Following Landau's Classical Theory of fields: **Covariant differentiation**

Summation over repeated indices is assumed $A^{..i..}_{.....}B^{.....}_{....i.} \equiv \sum_{n=1}^{N} A^{..i..}_{.....}B^{.....}_{....i.}$ where N is the dimension of the space we are using. Let's consider an arbitrary non-degenerated (e.g. inversible) non-linear transformation of coordinates. Contravariant vector transforms as differentials of the contravariant components of the coordinated (it is a definition)

$$x^i = x^i\left(x' = \left\{x'^k\right\}\right); \vec{x} = \vec{x}\left(\vec{x}'\right); dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k;$$

$$def:\ A^i = \frac{\partial x^i}{\partial x'^k} A'^k \tag{1}$$

$$A^{in} = \frac{\partial x^i}{\partial x'^k}\frac{\partial x^n}{\partial x'^m} A'^{km}$$

The contravariant components transform as partial differentials of a scalar function, φ

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial x'^k}{\partial x^i}\cdot\frac{\partial \varphi}{\partial x'^k};\quad \frac{\partial x'^k}{\partial x^i}\frac{\partial x^i}{\partial x'^j} = \delta^k_j$$

$$def:\ A_i = \frac{\partial x'^k}{\partial x^i} A'_k;\ A_{in} = \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^m}{\partial x^n} A'_{km} \tag{2}$$

Such definition preserves scalar product of two vectors

$$def:\ \vec{A}\cdot\vec{B} = A^i B_i = \frac{\partial x^i}{\partial x'^k}\frac{\partial x'^j}{\partial x^i} A'^k B'_j = \delta^j_k A'^k B'_j = A'^k B'_k \tag{3}$$

Length of an element is defined by symmetric tensor g_{ik} :

$$def:\ ds^2 = g_{ik} dx^i dx^k;\ g_{ik} = g_{ki};$$

$$def: \left[g^{ik}\right] \equiv \left[g_{ki}\right]^{-1} \to g^{ik} g_{kj} = \delta^i_j \tag{4}$$

since any antisymmetric tensor convolved with symmetric tensor $dx_k dx_j$ results in trivial zero. It is important to remember that metric tensor can be a function of coordinates $g_{ik} = g_{ik}\left(x\right)$!

It is true that metric tensor is the tensor

$$ds^2 = g_{ik} dx^i dx^k = g'_{nm} dx'^n dx'^m \to g_{ik} = \frac{\partial x'^n}{\partial x^i}\frac{\partial x'^m}{\partial x^k} g'_{nm} \to$$

$$\frac{\partial x'^n}{\partial x^i}\frac{\partial x'^m}{\partial x^k} g'_{nm} dx^i dx^k = \frac{\partial x'^n}{\partial x^i}\frac{\partial x'^m}{\partial x^k}\frac{\partial x^i}{\partial x'^j}\frac{\partial x^k}{\partial x'^l} g'_{nm} dx'^j dx'^l = \tag{4}$$

$$\delta^n_j \delta^m_l g'_{nm} dx'^j dx'^l = g'_{nm} dx'^n dx'^m \ \#$$

If we considering transformation from Cartesian coordinates where metric tensor is diagonal matrix, then

$$g_{ik}=\frac{\partial x'^n}{\partial x^i}\frac{\partial x'^m}{\partial x^k}\delta^n_m=\frac{\partial x'^n}{\partial x^i}\frac{\partial x'^n}{\partial x^k}\equiv\sum_n\frac{\partial x'^n}{\partial x^i}\frac{\partial x'^n}{\partial x^k} \tag{5}$$

which is obviously is the symmetric tensor.

Lowering and raising indices is accomplished by the metric tensors:

$$\begin{aligned}&A_i=g_{ik}A^k;A_{ik}=g_{in}A^n{}_k=g_{km}A_i^m=g_{in}g_{km}A^{nm}\\&A^k=g^{ki}A_i;A^{nm}=g^{nk}A_k{}^m=g^{mk}A^n{}_k=g^{ni}g^{mk}A_{ik};\end{aligned} \tag{6}$$

Which again coming from testing transform from Cartesian coordinates where $A'_k=A'^k$:

$$\begin{aligned}&A_i=\frac{\partial x'^k}{\partial x^i}A'_k;A^j=\frac{\partial x^j}{\partial x'^l}A'^l;A'^l=\frac{\partial x'^l}{\partial x^j}A^j\\&A'_k=A'^k\rightarrow A_i=\frac{\partial x'^k}{\partial x^i}\frac{\partial x'^k}{\partial x^j}A^j=g_{ij}A^j;\\&\Rightarrow A^j=\left[g_{ij}\right]^{-1}A_i=g^{ji}A_i\end{aligned} \tag{7}$$

Similar conclusions can be derived for tensors.

The totally asymmetric tensor $e^{ijklm....}$ is defined in the Cartesian system as

$$e^{12...n}=1;e^{..ik..}=-e^{..ki..}. \tag{8}$$

Curvilinear coordinate system does changes it and to distinguish it from that in the Cartesian system we will call it $\varepsilon^{ijklm....}$. Again, let's consider transformation from a Cartesian system with

$$\varepsilon^{ik...}=\frac{\partial x^i}{\partial x'^j}\frac{\partial x^k}{\partial x'^l}...e^{jl...}=Je^{ik...};\ J=\det\left[\frac{\partial x^i}{\partial x'^j}\right] \tag{9}$$

where we used relation derived in intro to the Linear algebra. Eq. (9) is actually one of standard definitions of matrix' determinant. The determinant of the metric tensor is connected with J. It is transformed as

$$\begin{aligned}&g_{ik}=\frac{\partial x'^n}{\partial x^i}\frac{\partial x'^m}{\partial x^k}g'_{nm};g=\det\left[g_{ik}\right]\\&g^{ik}=\frac{\partial x^i}{\partial x'^n}\frac{\partial x^k}{\partial x'^m}g'^{nm};\det\left[g^{ik}\right]=\det\left[g_{ik}\right]^{-1}=g^{-1}\\&g^{-1}=\det\left[g^{ik}\right]=\det\left[\frac{\partial x^i}{\partial x'^j}\right]^2\det\left[g'^{nm}\right]=J^2\det\left[g'^{nm}\right]\end{aligned} \tag{10}$$

and since we define transformation of totally asymmetric tensor from the Cartesian system where metric tensor is the unit matrix with unit determinant, eqs. (9) and (10) produce a simple[1]

[1] It is worth to note that in special relativity determinant of the flat space and time metric is equal -1. In this case $J=\sqrt{-g^{-1}}$.

$$\varepsilon^{ik\ldots} = \frac{e^{ik\ldots}}{\sqrt{g}}; \tag{11}$$

Because coordinate transformation is a function of location, the Covariant differentiations varies significantly from what we used in flat Cartesian space. First and foremost, as simple deferential of the vector is no longer is a vector:

$$A_i = \frac{\partial x'^k}{\partial x^i} A_k' \to dA_i = \frac{\partial x'^k}{\partial x^i} dA_k' + A_k' \frac{\partial^2 x'^k}{\partial x^i \partial x^j} dx^j \tag{12}$$

has and extra term which violates the transformation rule for vectors (2).

Let's find a tensor which plays role of tensor $\frac{\partial A_i}{\partial x^k}$ in Cartesian coordiantes. We need to transform this tensor to curvelinear corrdinates – to have such trasformation we need to figure out how to translate vector from one point to another which is separated bu infitezimally small distance. Such transformation should not affect dA_i in the Cartesian system.

Moving from point x^i to $x^i + dx^i$ result in changes vector A^i to $A^i + dA^i$. Let make a parallel translation of vector A^i to point $x^i + dx^i$ and define change resulting from this translation as δA^i . The remaining deferens is

$$DA_i = dA_i - \delta A_i; \tag{12}$$

This translational change should change sum of vectors in the sum and both vectors, e.g. it has to be linear:

$$\delta A_i = -\Gamma^i{}_{kl} A^k dx^l \tag{13}$$

where $\Gamma^i{}_{kl}$ are so-called Christoffel symbols functions of coordinates which depend on the system. By definition, they are zero in Cartesian system – which meant that <u>they are not tensors (hence they are called symbols!)</u>: zero tensor or vectors remain zero in all coordinate systems. We can also define low-index called Christoffel symbols as

$$\Gamma_{i,kl} \overset{def}{=} g_{im} \Gamma^m{}_{kl} \Rightarrow \Gamma^m{}_{kl} = g^{mi} \Gamma_{i,kl} \tag{14}$$

Scalars are by definition do not change under parallel translation, e.g. the convolution of the scalar vector product should not change:

$$\begin{aligned} &\delta\left(B^i A_i\right) = A_i \delta B^i + B^i \delta A_i = 0 \to B^i \delta A_i = -A_i \delta B^i = \Gamma^i{}_{kl} B^k A_i dx^l; \\ &B^i \delta A_i = -A_i \delta B^i = \Gamma^k{}_{il} B^i A_k dx^l \to \delta A_i = \Gamma^k{}_{il} A_k dx^l \end{aligned} \tag{14}$$

which defined change of the covariant vector under parallel translation. Thus, we found expression for covariant differentials:

$$dA^i = \frac{\partial A^i}{\partial x^l} dx^l; dA_i = \frac{\partial A_i}{\partial x^l} dx^l;$$
$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^k{}_{il} A^k \right) dx^l; DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k{}_{il} A_k \right) dx^l; \tag{15}$$

which now transform as vectors – e.g. the expression is the brackets are tensors which we will write as and call "covariant derivatives"

$$DA^i = A^i{}_{;l} dx^l; DA_i = A_{i;l} dx^l;$$
$$A^i{}_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i{}_{kl} A^k; \; A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k{}_{il} A_k; \tag{16}$$

Similarly, we can define covariant derivatives of a tensor

$$\delta\left(A^i B^k\right) = \delta A^i B^k + A^i \delta B^k = -A^i \Gamma^k{}_{lm} B^l dx^m - B^k \Gamma^i{}_{lm} A^l dx^m$$
$$\delta\left(A^{ik}\right) = -\left(A^{im}\Gamma^k{}_{ml} + A^{mk}\Gamma^i{}_{lm}\right) dx^l$$
$$DA^{ik} = dA^{ik} + \Gamma^i{}_{ml} A^{mk} + \Gamma^k{}_{lm} A^{im} \tag{17}$$
$$A^{ik}{}_{;l} = \frac{\partial A^{ik}}{\partial x^l} + \Gamma^i{}_{ml} A^{mk} + \Gamma^k{}_{ml} A^{im};$$
$$A_{ik;l} = \frac{\partial A_{ik}}{\partial x^l} - \Gamma^m{}_{il} A_{mk} - \Gamma^m{}_{kl} A_{im}; A^i{}_{k;l} = \frac{\partial A^i{}_k}{\partial x^l} - \Gamma^m{}_{kl} A^i{}_m + \Gamma^i{}_{ml} A^m{}_k;$$

For scalars $\delta\varphi = 0 \to D\varphi = d\varphi;$ Since covariant derivative is a linear operator, it is easy to show that covariant derivative of a product behaves the same way as regular derivative:

$$D\left(A_i B_k\right) = D\left(A_i\right) B_k + A_i D\left(B_k\right);$$
$$\left(A_i B_k\right)_{;l} = A_{i;l} B_k + A_{i;l} B_k \tag{18}$$

Since the newly created object behave as tensors, we can define contravariant derivatives using metric tensor:

$$A_i^{;k} = g^{kl} A_{i;l}; A^{i;k} = g^{kl} A^i{}_{;l}, \; etc.. \tag{19}$$

Christoffel symbols $\Gamma^i{}_{kl}$ are symmetric with respect to low indices, e.g. $\Gamma^i{}_{kl} = \Gamma^i{}_{lk}$. The easiest way to prove it is to use a vector which is gradient of a scalar function

$$A_i = \frac{\partial\varphi}{\partial x^i}; A_{k;i} = \frac{\partial A_k}{\partial x^i} - \Gamma^j{}_{ki} A_j;_{k;i} \; A_{i;k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j{}_{ik} A_j$$
$$A_{k;i} - A_{i;k} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} + \left(\Gamma^j{}_{ik} - \Gamma^j{}_{ki}\right) A_j \tag{20}$$
$$\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} = \frac{\partial^2\varphi}{\partial x^i \partial x^k} - \frac{\partial^2\varphi}{\partial x^k \partial x^i} = 0; \; B_{ik} = A_{k;i} - A_{i;k} = \left(\Gamma^j{}_{ik} - \Gamma^j{}_{ki}\right)\frac{\partial\varphi}{\partial x^j}.$$

B_{ik} is the difference of two tensors and therefore is a tensor. But this tensor es equal zero in Cartesian coordinate system – hence it is zero everywhere! Hence,

$$\Gamma^{j}{}_{ik} = \Gamma^{j}{}_{ki}; \Gamma_{j,ik} = \Gamma_{j,ki}. \tag{21}$$

and for 3D space there are 18 independent indices (40 in 4D space).

A simple by tedious calculations using fact that in eq. (16) both sides have to transform as vectors will result in transformation rule for the Christoffel symbols:

$$\Gamma^{i}{}_{kl} = \Gamma'^{m}_{nj} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^j}{\partial x^l} + \frac{\partial^2 x'^j}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^j}. \tag{22}$$

In addition to defining transformation of the Christoffel symbols it also uniquely defines it for a transformation from Cartesian system where Γ'^{m}_{nj} =0 as

$$\Gamma^{i}{}_{kl} = \frac{\partial^2 x'^j_c}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^j_c}; \tag{22-Cart}$$

which clearly shows that the symbols are symmetric with respect to the low indices.

Now we have all necessary instruments to connect the Christoffel symbols with the metric tensor. First let's show that covariant derivative of g_{ik} is equal zero: we will use the method of lowering indices of vectors:

$$\begin{gathered} DA_i = g_{ik} DA^k \ and \ A_i = g_{ik} A^k \Rightarrow \\ D\left(g_{ik} A^k\right) = A^k D\left(g_{ik}\right) + g_{ik} DA^k = g_{ik} DA^k \to D\left(g_{ik}\right) = 0; \\ g_{ik;l} = 0; \end{gathered} \tag{23}$$

Last equation gives us necessary connection:

$$\begin{gathered} g_{ik;l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk} \Gamma^{m}{}_{il} - g_{im} \Gamma^{m}{}_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k;il} - \Gamma_{i;kl} = 0; \\ \frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k.il} + \Gamma_{i,kl}; \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik}; -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,li}; \Rightarrow \\ \frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i} = 2\Gamma_{i;kl} \Rightarrow \Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i} \right); \\ \Gamma^{i}{}_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right); \end{gathered} \tag{23}$$

where we used symmetry of the symbols to find the ratio: red and blue terms cancel each other in the sum.

Now we are close to find a way to define differential operators in curvilinear coordinates.

Let's start from defining the differential of the metric tensor determinant

$$g = \det\left[g_{ik}\right]; dg = g_{ik}{}^{\text{minor}} dg_{ik} + O\left(dg_{ik}{}^{2}\right)$$

$$g_{ik}{}^{\text{minor}} = \frac{1}{g} g^{ik}; dg = gg^{ik} dg_{ik} \to \frac{\partial g}{\partial x^m} dx^m = gg^{ik} \frac{\partial g_{ik}}{\partial x^m} dx^m$$

$$\Gamma^i{}_{ki} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{mi}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^m} \right); \tag{24}$$

$$g^{im} \frac{\partial g_{mk}}{\partial x^l} - g^{im} \frac{\partial g_{ki}}{\partial x^m} = g^{im} \frac{\partial g_{mk}}{\partial x^l} - g^{mi} \frac{\partial g_{km}}{\partial x^i} = 0;$$

$$\Gamma^i{}_{ki} = \frac{1}{2} g^{im} \frac{\partial g_{im}}{\partial x^k} = \frac{1}{2g} \frac{\partial g}{\partial x^k} = \frac{\partial \ln \sqrt{g}}{\partial x^k}$$

Similarly

$$g^{lk}\Gamma^i{}_{kl} = \frac{1}{2} g^{kl} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{mi}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^m} \right) \equiv g^{kl} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{mi}}{\partial x^k} - \frac{1}{2} \frac{\partial g_{ki}}{\partial x^m} \right)$$

$$\Downarrow \tag{25}$$

$$g^{lk}\Gamma^i{}_{kl} = -\frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} g^{ik}\right)}{\partial x^k}$$

Eq. (25) allows us to calculate divergence of vector in curvilinear coordinates as:

$$div\vec{A} = A^i_{;i} = \frac{\partial A^i}{\partial x^i} + \Gamma^i{}_{ki} A^k = \frac{\partial A^i}{\partial x^i} + A^k \frac{\partial \ln \sqrt{g}}{\partial x^k} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} A^k\right)}{\partial x^k}; \tag{26}$$

Gradient is the most trivial differential operator since it does not change

$$grad\varphi \equiv \vec{\nabla}\varphi = \hat{e}^k \cdot \frac{\partial \varphi}{\partial x'^k};\ \hat{e}^k = \frac{\partial x'^k}{\partial x^i} \vec{g}^i; \vec{g}^i = \frac{\partial x^i}{\partial x'^k} \hat{e}^k; \frac{\partial \varphi}{\partial x'^k} = \frac{\partial \varphi}{\partial x^j} \frac{\partial x^j}{\partial x'^k};$$

$$\vec{\nabla}\varphi = \hat{e}^k \frac{\partial \varphi}{\partial x'^k} = \frac{\partial x'^k}{\partial x^i} \frac{\partial x^j}{\partial x'^k} \vec{g}^i \frac{\partial \varphi}{\partial x^j} = \delta_i^j \vec{g}^i \frac{\partial \varphi}{\partial x^j} = \vec{g}^i \frac{\partial \varphi}{\partial x^i}; \tag{27}$$

$$\vec{\nabla}\varphi = \vec{g}_i \frac{\partial \varphi}{\partial x_i}; \vec{g}_i = g_{ik} \vec{g}^k;$$

Now we can also define the Laplacian as

$$\Delta\varphi = div\left(\vec{\nabla}\varphi\right); \vec{\nabla}\varphi = \vec{g}^i \frac{\partial \varphi}{\partial x^i}; \left(\vec{\nabla}\varphi\right)_k = \vec{g}_k \vec{g}^i \frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x^k}; \left(\vec{\nabla}\varphi\right)^k = g^{ki} \frac{\partial \varphi}{\partial x^i}$$

$$div\left(\vec{\nabla}\varphi\right) == \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\left(\vec{\nabla}\varphi\right)^k\right)}{\partial x^k} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g} g^{ki} \frac{\partial \varphi}{\partial x^i}\right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g} \frac{\partial \varphi}{\partial x_k}\right) \tag{28}$$

The only remaining operator is *curl*, which is a pseudovector. In Cartesian system it is

$$curl\vec{A} = \vec{\nabla} \times \vec{A} = \hat{e}_i e^{ikl} \frac{\partial A'_k}{\partial x'_l} \tag{29}$$

With all components in the right being either vectors or tensors, we can transfer them into the curvilinear system as

$$curl\vec{A} == \vec{g}_i \varepsilon^{ikl} \frac{\partial A_k}{\partial x_l} = \vec{g}_i \frac{e^{ikl}}{\sqrt{g}} \frac{\partial A_k}{\partial x_l} \tag{30}$$

This gives us all necessary differential operators we will use in this portion of the course.

Additional notes: Co- and contravariant systems, transformations and the gradient

Start from metric tensor in an arbitrary curvilinear system of coordinates:

$$
\begin{gathered}
ds^2 \equiv dq^k dq_k \equiv g_{ik} dq^i dq^k \equiv g^{ik} dq_i dq_k; \\
\vec{g}^k = \frac{d\vec{r}}{dq_k}; \vec{g}_k = \frac{d\vec{r}}{dq^k} \\
A^i = g^{ik} A_k; A_i \equiv g_{ik} A^k; g^{ik} g_{kj} = \delta^i_j; \\
\vec{A} = \vec{g}^k A_k \equiv \vec{g}_k A^k; \\
g^{ik} = \vec{g}^i \cdot \vec{g}^k; g_{ik} = \vec{g}_i \cdot \vec{g}_k; \vec{g}^i \cdot \vec{g}_k = \delta^i_k
\end{gathered}
\tag{1}
$$

We can start from Cartesian system with unit diagonal metric and orthonormal basis

$$
\begin{gathered}
ds^2 \equiv dx^k dx_k; dx_k = dx^k \\
\vec{e}^k = \vec{e}_k = \hat{e}_k; \hat{e}_k \cdot \hat{e}_j = \delta_{kj} \\
\vec{A} = \hat{e}_k a_k; a_k = a^k \\
q^i = q^i(x); dq^i = \left[\frac{\partial q^i}{\partial x^k}\right] dx^k = J^i_k dx^k; J^i_k \equiv \frac{\partial q^i}{\partial x^k}; \\
dq_i = \left[\frac{\partial x^k}{\partial q^i}\right] dx_k = J^{-1i}{}_k dx_k \\
A^i = \left[\frac{\partial q^i}{\partial x^k}\right] a^k; A_i = \left[\frac{\partial x^k}{\partial q^i}\right] a_k
\end{gathered}
\tag{2}
$$

The gradient transformation is the easy one:

$$
\begin{gathered}
grad\varphi \equiv \vec{\nabla}\varphi = \hat{e}^k \cdot \frac{\partial \varphi}{\partial x^k};\ \hat{e}^k = \frac{\partial x^k}{\partial q^i} \vec{g}^i; \frac{\partial \varphi}{\partial x^k} = \frac{\partial \varphi}{\partial q^j} \frac{\partial q^j}{\partial x^k}; \\
\vec{\nabla}\varphi = \vec{g}^i \left[\frac{\partial q^j}{\partial x^k} \cdot \frac{\partial x^k}{\partial q^i}\right] \frac{\partial \varphi}{\partial q^j}; \left[\frac{\partial q^j}{\partial x^k} \cdot \frac{\partial x^k}{\partial q^i}\right] = \delta^j_i \vec{g}^i \frac{\partial \varphi}{\partial q^j} = \vec{g}^i \frac{\partial \varphi}{\partial q^i}
\end{gathered}
\tag{3}
$$