)\hat{\varepsilon}_1 + \sin(\omega_0 t_r)\hat{\varepsilon}_2\right] \approx \beta\hat{\varepsilon}_1 + \beta\omega_0 t_r \hat{\varepsilon}_2 \, . \qquad (203)$$

It follows from eq. (202) and (203) that

$$\begin{aligned}
&\vec{n}(t_r)\times\left[\vec{n}(t_r)\times\vec{\beta}(t_r)\right] = \vec{n}(t_r)\times\left[\vec{n}(t_r)\times\left(\beta\hat{\varepsilon}_1 + \beta\omega_0 t_r \hat{\varepsilon}_2\right)\right] \\
&= \vec{n}(t_r)\times\left[\beta\vec{n}(t_r)\times\hat{\varepsilon}_1 + \beta\omega_0 t_r \vec{n}(t_r)\times\hat{\varepsilon}_2\right] \\
&= \vec{n}(t_r)\times\left(\beta\sin\theta\hat{\varepsilon}_2 + \beta\omega_0 t_r \hat{\varepsilon}_\perp\right) \\
&= \beta\sin\theta\hat{\varepsilon}_\perp - \beta\omega_0 t_r \hat{\varepsilon}_{//}
\end{aligned} \ . \qquad (204)$$

Now we assume

$$\theta : \frac{1}{\gamma} << 1 \, , \qquad (205)$$

and replace $\sin\theta$ with θ in eq. (204) to get

$$\vec{n}(t_r)\times\left[\vec{n}(t_r)\times\vec{\beta}(t_r)\right] = \beta\theta\hat{\varepsilon}_\perp - \beta\omega_0 t_r \hat{\varepsilon}_{//} \, . \qquad (206)$$

Inserting eq. (206) into eq. (201) leads to

$$\frac{dI(\omega)}{d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2e^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}\left(\beta\theta\hat{\varepsilon}_{\perp}-\beta\omega_0t_r\hat{\varepsilon}_{//}\right)e^{i\omega\left(t_r-\vec{n}\cdot\vec{r}\left(t_r\right)/c\right)}\,dt_r\right|^2 . \qquad (207)$$

Fig. 6 suggests that

$$\begin{aligned}\vec{r}\left(t_r\right)&=c\int\vec{\beta}\left(t_r\right)dt_r+C_1\\&=c\int\beta\left[\cos\left(\omega_0t_r\right)\hat{\varepsilon}_1+\sin\left(\omega_0t_r\right)\hat{\varepsilon}_2\right]dt_r+C_1 ,\\&=\frac{\beta c}{\omega_0}\left[\sin\left(\omega_0t_r\right)\hat{\varepsilon}_1-\cos\left(\omega_0t_r\right)\hat{\varepsilon}_2\right]+\frac{\beta c}{\omega_0}\hat{\varepsilon}_2\end{aligned} \qquad (208)$$

where the integration constant C_1 is chosen so that

$$\vec{r}\left(0\right)=0 . \qquad (209)$$

For $\omega_0t_r<<1$, we keep up to the third power of ω_0t_r since it is in the phase part of eq. (207) and obtain

$$\vec{r}\left(t_r\right)\approx\frac{\beta c}{\omega_0}\left(\omega_0t_r-\frac{\left(\omega_0t_r\right)^3}{6}\right)\hat{\varepsilon}_1+\frac{\beta c}{\omega_0}\frac{\left(\omega_0t_r\right)^2}{2}\hat{\varepsilon}_2 . \qquad (210)$$

It follows from eq. (210) and (202) that

$$\vec{n}\left(t_r\right)\cdot\vec{r}\left(t_r\right)\approx\frac{\beta c\cos\theta}{\omega_0}\left(\omega_0t_r-\frac{\left(\omega_0t_r\right)^3}{6}\right)\approx\frac{\beta c}{\omega_0}\left(1-\frac{\theta^2}{2}\right)\left(\omega_0t_r-\frac{\left(\omega_0t_r\right)^3}{6}\right) . \qquad (211)$$

Therefore, the phase factor in eq. (207) reads

$$
\begin{aligned}
\omega\left[t_r - \vec{n}(t_r)\cdot\vec{r}(t_r)/c\right] &= \omega t_r - \frac{\omega\beta}{\omega_0}\left(1-\frac{\theta^2}{2}\right)\left(\omega_0 t_r - \frac{(\omega_0 t_r)^3}{6}\right) \\
&= \omega t_r - \omega\beta t_r\left(1-\frac{\theta^2}{2}\right) + \frac{\omega\beta}{\omega_0}\left(1-\frac{\theta^2}{2}\right)\frac{(\omega_0 t_r)^3}{6} \\
&= \omega t_r\left(1-\beta+\beta\frac{\theta^2}{2}\right) + \frac{\omega\beta}{\omega_0}\left(1-\frac{\theta^2}{2}\right)\frac{(\omega_0 t_r)^3}{6} \qquad . \qquad (212) \\
&\approx \omega t_r\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right) + \frac{\omega(\omega_0 t_r)^3}{6\omega_0} \\
&= \frac{\omega}{2\omega_0}\left[\omega_0 t_r\left(\frac{1}{\gamma^2}+\theta^2\right) + \frac{(\omega_0 t_r)^3}{3}\right]
\end{aligned}
$$

Inserting eq. (212) into eq. (207) yields

$$
\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}\left(\beta\theta\hat{\varepsilon}_\perp - \beta\omega_0 t_r\hat{\varepsilon}_{//}\right)\exp\left\{i\frac{\omega}{2\omega_0}\left[\omega_0 t_r\left(\frac{1}{\gamma^2}+\theta^2\right)+\frac{(\omega_0 t_r)^3}{3}\right]\right\}dt_r\right|^2 . \quad (213)
$$

It is worth noting that the expansion assumes $\omega_0 t_r << 1$ and hence the integral limit of infinity violates the assumption. However, cubic power in the phase factor make large t_r not contribute and hence we can still do the expansion. Changing the integration variable in eq. (213) to

$$
x = \frac{\omega_0}{\sqrt{\frac{1}{\gamma^2}+\theta^2}}t_r , \qquad (214)
$$

i.e.

$$
t_r = \sqrt{\frac{1}{\gamma^2}+\theta^2}\,\frac{x}{\omega_0} , \qquad (215)
$$

eq. (213) becomes

$$\frac{dI(\omega)}{d\Omega}=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2}{4\pi^2 c\omega_0^2}$$

$$\times\left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\int_{-\infty}^{\infty}\left(\beta\theta\hat{\varepsilon}_{\perp}-\beta\omega_0 t_r\hat{\varepsilon}_{//}\right)\exp\left\{\frac{i}{2}\left[x\frac{\omega}{\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{\frac{3}{2}}+\frac{1}{3}\frac{\omega}{\omega_0}x^3\left(\frac{1}{\gamma^2}+\theta^2\right)^{\frac{3}{2}}\right]\right\}dx\right|^2$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2}{4\pi^2 c\omega_0^2}\left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\int_{-\infty}^{\infty}\left(\beta\theta\hat{\varepsilon}_{\perp}-\beta\omega_0 t_r\hat{\varepsilon}_{//}\right)\exp\left[i\frac{\eta}{2}\left(3x+x^3\right)\right]dx\right|^2$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2}{4\pi^2 c\omega_0^2}\left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\beta\theta\hat{\varepsilon}_{\perp}\int_{-\infty}^{\infty}\exp\left[i\frac{\eta}{2}\left(3x+x^3\right)\right]dx\right.$$

$$\left.-\left(\frac{1}{\gamma^2}+\theta^2\right)\beta\hat{\varepsilon}_{//}\int_{-\infty}^{\infty}x\exp\left[i\frac{\eta}{2}\left(3x+x^3\right)\right]dx\right|^2$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2}{4\pi^2 c\omega_0^2}\left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\beta\theta\hat{\varepsilon}_{\perp}I_{\perp}(\eta)-\left(\frac{1}{\gamma^2}+\theta^2\right)\beta\hat{\varepsilon}_{//}I_{//}(\eta)\right|^2 \quad ,\ (216)$$

with

$$\eta=\frac{1}{3}\frac{\omega}{\omega_0}\left(\frac{1}{\gamma^2}+\theta^2\right)^{\frac{3}{2}}, \quad (217)$$

$$I_{\perp}(\eta)\equiv\int_{-\infty}^{\infty}\exp\left[i\frac{\eta}{2}\left(3x+x^3\right)\right]dx\ , \quad (218)$$

and

$$I_{//}(\eta)\equiv\int_{-\infty}^{\infty}x\exp\left[i\frac{\eta}{2}\left(3x+x^3\right)\right]dx\ . \quad (219)$$

Using the following relation found in Table of integral, series and products,

$$\int_0^{\infty}\sin\left(a^3x^3\right)\sin(bx)dx=\frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)-\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$

and

$$\int_0^{\infty}\cos\left(a^3x^3\right)\cos(bx)dx=\frac{\pi}{6a}\sqrt{\frac{b}{3a}}\left\{J_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+J_{-\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)+\frac{\sqrt{3}}{\pi}K_{\frac{1}{3}}\left(\frac{2b}{3a}\sqrt{\frac{b}{3a}}\right)\right\}$$

the integral in eq. (218) can be written as

$$
\begin{aligned}
I_{\perp}(\eta) &\equiv \int_{-\infty}^{\infty} \exp\left[i\left(bx + a^3x^3\right)\right]dx \\
&= \int_{0}^{\infty} \exp\left[i\left(bx + a^3x^3\right)\right]dx + \int_{-\infty}^{0} \exp\left[i\left(bx + a^3x^3\right)\right]dx \\
&= \int_{0}^{\infty} \exp\left[i\left(bx + a^3x^3\right)\right]dx + \int_{0}^{\infty} \exp\left[-i\left(bx + a^3x^3\right)\right]dx \\
&= 2\int_{0}^{\infty} \cos\left[\left(bx + a^3x^3\right)\right]dx \\
&= 2\left\{\int_{0}^{\infty} \cos(bx)\cos\left(a^3x^3\right)dx - \int_{0}^{\infty} \sin(bx)\sin\left(a^3x^3\right)dx\right\} \\
&= \frac{2}{3}\sqrt{\frac{b}{a^3}}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)
\end{aligned}
, \qquad (220)
$$

with

$$
a = \left(\frac{\eta}{2}\right)^{1/3}, \qquad (221)
$$

and

$$
b = \frac{3\eta}{2} . \qquad (222)
$$

Inserting eq. (221) and eq. (222) into eq. (220) yields

$$
I_{\perp}(\eta) = \frac{2}{\sqrt{3}} K_{\frac{1}{3}}(\eta) . \qquad (223)
$$

Similarly, the integral in eq. (219) can be written as

$$
\begin{aligned}
I_{//}(\eta) &\equiv \int_{-\infty}^{\infty} x \exp\left[i\left(bx+a^3x^3\right)\right]dx \\
&= \frac{d}{idb}\int_{-\infty}^{\infty} \exp\left[i\left(bx+a^3x^3\right)\right]dx \\
&= \frac{2}{i3}\frac{d}{db}\left\{\sqrt{\frac{b}{a^3}}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)\right\} \\
&= \frac{2}{i3}\left\{K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)\frac{d}{db}\sqrt{\frac{b}{a^3}}+\sqrt{\frac{b}{a^3}}\frac{d}{db}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)\right\} \\
&= \frac{2}{i3}\left\{\frac{1}{2}\sqrt{\frac{1}{a^3b}}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)+\sqrt{\frac{b}{a^3}}\frac{d}{db}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)\left(\frac{d}{dX}K_{\frac{1}{3}}(X)\right)_{X=\frac{2b}{3}\sqrt{\frac{b}{3a^3}}}\right\} \\
&= \frac{2}{i3}\left\{\frac{1}{2}\sqrt{\frac{1}{a^3b}}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)+\frac{b}{\sqrt{3}a^3}\left(\frac{d}{dX}K_{\frac{1}{3}}(X)\right)_{X=\frac{2b}{3}\sqrt{\frac{b}{3a^3}}}\right\}
\end{aligned} \quad . \text{(224)}
$$

Using the relation

$$
z\frac{d}{dz}K_\nu(z)+\nu K_\nu(z)=-zK_{\nu-1}(z), \quad \text{(225)}
$$

and

$$
K_{-\nu}(z)=K_\nu(z) \ , \quad \text{(226)}
$$

it follows

$$
\frac{d}{dz}K_\nu(z)=-\frac{\nu}{z}K_\nu(z)-K_{1-\nu}(z) \ . \quad \text{(227)}
$$

Inserting eq. (227) into eq. (224) yields

$$
\begin{aligned}
I_{//}(\eta) &= \frac{2}{i3}\left\{\frac{1}{2}\sqrt{\frac{1}{a^3b}}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)+\frac{b}{\sqrt{3}a^3}\left(-\frac{1}{3X}K_{\frac{1}{3}}(X)-K_{\frac{2}{3}}(X)\right)_{X=\frac{2b}{3}\sqrt{\frac{b}{3a^3}}}\right\} \\
&= \frac{2}{i3}\left\{\frac{1}{2}\sqrt{\frac{1}{a^3b}}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)-\frac{1}{2\sqrt{a^3b}}K_{\frac{1}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)-\frac{b}{\sqrt{3}a^3}K_{\frac{2}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)\right\} \ . \quad \text{(228)} \\
&= -\frac{2}{i3}\frac{b}{\sqrt{3}a^3}K_{\frac{2}{3}}\left(\frac{2b}{3}\sqrt{\frac{b}{3a^3}}\right)
\end{aligned}
$$

Inserting eq. (221) and (222) into eq. (228) yields

$$I_{//}(\eta) = -\frac{2}{i\sqrt{3}} K_{\frac{2}{3}}(\eta) \ . \tag{229}$$

Inserting eq. (223) and eq. (229) into eq. (216) yields

$$\begin{aligned}
\frac{dI(\omega)}{d\Omega} &= \frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2 \beta^2}{4\pi^2 c\omega_0^2}\left|\sqrt{\frac{1}{\gamma^2}+\theta^2}\,\theta\frac{2}{\sqrt{3}}K_{\frac{1}{3}}(\eta)\hat{\varepsilon}_{\perp} + \left(\frac{1}{\gamma^2}+\theta^2\right)\frac{2}{i\sqrt{3}}K_{\frac{2}{3}}(\eta)\hat{\varepsilon}_{//}\right|^2 \\
&= \frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2 \beta^2}{4\pi^2 c\omega_0^2}\left\{\left[\sqrt{\frac{1}{\gamma^2}+\theta^2}\,\theta\frac{2}{\sqrt{3}}K_{\frac{1}{3}}(\eta)\right]^2 + \left[\left(\frac{1}{\gamma^2}+\theta^2\right)\frac{2}{\sqrt{3}}K_{\frac{2}{3}}(\eta)\right]^2\right\} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2 \beta^2}{4\pi^2 c\omega_0^2}\left(\frac{1}{\gamma^2}+\theta^2\right)^2 \frac{4}{3}\left\{\frac{\theta^2}{\left(\frac{1}{\gamma^2}+\theta^2\right)}K_{\frac{1}{3}}{}^2(\eta) + K_{\frac{2}{3}}{}^2(\eta)\right\} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{\omega^2 e^2 \beta^2}{3\pi^2 c\omega_0^2\gamma^4}\left(1+\theta^2\gamma^2\right)^2\left\{K_{\frac{2}{3}}{}^2(\eta) + \frac{\theta^2\gamma^2}{\left(1+\theta^2\gamma^2\right)}K_{\frac{1}{3}}{}^2(\eta)\right\}
\end{aligned} \ . \tag{230}$$

APPENDIX: the trajectory of particle can only intersects with light-cone once.

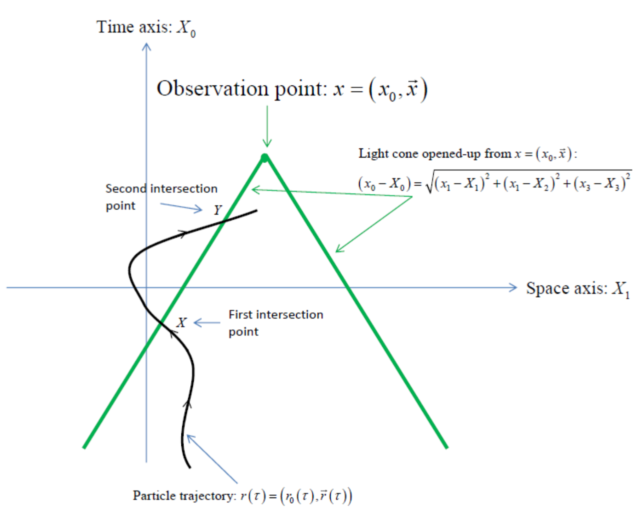


Figure A1: illustration of why it is not possible for the trajectory of a particle to intersect with light-cone twice.

Since the two intersection points are on the light-cone opened-up by $x=\left(x_0,\vec{x}\right)$, they satisfy the following equation:

$$\left(x_0-X_0\right)-\sqrt{\left(X_1-x_1\right)^2+\left(X_2-x_3\right)^2+\left(X_3-x_3\right)^2}=0\,, \qquad (231)$$

and

$$\left(x_0-Y_0\right)-\sqrt{\left(Y_1-x_1\right)^2+\left(Y_2-x_3\right)^2+\left(Y_3-x_3\right)^2}=0\,. \qquad (232)$$

Subtracting eq. (232) with eq. (231) yields

$$Y_0-X_0=\sqrt{\left(X_1-x_1\right)^2+\left(X_2-x_2\right)^2+\left(X_3-x_3\right)^2}-\sqrt{\left(Y_1-x_1\right)^2+\left(Y_2-x_2\right)^2+\left(Y_3-x_3\right)^2}\,. \qquad (233)$$

The three points $\vec{x}$, $\vec{X}$ and $\vec{Y}$ form a triangle and since the difference in the length of any two sides of a triangle is always smaller than the length of the third side, it follows from eq. (233)

$$Y_0-X_0\leq\sqrt{\left(X_1-Y_1\right)^2+\left(X_2-Y_2\right)^2+\left(X_3-Y_3\right)^2}\,. \qquad (234)$$

The time it takes for the particle to get from $\vec{X}$ to $\vec{Y}$ is given by

$$\Delta t=\frac{Y_0-X_0}{c}\,,$$

and hence the average velocity of the particle during its travelling from $\vec{X}$ to $\vec{Y}$ is

$$\left\langle v_{particle}\right\rangle=\frac{\sqrt{\left(X_1-Y_1\right)^2+\left(X_2-Y_2\right)^2+\left(X_3-Y_3\right)^2}}{\Delta t}=\frac{c\sqrt{\left(X_1-Y_1\right)^2+\left(X_2-Y_2\right)^2+\left(X_3-Y_3\right)^2}}{Y_0-X_0}\,. \qquad (235)$$

According to eq. (234) , the following relation holds

$$\frac{\sqrt{\left(X_1-Y_1\right)^2+\left(X_2-Y_2\right)^2+\left(X_3-Y_3\right)^2}}{Y_0-X_0}\geq 1\,, \qquad (236)$$

and inserting eq. (236) into eq. (235) yields

$$\left\langle v_{particle} \right\rangle = c\frac{\sqrt{\left(X_1 - Y_1\right)^2 + \left(X_2 - Y_2\right)^2 + \left(X_3 - Y_3\right)^2}}{Y_0 - X_0} \geq c \ . \qquad (237)$$

Eq. (237) violates special relativity and hence the trajectory of a particle cannot intersect a light-cone twice.

and the observation point must share the same world line.

For any observation point in 4-D space, $x = \left(\vec{x}, x_0\right)$, there are only one τ value contributing to the integral in eq. (92) (since light go straight and particles move slower than light, the particle will never see the photon radiated by itself at an earlier time. Consequently, the 4-D trajectory). We denote these contributing τ values

Using eq. (67) and (68), eq. (59) can be written as

$$\begin{aligned} A^{\alpha}\left(\bar{\vec{x}}, \bar{x}_0\right) &= \int_0^{x_0} d\left(\bar{x}'_0 - \tau_0\right) \int_{-\infty}^{\infty} \frac{\delta\left(\bar{x}_0 - \tau_0 - \left(\bar{x}'_0 - \tau_0\right) - \left|\bar{\vec{x}} - \bar{\vec{x}}'\right|\right)}{4\pi\left|\bar{\vec{x}} - \bar{\vec{x}}'\right|} J^{\alpha}\left(\bar{\vec{x}}', \bar{x}'_0\right) d^3\bar{x}' \\ &= \int_{\tau_0}^{\bar{x}_0} d\bar{x}'_0 \int_{-\infty}^{\infty} \frac{\delta\left(\bar{x}_0 - \bar{x}'_0 - \left|\bar{\vec{x}} - \bar{\vec{x}}'\right|\right)}{4\pi\left|\bar{\vec{x}} - \bar{\vec{x}}'\right|} J^{\alpha}\left(\bar{\vec{x}}', \bar{x}'_0\right) d^3\bar{x}' \end{aligned} \quad , \quad (238)$$

which, after removing the bar from all variables and taking the limit of $\tau_0 \to -\infty$, becomes

$$A^{\alpha}\left(\vec{x}, x_0\right) = \int_{-\infty}^{x_0} dx'_0 \int_{-\infty}^{\infty} \frac{\delta\left(x_0 - x'_0 - \left|\vec{x} - \vec{x}'\right|\right)}{4\pi\left|\vec{x} - \vec{x}'\right|} J^{\alpha}\left(\vec{x}', x'_0\right) d^3x' . \qquad (239)$$