Equation of motion

Consider a particle moving through a magnet w/ gradient

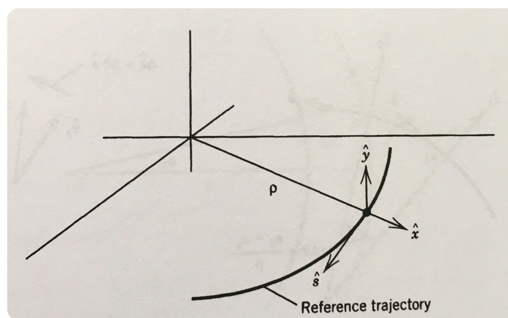
$B' = \frac{\partial B_y}{\partial x}$ over a distance of Δs,

$$\Delta x' = \frac{dx}{ds} = -\frac{B' \Delta s}{(B\rho)} x \quad \text{(see Last Lecture)}$$

$$\frac{\Delta x'}{\Delta s} = -\frac{B'(s)}{(B\rho)} x$$

$$\Rightarrow \boxed{x'' + \frac{B'(s)}{(B\rho)} x = 0}$$

Recall that this formalism came from the assumption that there was no magnetic field on the axis. If there is a nonzero magnetic field on the design trajectory, this equation simply represents a difference between the slope change of particle in question and the ideal particle.



In the case of non-zero magnetic field (e.g. synchrotrons), it can be show that the equations of motion are:

$$\therefore \begin{cases} \frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{(B\rho)} \frac{\partial B_y(s)}{\partial x} \right] x = 0 & (3.46) \\ \frac{d^2y}{ds^2} - \frac{1}{(B\rho)} \frac{\partial B_y(s)}{\partial x} y = 0 & (3.4) \end{cases}$$

Centripital term

Note that the added term to 'x' equation represents the centripetal motion. For large accelerators, the centripetal term is usually small in comparison

with the gradient term. The difference in sign represents the fact that a single magnetic gradient cannot provide focusing forces in both of the transverse dimensions.
These equations of motion have the form:

$$x'' + K(s)\,x = 0$$

Two methods of solution:

- K is normally constant in an accelerator section by design, so within each component, we use harmonic oscillator solution and piece them together at interfaces.
- We can get a closed form solution using the properties of Hill's equation.

<u>Piecewise method of solution</u>

Just like last section, here we can describe the motion of a particle by a 2x2 matrix. There are three cases to consider:
<u>K=0:</u> this case is the same as having a drift space L:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

In y (vertical) plane, this corresponds to either a drift between the magnets or propagating through a magnet with constant By
In the x (horizontal) plane, this corresponds to a drift space between the magnets or if the centripetal term is exactly balanced by the field gradient. The latter case is unusual. Frequently, for other than exact calculation, the radius of curvature of a high energy accelerator is so large that the centripetal term maybe neglected.
<u>K>0:</u> over distance l, the equation of motion is just a simple harmonic oscillator, and in matrix term,

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{out} = \begin{bmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{in}$$

<u>K<0:</u> over this range, the solution is hyperbolic sine functions

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{out} = \begin{bmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{K}L) & \cosh(\sqrt{|K|}L) \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{in}$$

in thin lens limit $\cos x \to 1 + \frac{x^2}{2}$, $\sin x \to x + \frac{1}{x} + \cdots$

so, in the case of say $K > 0$

$$\therefore \begin{bmatrix} x \\ x' \end{bmatrix}_{out} = \begin{bmatrix} 1 & \frac{\sqrt{K}L}{\sqrt{K}} \\ -\sqrt{K}\sqrt{K}L & 1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ -KL & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$L \to 0, \; KL: \text{finite} \Rightarrow \begin{bmatrix} x \\ x' \end{bmatrix}_{out} = \begin{bmatrix} 1 & 0 \\ -KL & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}_{in}$$

Compared to the transfer matrix for a thin lens:

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

the equation for the focal length is $f = \frac{1}{KL}$

Similar to previous case, for a series of elements: $M_1, M_2, \ldots,$

input and output conditions are related by

$$M = M_n M_{n-1} \cdots M_2 M_1$$

Closed form solution

Second solution is based on Hill's equation: a differential equation studied extensively in the nineteenth century.

The equation of motion $x'' + K(s)\,x = 0$

Has the property that for an important class of accelerators, K is periodic, I.e.

$$K(s + C) = K(s)$$

The repeat distance, C, may be as large as the circumference of a synchrotron. The result of nineteenth century mathematics can be expressed in the form

$$x = A\,w(s)\cos[\psi(s) + \delta]$$

Constants of integration

$w(s)$: periodic function with periodicity C.

If K was positive everywhere, we would get

$$x = A\cos\left[\psi(s) + \delta\right],$$

$\psi = \sqrt{K}\, s$ & A & δ are constants of integration.

When K is a periodic function of position, the solution will differ from that of the simple harmonic oscillator by a spatially varying amplitude, I.e. w(s), and the phase will no longer be linear in 's'.

To find $w(s)$ and $\psi(s)$, substitute the general solution into the eqn:

$$x'' + Kx = 0 \Rightarrow -A\left(w\psi'' + 2w'\psi'\right)\sin\left[\psi + \delta\right]$$

$$+A\left(w'' - w\psi'^2 + Kw\right)\cos\left[\psi + \delta\right] = 0$$

These two coefficients have to go to zero.

$w \times \sin$ term: $2ww'\psi' + w^2\psi'' = (w^2\psi')' = 0$

$$\therefore \psi' = \frac{k}{w(s)^2}$$

little k, distinc from big K arbitrary constant

the cosine term turns into:

$$w^3(w'' + \underline{K}w) = k^2 \quad (3.59)$$

strickly speaking, $w(s)$ need not be periodic. But we are Looking fo solutions to synchrotron, so, we will look for solutions that have a periodicity C

Now, we can represent the same matrix components in terms of parameters introduced here:

i.e.

$$x = A\,w(s)\cos\left[\psi(s) + \delta\right]$$

Rewrite as:

$$x = w(s)\left(A_1 \cos\psi + A_2 \sin\psi\right)$$

then,

little k

$$x' = \left(A_1 w' + \frac{A_2 k}{w}\right)\cos\psi + \left(A_2 w' - \frac{A_1 k}{w}\right)\sin\psi$$

then given initial conditions x_0, x'_0 at $s = s_0$, the constants are

$$A_1 = \frac{x_0}{w},$$

$$A_2 = \frac{x'_0 w - x_0 w'}{k}$$

Now, by requiring that the function be periodic in C, we may write the matrix propagation from $s_0 \longrightarrow s_0 + C$

$$\begin{bmatrix} x \\ x' \end{bmatrix}_{s_0+C} = \begin{bmatrix} \cos\Delta\psi_c - \frac{ww'}{k}\sin\Delta\psi_c & \frac{w^2}{k}\sin\Delta\psi_c \\ -\frac{1+(ww'/k)^2}{w^2/k}\sin\Delta\psi_c & \cos\Delta\psi_c + \frac{ww'}{k}\sin(\Delta\psi_c) \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}_{s_0} \quad (3.64)$$

the phase of particle oscillation advances by

$$\psi(s_0 \longrightarrow s_0 + C) \equiv \Delta\psi_c = \int_{s_0}^{s_0+C} \frac{k\,ds}{w^2(s)}$$

because $w(s)$ is periodic, this integral is independant of s_0.