PHY 564
Advanced Accelerator Physics
Lecture 8

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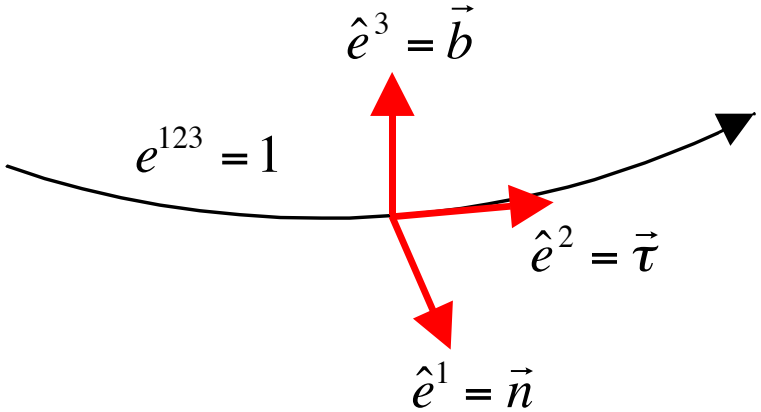
Recap of previous lectures

Accelerator Hamiltonian: continued

Figures 2 and 3 illustrate the Frenet-Serret coordinate system and define 3 orthogonal unit vectors: Normal $\hat{e}^1 = \vec{n}(s)$, tangent $\hat{e}^2 = \vec{\tau}(s)$, and normal and bi-normal $\hat{e}^3 = \vec{b}(s) = \left[\vec{n} \times \vec{\tau}\right]$:

$$\left(\vec{n} \cdot \vec{\tau}\right) = \left(\vec{b} \cdot \vec{n}\right) = \left(\vec{b} \cdot \vec{\tau}\right) = 0.$$

Coordinates



$$\vec{\tau} = \frac{d\vec{r}_o(s)}{ds} = \vec{r}_o{}'$$

$$\vec{n} = -\frac{\vec{r}_o{}''}{\left|\vec{r}_o{}''\right|}$$

$$\vec{b} = \left[\vec{n} \times \vec{\tau}\right]$$

Fig. 3. Unit vectors in the Frenet-Serret coordinate system and their definitions

From least action we get the Most General Form of the Accelerator Hamiltonian

$$h^* = -(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2}$$
$$+\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)$$

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$

$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \equiv -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \rightarrow \frac{dH}{ds} = \frac{\partial h^*}{\partial t}$$

with s as independent variable.

And $t(s)$ is arrival time of a particle to the azimuth s.

After a one-page-long exercise, using the first pair of Maxwell equations and conditions (116), one can express the 4-potential in this gauge though the components of the magnetic- and electric- fields, in other words, make an unique vector potential:

$$A_1 = \frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\big|_{ro} \frac{x^k}{k!}\frac{y^{n+1}}{(n+1)!};\ A_3 = -\frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\big|_{ro} \frac{x^{k+1}}{(k+1)!}\frac{y^n}{n!}$$

$$A_2 = \sum_{n=1}^{\infty}\left\{\partial_x^{n-1}\Big(\big(1+Kx\big)B_y + \kappa x B_s\Big)_{ro}\frac{x^n}{n!} - \partial_y^{n-1}\Big(\big(1+Kx\big)B_x - \kappa y B_s\Big)_{ro}\frac{y^n}{n!}\right\} +$$

$$+\frac{1}{2}\sum_{n.k=1}^{\infty}\left\{\partial_x^{n-1}\partial_y^k\Big(\big(1+Kx\big)B_y + \kappa x B_s\Big)_{ro}\frac{x^n}{n!}\frac{y^k}{k!} - \partial_x^{n}\partial_y^{k-1}\Big(\big(1+Kx\big)B_x - \kappa y B_s\Big)_{ro}\frac{x^n}{n!}\frac{y^k}{k!}\right\};$$

$$\varphi = \varphi_o\big(s,t\big) - \sum_{n=1}^{\infty}\partial_x^{n-1}E_x\big|_{ro}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\partial_y^{n-1}E_y\big|_{ro}\frac{y^n}{n!} - \frac{1}{2}\sum_{n.k=1}^{\infty}\left(\partial_x^{n-1}\partial_y^k E_x\big|_{ro} + \partial_x^n\partial_y^{k-1}E_y\big|_{ro}\right)\frac{x^n}{n!}$$

(118)

where $f\big|_{ro}$; $\big(f\big)_{ro}$ denotes that the value of the function ***f*** is taken at the reference orbit $r_o(s)$: i.e., at $x=0;\ \ y=0$, but in an arbitrary moment of time ***t***.

We also should select variables that are zero at the reference orbit. The following pair is one of better choices:

$$\left\{\tau = -c(t - t_o(s)),\ \delta = \left(H - E_o(s) - e\varphi_o(s,t)\right)/c\right\}, \tag{128}$$

which are zero for the reference particle. Generation function is easily to come with:

$$\Phi(q,\tilde{P},s) = \tilde{P}_1 x + \tilde{P}_3 y - \left(E_o(s) + c\delta\right)\left(t - t_o(s)\right) - e\int^t \varphi_o(s,t_1)dt_1\,, \tag{129}$$

and it produces what is desired:

$$P_1 = \frac{\partial\Phi}{\partial x} = \tilde{P}_1;\ P_3 = \frac{\partial\Phi}{\partial y} = \tilde{P}_3; H = \frac{\partial\Phi}{\partial(-t)} = E_o + c\delta + e\varphi_o(s,t);$$

$$\tilde{q}_1 = \frac{\partial\Phi}{\partial\tilde{P}_1} = x;\ \ \tilde{q}_3 = \frac{\partial\Phi}{\partial\tilde{P}_3} = y;\ \tilde{q}_\delta = \frac{\partial\Phi}{\partial\delta} = -c\left(t - t_o(s)\right) = \tau$$

$$\tilde{h} = h + \frac{\partial\Phi}{\partial s} = h + \frac{E_o(s) + c\delta}{\mathrm{v_o}(s)} + E_o'(s)\tau/c - e\int^t \varphi_o'(s,t_1)dt_1$$

Now the only remaining task is to express the new Hamiltonian function with an updated canonical pair (130) and (115):

$$\tilde{h} = -(1+Kx)\sqrt{p_o{}^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2} + \\ + \frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) + \frac{c}{\mathrm{v_o}}\delta - \frac{e}{c}\varphi_{//}(s,\tau) \tag{132}$$

where we used following trivial expansion and definition:

$$\frac{\left(E_o + c\delta + e\varphi_o(s,t) - e\varphi\right)^2}{c^2} - m^2c^2 = p_o{}^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2;$$

$$\varphi_{\perp def} = \varphi(s,x,y,t) - \varphi_o(s,t) \equiv \varphi(s,x,y,t) - \varphi(s,0,0,t)$$

Scaling variables.

Frequently, it is useful to scale one of canonical variables. Typical scaling in accelerator physics involves dividing the canonical momenta P_1, P_3, δ by the momentum of the reference particle:

$$\pi_1 = \frac{P_1}{p_o}; \ \pi_3 = \frac{P_3}{p_o}; \ \ \pi_o = \frac{\delta}{p_o}.$$

These variables are dimensionless and also are close to $x', y', \ \delta E / p_o c$ for small deviations. Such scaling only is allowed in Hamiltonian mechanics when the scaling parameter is constant, i.e., is not function of s. Thus, scaling by the particle's momentum remains within the framework of Hamiltonian mechanics only if the reference particle's momentum is constant, that is, when the longitudinal electric field is zero along the reference particle's trajectory (i.e. at moment $t=t_o$(s)). One similarly can scale the coordinates by a constant.

$$\xi_1 = \frac{x}{L}; \ \xi_3 = \frac{y}{L}; \ \ \xi_o = \frac{\tau}{L}.$$

Scaling by a constant is easy; divide the Hamiltonian by the constant and rename the variables. Hence, transforming (134) with constant, called $p_{o,}$ will make Hamiltonian (132) into

$$\tilde{h} = -(1+Kx)\sqrt{1+\frac{2E_o}{p_o c}\left(\delta-\frac{e}{p_o c}\varphi_\perp\right)+\left(\delta-\frac{e}{p_o c}\varphi_\perp\right)^2-\left(\pi_1-\frac{e}{p_o c}A_1\right)^2-\left(\pi_3-\frac{e}{p_o c}A_3\right)^2}+$$

$$+\frac{e}{p_o c}A_2+\kappa x\left(\pi_3-\frac{e}{p_o c}A_3\right)-\kappa y\left(\pi_1-\frac{e}{p_o c}A_1\right)+\frac{c}{\mathrm{v_o}}\delta-\frac{e}{p_o c}\varphi_{//}(s,\tau)$$

(132 @ **<u>constant energy</u>**)

Usage of this Hamiltonian is very popular for storage rings or transport channels, wherein the energy of the particles remains constant in time. It should not be employed for particles undergoing an acceleration.

Expanding the Hamiltonian.

Expanding the Hamiltonian (132) is a nominal tool in accelerator physics that allows the separation of the effects of various orders and sometimes the use of perturbation-theory approaches. Having completed the process of creating canonical variables, which are zero for the reference particle, the next step is to assume (which is true for operational accelerators) that the relative deviations of momenta are small

$$\left|\frac{P_1}{p_o}\right| << 1; \left|\frac{P_3}{p_o}\right| << 1; \left|\frac{\delta}{p_o}\right| << 1;$$

and that the EM fields are sufficiently smooth around the reference trajectory to allow expansion in terms of x; y; τ. We will consider1 all six variables to be of the same order (of infinitesimally, α). We call the order of expansion to be the maximum total power in a product that is any combination of $x, y, \tau, P_1, P_2, \delta$. Unless there is a good reason not to do so, we truncate the series using this rule.

$$X^T = \begin{bmatrix} q^1 & P_1 & ..... & .... & q^n & P_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & ..... & .... & x_{2n-1} & x_{2n} \end{bmatrix}, \tag{138}$$

where ^{T} stands for "transposed". Using this notion, Hamiltonian equations can be written as one

$$\frac{dX}{ds} = S \cdot \frac{\partial H}{\partial X} \qquad \Leftrightarrow \frac{dx_i}{ds} = S_{ij} \cdot \frac{\partial H}{\partial x_j} \equiv \sum_{j=1}^{2n} S_{ij} \cdot \frac{\partial H}{\partial x_j} \tag{139}$$

wherein we introduce matrix S – a generator of the symplectic group (see further). The matrix, S, is asymmetric, with $S_{2m-1,2m} = 1 = -S_{2m,2m-1}$, $m = 1,...,n$, and other elements are zero.

In matrix form **S** has n diagonal blocks with a 2x2 matrix σ, and the rest is the field of zeros:

$$S = \begin{vmatrix} \sigma & 0 & .... & 0 \\ 0 & \sigma & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & \sigma \end{vmatrix}_{2nx2n} \quad ; \quad \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{140}$$

This is one of the very important objects in Hamiltonian mechanics.

We finish this section with the explicit form of the first non-trivial term in the expansion of (135):

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L\left(xP_3 - yP_1\right) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau \quad ; \tag{144}$$

with

$$\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o \mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2;$$

$$\frac{G}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2; \quad ;$$

$$\frac{2N}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_o c}B_x - \frac{e}{p_o \mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o \mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right) \tag{145}$$

$$L = \kappa + \frac{e}{2p_o c}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3};$$

$$F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

We finish this section with the explicit form of the first non-trivial term in the expansion of (135):

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L\left(xP_3 - yP_1\right) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau \quad ; \tag{143}$$

with

$$\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o \mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2;$$

$$\frac{G}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2; \quad ;$$

$$\frac{2N}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_o c}B_x - \frac{e}{p_o \mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o \mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right) \tag{144}$$

$$L = \kappa + \frac{e}{2p_o c}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3};$$

$$F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

If momentum p_o is constant, we can use (134) and rewrite Hamiltonian of the linearized motion (143) as

$$\tilde{h}_n = \frac{\pi_1^{\,2} + \pi_3^{\,2}}{2} + f\frac{x^2}{2} + n \cdot xy + g\frac{y^2}{2} + L\left(x\pi_3 - y\pi_1\right) + \frac{\pi_o^{\,2}}{2} \cdot \frac{m^2c^2}{p_o^{\,2}} + u\frac{\tau^2}{2} + g_x x\pi_o + g_y y\pi_o + f_x x\tau + f_y y\tau \quad ; \qquad (144\text{-n})$$

with

$$f = \frac{F}{p_o};\ n = \frac{N}{p_o};\ g = \frac{G}{p_o};\ u = \frac{U}{p_o};\ f_x = \frac{F_x}{p_o};\ f_y = \frac{F_y}{p_o}; \qquad (145\text{-n})$$

Note that

$$x' = \frac{\partial h_n}{\partial \pi_1} = \pi_1 - Ly; \quad y' = \frac{\partial h_n}{\partial \pi_3} = \pi_3 + Lx; \quad ; \qquad (146)$$

i.e. as soon as L=0, we can use traditional x’ and y’ as reduced momenta.

For flat reference orbit - $\kappa = 0$, in the absence of transverse coupling (L=0, N=0) and transverse electric fields, DC magnetic fields it becomes a much simpler and much more familiar form:

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + G\frac{y^2}{2} + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta$$

$$or \qquad ; \qquad (146)$$

$$\tilde{h}_n = \frac{x'^2 + y'^2}{2p_o} + f\frac{x^2}{2} + g\frac{y^2}{2} + \frac{\pi_o^2}{2}\cdot\frac{m^2c^2}{p_o^2} + u\frac{\tau^2}{2} + g_x x\delta$$

with

$$f = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x}\right];\ g = \frac{e}{p_o c}\frac{\partial B_x}{\partial y};\ \ u = \frac{e}{pc^2}\frac{\partial E_s}{\partial t};\ \ g_x = -K\frac{c}{\mathrm{v}_o}. \qquad (147)$$

Finally, see an Appendix where Mathematica tool allowing expansion of the accelerator to an arbitrary order is presented.

Lecture 8

Power of Hamiltonian method: Phase space and invariants

As we discussed before, there can be invariants of motion. For example, when the Hamiltonian of the system does not depend on either the coordinates or momenta, we automatically gain "for-free" an integral of motion of the invariant. There are more general invariants of motion that exist for all Hamiltonian systems – the Poincaré invariants. Let' us consider a Hamiltonian system that is described by the set of coordinates, Canonical momenta, and the independent variable *s*. In matrix form, the Hamiltonian equations are written as

$$H = H(Q,P,s); X = \begin{bmatrix} Q_1 \\ P_1 \\ \dots \\ Q_n \\ P_n \end{bmatrix}; \frac{dX}{ds} = S\frac{\partial H}{\partial X}; S = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{bmatrix}; \tag{148}$$

where S is a generator (norm) of a symplectic group of matrixes (two different but closely related types of mathematical groups). The space of coordinates and momenta is called phase space of the system with dimension ***2n***.

Note that we call independent variable *s*, but it can be any monotonic variable, like the time *t* or anything else. It really does not matter too much – it will only affect how Hamiltonian looks, but not that system remain Hamiltonian.

Here, consider an ensemble of particles in this phase space whose motions satisfy the Hamiltonian equations; then, their motion is completely determined by their initial position in the phase space. This means that in the Hamiltonian system the phase-space trajectories of particles, which initially were separated, will never cross! Consider one trajectory in the phase space $X_o(s)$, which satisfies the Hamiltonian equation (148) and another trajectory with an infinitesimally small deviation from X_o

$$X_1 = X_o(s) + \delta X;$$

$$\frac{dX_o}{dz} = S\frac{\partial H(X)}{\partial X}\bigg|_{X=X_o}; \; H(X+\Delta X) = H(X) + \frac{\partial H(X)}{\partial X}\Delta X + \frac{1}{2}\frac{\partial^2 H(X)}{\partial X^2}\Delta X^2 + O(\Delta X^3);$$

$$\frac{d\left(X_o(s)+\delta X\right)}{ds} = S\frac{\partial H(X)}{\partial X}\bigg|_{X=X_o(s)+\delta X} \cong S\frac{\partial H(X)}{\partial X}\bigg|_{X=X_o} + S\mathrm{H(s)}\delta X + O(\delta X^2);$$

$$\mathrm{H(s)} = \frac{1}{2}\frac{\partial^2 H(X)}{\partial X^2}\bigg|_{X=X_o(s)};$$

$$\left[\mathrm{H(s)}\right]_{ij} = \left[\frac{\partial^2 H}{\partial x_i \partial x_j}\right]\bigg|_{X=X_o(s)} \Rightarrow \left[\mathrm{H(s)}\right]_{ij} \equiv \left[\mathrm{H(s)}\right]_{ji} \Rightarrow \mathrm{H^T(s)=H(s)} \tag{149}$$

with a symmetric *2n* x *2n* matrix **H(s)**.

Thus, the equation of motion for a small deviations about the known trajectory are linear but s-dependent, and can be expressed via the linear transform matrix *M(s)*:

$$\frac{d\delta X}{dz} = S \cdot \mathrm{H(s)} \cdot \delta X$$

$$\delta X(s) = M(s) \cdot \delta X(0);$$

$$\frac{dM}{ds} = S \cdot \mathrm{H(s)} \cdot M \qquad \Rightarrow \tag{150}$$

$$\frac{d\delta X}{ds} = \frac{dM(s)}{ds} \cdot \delta X(0) = S \cdot \mathrm{H(s)} \cdot M \cdot \delta X(0) = S \cdot \mathrm{H(s)} \cdot \delta X \#$$

The matrix M(s) is symplectic, which is proven as follows:

$$M^T(z)\cdot S\cdot M(z)=S \quad -\textit{symplectic condition};\ M(0)=\hat{1}\ \Rightarrow M^T(0)\cdot S\cdot M(0)=S;$$

$$\frac{d\left(M^T SM\right)}{ds}=\frac{dM^T}{ds}SM+M^T S\frac{dM}{ds}$$

$$\frac{dM^T}{ds}=\left(S\cdot \mathrm{H}\cdot M\right)^T=M^T\mathrm{H}^{\mathrm{T}}S^T=-M^T\mathrm{H}S$$

$$\textit{because}\ \ \mathrm{H}^{\mathrm{T}}=\mathrm{H};\ \ S^T=-S;\ \ S^2=-\hat{1}$$

$$\frac{d\left(M^T SM\right)}{ds}=-M^T\mathrm{H}SSM+M^T SS\mathrm{H}M=M^T\mathrm{H}M-M^T\mathrm{H}M\equiv 0$$

$$\Rightarrow M^T(s)\cdot S\cdot M(s)=conts=M^T(0)\cdot S\cdot M(0)=S \quad \#$$

The symplectic condition has two asymmetric *2n* x *2n* matrixes on both sides

$$M^T(s)\cdot S\cdot M(s)=S \tag{151}$$

and imposes n(2n-1) conditions on the matrix M.

These conditions result in invariants of motion for the ensembles of particles, called Poincaré invariants. Accordingly, for 3-D motion, there are 15 Poincaré invariants! The most well-known one, the conservation of the phase space volume (Liouville's theorem), is a consequence of the unit determinant of the matrix M:

$$\begin{aligned}&\det\left[M^T(s)\cdot S\cdot M(s)\right]=\det S\ \rightarrow\\&\left(\det M(s)\right)^2=1\rightarrow\det M=\pm 1;\\&but\,\det M(0)=1\rightarrow\det M=1\ \#\end{aligned} \tag{152}$$

In last class you learned that determinate of symplectic matrix equal +1.

Next, we consider an infinitesimally small phase-space volume ΔV_{2n} around a known trajectory and its transformation:

$$\begin{aligned} &\Delta V_{2n}(s) = \det\left|\frac{\partial \Delta X(s)}{\partial \Delta X(0)}\right| \Delta V_{2n}(0) = \det M(s) \cdot \Delta V_{2n}(0) \rightarrow \\ &\Delta V_{2n}(s) = \Delta V_{2n}(0) = const \end{aligned} . \tag{153}$$

The 3-dimensional volume occupied by the particles often is termed 3-D beam emittance. The rest of the Poincaré invariants represent similar conservation laws for the sum of projections on hyper-surfaces in *2n*-phase space.

But there is one nice instance wherein the Hamiltonian is decoupled, i.e., it is the direct sum of individual Hamiltonians for each canonical pair.

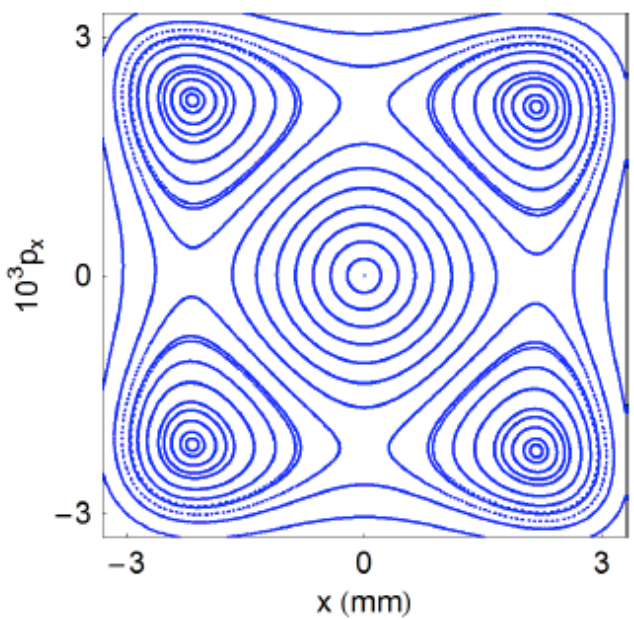
$$H = H(Q,P,s) = \sum_{k=1}^{3} H_k(Q_k, P_k, s) \tag{154}$$

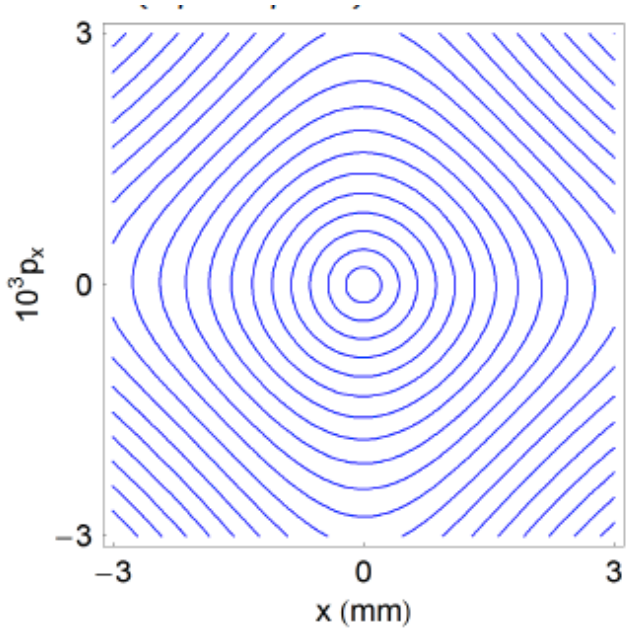
Then, the phase space is two-dimensional and the area of the space phase occupied by the beam is called beam emittance for a specific dimension – horizontal (x, Px), vertical (y, Py) or longitudinal (-t,E). All three emittances are constants (integrals) of motion.

Phase space

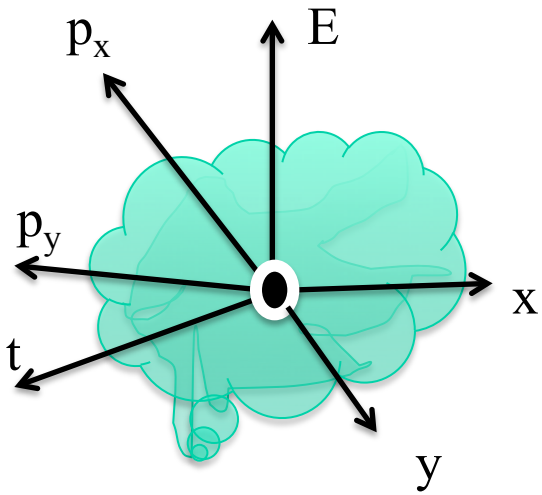
The full set of coordinates and momenta of particle (or a ensemble of particles) $\{q_i,P_i\}$ is called phase space. Naturally dimension of the phase space is always even: 2,4,6.., 2n. While motion in the coordinate space $\{q_i\}$ can be rather arbitrary, the same motion in the phase space satisfies a number of very strong constrains, e.g. there is a number of invariants.

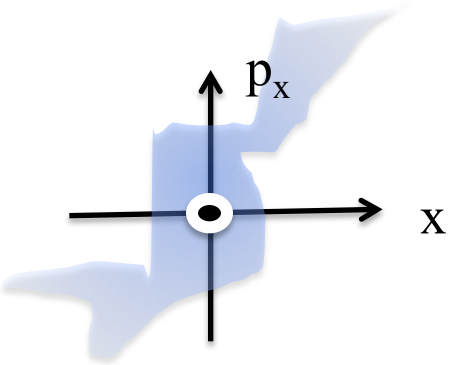
Location or motion of particles in the phase space are called phase-space plots or phase-space diagrams. Naturally we usually can plot on the paper or show on the screen only one coordinate and one momentum – hence, you usually see phase plot for 1D case, or for projections of multi-dimensional phases space plot on one plane.

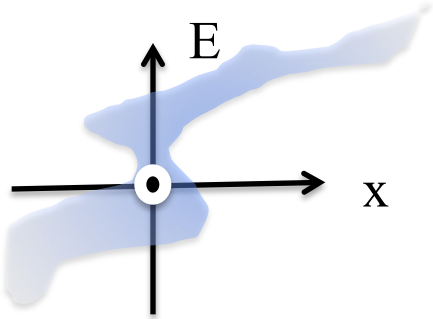


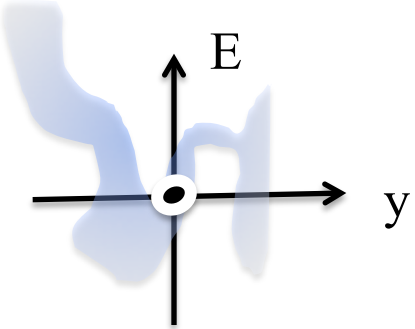


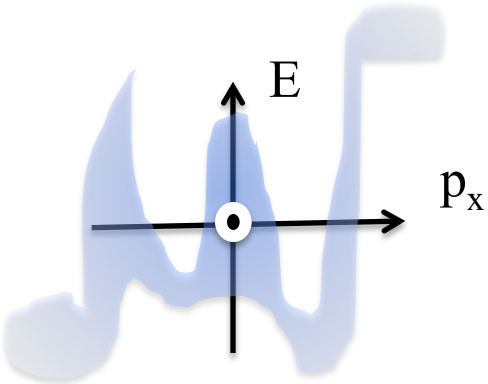
It is hard to draw 6D phase space distribution in PPT.. but it is easy to draw 2D projections

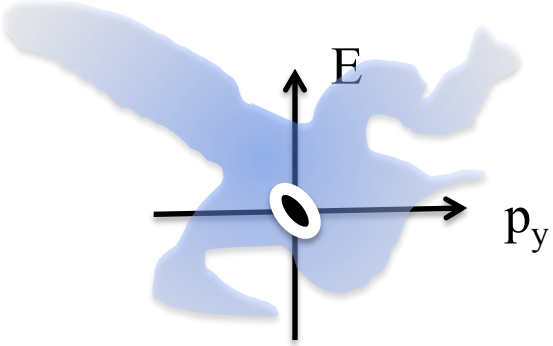








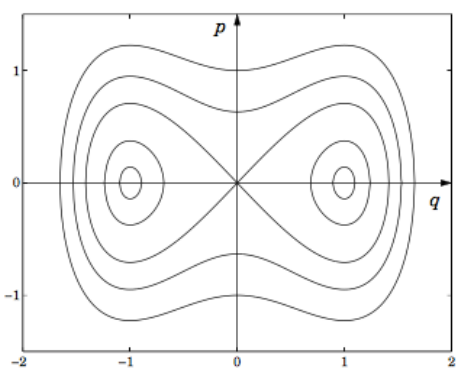


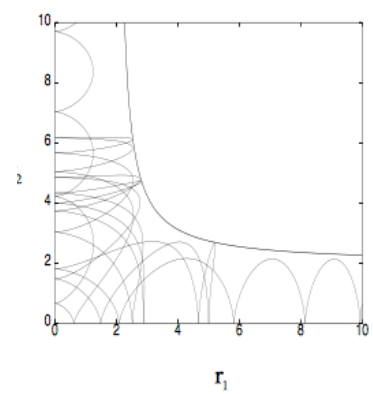


Example of {x,Px} phase-space diagram showing trace of the particles motion in accelerators: a set of particles with initial coordinate were seeded n the plot and then traced for a large number of turns. Stable motion results in periodic and semi-periodic results in “orbits – semi-closed trajectories ” in the phase space.

One of very important featured of the particle’s trajectories in the phase space that they cannot cross. It comes from a simple observation that to particles having the same values of coordinates and momenta $\{q_i,P_i\}$ at the same moment of time, will follow identical trajectories! Note, that this is very general statement – it does not rely on Hamiltonian mechanics, but only on the assumption that full set of coordinates and momenta $\{q_i,P_i\}$ fully describes the initial conditions for a particle.

It not true for motion in coordinate space – particle’s trajectory can cross since at the same point they may have different momenta. The same is true for projection of phase diagram for 2D or 3D motion on any subset of coordinate and momenta {x,Px,y,Py}->{x,y} or {x,Px} or {y,Px}…





(a) Phase plot of decoupled motion – no crossing. A special unstable point at zero correspond to a stopping point – e.g. two trajectories approach each other but never cross! ; (b) Projection of 4D phase-space trajectories on (q1,q2) coordinates – naturally they can cross.

Only if the Hamiltonian is direct sum of those for Canonical pairs (like in eq. (154)) projection on {qi,Pi} will be also fully independent and will have properties of complete phases space diagrams – no crossing, etc. Hence, a transformation from s_1 to s_2 can be described by a map:

$$X(s_2) = \mathtt{M}_{(s_1|s_2)}\left(X(s_1)\right) \equiv \mathtt{M} : X(s_1) \tag{155}$$

which can be locally linearized in proximity of any trajectory $X_o(s)$:

$$\begin{gathered} \delta X(s_2) = M_{X_o}(s_1|s_2) \cdot \delta X(s_1) + O\left(\delta^2\right) \\ M_{X_o}(s_1|s_2) = \left.\frac{\partial \mathtt{M}_{(s_1|s_2)}(X)}{\partial X}\right|_{X=X_o} \end{gathered} \tag{156}$$

As we discussed above, this matrix is symplectic. We will call map (155) symplectic if it is locally symplectic, e.g.

$$M^T_{X_o}(s_1|s_2) \cdot S \cdot M_{X_o}(s_1|s_2) = S \quad \forall X_o, s_1, s_2 \tag{157}$$

Just to reinforce – any map generated by Hamiltonian motion, is symplectic.

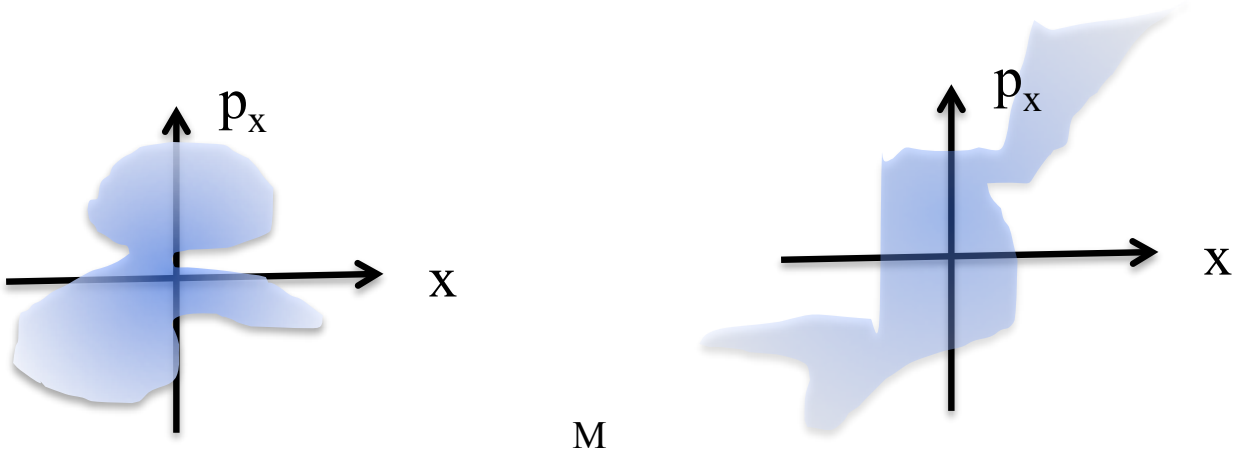
Now, instead of talking about particle motion, we can consider transformation of various volumes in the phase space or transformation of functions, such as particle's density. First, let's consider a space phase volume (dimension 2n) occupied by particles having an arbitrary hyper-surface Ω. Then the hyper-surface can undergo and transformation, but it's the value of the volume inside $\int_{\Omega} \prod_{i=1}^{n} dq_i \, dP^i = inv$ (158)

would not change – this is know as Liouville theorem. The prove is easy

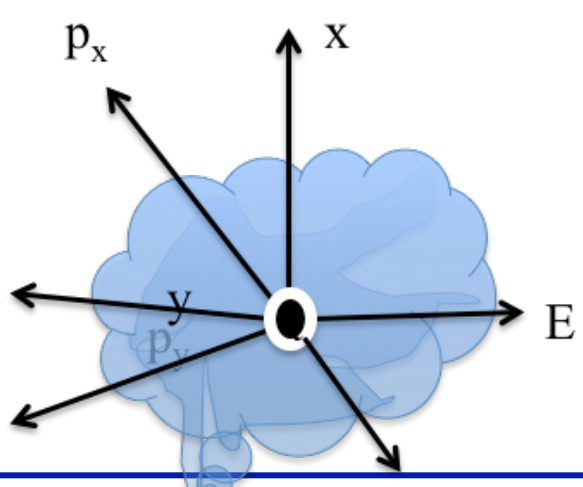
$$\begin{aligned} V(s) &= \int_{\Omega} \prod_{i=1}^{n} dq_i \, dP^i \equiv \int_{\Omega} dX(s) \equiv \int_{\Omega} dV(s) \\ V(s_2) &= \int_{\Omega} dX(s_2) \equiv \int_{\Omega} \det M(s_1 | s_2) \cdot dX(s_1) = \int_{\Omega} dX(s_1) = V(s_1) \end{aligned} \tag{159}$$

where use the fact that transformation is symplectic.

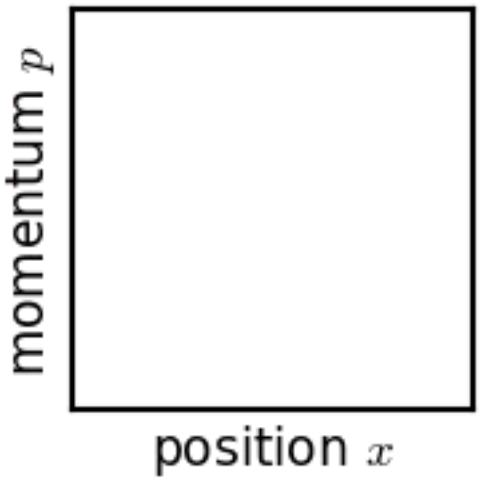
If particles do not decay or disappear in any other way (scatter on residual gas and fly away!), than number of particles inside any hyper-surface transforming according to the map (155) is preserved. Remember, that trajectories can not cross in the phase space – it also means that particle can not cross a boundary which moving according to the particle's motion. In accelerator physics it is called water-bag. You can deform it, twist and turn, but can not change its volume. The phase-space liquid is in-compressible.

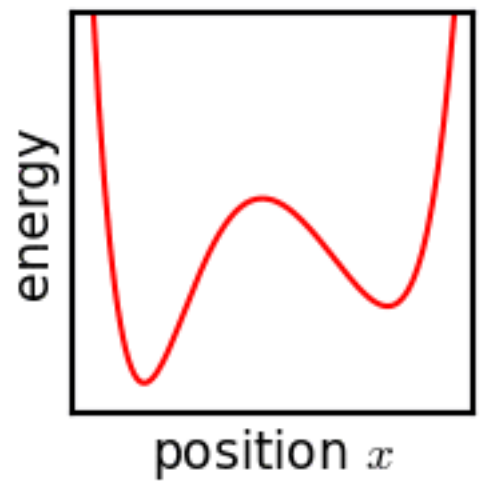


Totally non-linear map for 1D case – while boundary can change dramatically, the volume does not change.



From Wiki





While one can have a lot of fun with the phase space trajectories, invariants (a property of Hamiltonian systems!) are even more important. The most important is conservation of the phase space volume. As we discussed in one of our lectures, any motion of Hamiltonian system is an equivalent to a Canonical transformation.

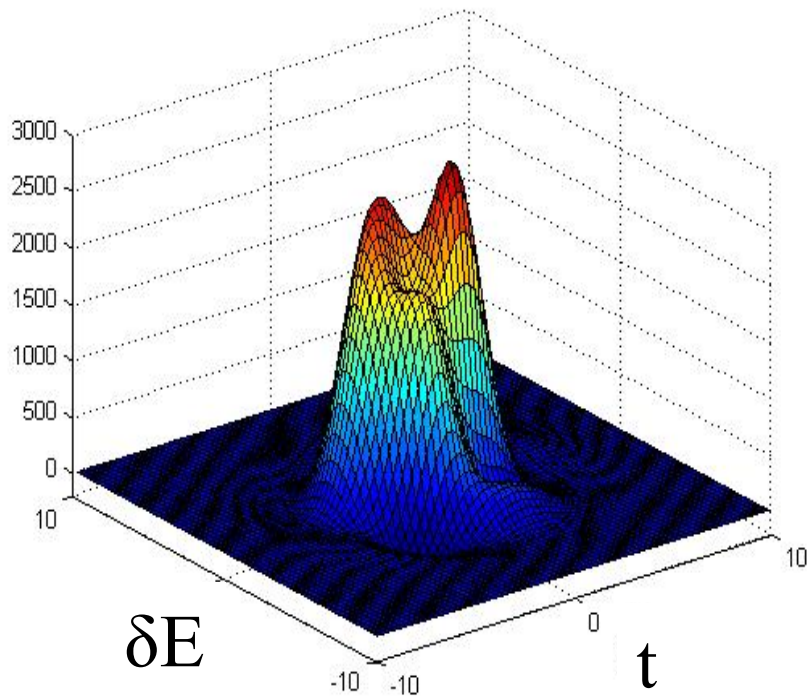
It means that phase space density of an ensemble of particles is particles is invariant:

$$f(X,s)_{def} = \frac{dN}{dX^{2n}} \Rightarrow f(\mathtt{M}:X) \equiv f(X)$$
$$f\left(\mathtt{M}_{(s_1|s_2)}\left(X(s_1)\right),s_2\right) \equiv f\left(X(s_1),s_1\right) \tag{160}$$

In other words, the phase space density is preserves along the trajectories. This is foundation for one of most used equation in accelerator and plasma physics – Vlasov equation:

$$\frac{df(X(s),s)}{ds} = \frac{\partial f(X,s)}{\partial s} + \frac{\partial f(X,s)}{\partial X}\frac{dX}{ds} = 0$$
$$\frac{dX}{ds} = S \cdot \frac{\partial H(X,s)}{\partial X} \tag{160eq}$$
$$\frac{\partial f(X,s)}{\partial s} + \frac{\partial f(X,s)}{\partial X} \cdot S \cdot \frac{\partial H(X,s)}{\partial X} = 0$$

It is also referred to as method of trajectories – now you know what it is about. We will return to this equation when we will study collective effects.



Distribution function of electron beam (in longitudinal plane) undergoing interaction in a storage ring FEL.

Since symplecticity of the map and corresponding matrices, there are n*(2n-1) total conditions. One of them is *detM=1* we already put in use. The rest of the invariants are called after French mathematician/physicist Poincaré.

The other invariants preserved by symplectic transformations were found by Poincaré and they are the sum of projections onto an appropriate manifold in two, four…. (2n-2) dimensions. In integral form it is

$$\sum_{i=1}^{n} \iint dq_i dP^i = inv; \sum_{i \neq j} \iint \iint dq_i dP^i dq_j dP^j = inv \ldots\ldots \tag{161}$$

If you count the number of Poincaré invariants (including Liouville!) you should not be surprised to find that there is n*(2n-1).

Why these invariants are important? is a very good question. The main reason is that frequently they can be useful to solve problem analytically – the same way as energy conservation completely solves problem in 1D potential. The other important reason is that they actually restrict what one can do with beams of particles, e.g. does not allows us to compress “waterbag”.

The look of these invariants is deceivingly simple. Let just discuss one of them – sum of the projections on 2D surfaces for n=2 case, e.g. a classical accelerator problem with couples transverse (x and y) motion:

$$\sum_{i=1}^{2} \iint dq_i dP^i = \iint dx dP_x + \iint dy dP_y = inv \tag{162}$$

It states that sum of projections of phase space volume onto two one dimensional "phase-plots" is invariant of motion. But in some cases one of the projection can have negative value…. We will discuss this in more details later when discussing linear coupling.

Linear equations of motion

We finished the accelerator Hamiltonian expansion by concluding that the first not-trivial term in the accelerator Hamiltonian expansion is a quadratic term of canonical momenta and coordinates. This Hamiltonian can be written in the matrix form (letting ***n*** be a dimension of the Hamiltonian system with n canonical pairs $\{q_i, P_i\}$)

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s) x_i x_j \equiv \frac{1}{2} X^T \cdot \mathbf{H}(s) \cdot X; \tag{163}$$

$$X^T = \begin{bmatrix} q^1 & P_1 & ..... & .... & q^n & P_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & ..... & .... & x_{2n-1} & x_{2n} \end{bmatrix},$$

with the self-evident feature that a symmetric matrix can be chosen

$$\mathbf{H}^{\mathbf{T}} = \mathbf{H} \tag{164}$$

(to be exact, a quadratic form with any asymmetric matrix has zero value). The equations of motion are just a set of ***2n*** linear ordinary differential equations with ***s***-dependent coefficients:

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X; \;\; \mathbf{D} = \mathbf{S} \cdot \mathbf{H}(s). \tag{165}$$

One important feature of this system is that

$$Trace[\mathbf{D}] = 0, \tag{166}$$

(the trivial proof is based on $Trace[\mathbf{AB}] = Trace[\mathbf{BA}]$; $Trace[\mathbf{A}^T] = Trace[\mathbf{A}]$ and $(\mathbf{SH})^T = -(\mathbf{HS})$). i.e., the Wronskian determinant of the system (http://en.wikipedia.org/wiki/Wronskian) is equal to one. The famous Liouville theorem comes from well-known operator formula $\frac{d\det[\mathbf{W}(s)]}{ds} = Trace[\mathbf{D}]$; we do not need it here because we will have an easier method of proof. You also have it as a homework problem.

The solution of any system of first-order linear differential equations can be expressed through its ***2n*** initial conditions $\boldsymbol{X_o}$ at azimuth $\boldsymbol{s_o}$

$$X(s_o) = X_o, \tag{167}$$

through the transport matrix $\boldsymbol{M(s_o/s)}$:

$$X(s) = \mathbf{M}\left(s_o \middle| s\right) \cdot X_o. \tag{168}$$

There are two simple proofs of this theorem. The first is an elegant one: Let us consider the matrix differential equation

$$\mathbf{M}' \equiv \frac{d\mathbf{M}}{ds} = \mathbf{D}(s) \cdot \mathbf{M}; \tag{169}$$

with a unit matrix as its initial condition at azimuth $\boldsymbol{s_o}$

$$\mathbf{M}(s_o) = \mathbf{I}. \tag{170}$$

Such solution exists and then we readily see that

$$X(s) = \mathbf{M}(s) \cdot X_o. \tag{169-1}$$

satisfies eq.(165):

$$\frac{dX}{ds} = \frac{d\mathbf{M}(s)}{ds} \cdot X_o = \mathbf{D}(s) \cdot \mathbf{M}(s) \cdot X_o \equiv \mathbf{D}(s) \cdot X^{\#}.$$

Mathematically, it is nothing else but $M(s) = \lim_{N \to \infty} \prod_{k=1}^{N} \left(\mathbf{I} + \mathbf{D}(s_k)\right) \Delta s;\ \Delta s = (s - s_o)/N;\ s_k = s_o + k \cdot \Delta s.$

A more traditional approach to the same solution is to use the facts that a) there exists a solution of equation (165) with arbitrary initial conditions (less-trivial statement); and, b) any linear combination of the solutions also is a solution of eq. (165) (very trivial one). Considering a set of solutions of eq.(165) $M_k(s)$, k=1,…2n, with initial conditions at azimuth $\boldsymbol{s_o}$,then

$$M_1(s_o)=\begin{vmatrix}1\\0\\...\\0\\0\end{vmatrix};M_2(s_o)=\begin{vmatrix}0\\1\\...\\0\\0\end{vmatrix};.....M_{2n}(s_o)=\begin{vmatrix}0\\0\\...\\0\\1\end{vmatrix};,\quad \frac{dM_k(s)}{ds}=\mathbf{D}(s)\cdot M_k(s); \tag{171}$$

and their linear combination

$$X(s)=\sum_{k=1}^{2n}x_{ko}\cdot M_k(s), \tag{172}$$

which satisfies the initial condition (167)

$$X(s_o)=\sum_{k=1}^{2n}x_{ko}\cdot M_k(s_o)=\begin{bmatrix}x_{1,0}\\x_{2,0}\\...\\x_{2n-1,0}\\x_{2n,0}\end{bmatrix}=X_o\cdot \tag{173}$$

Now, we recognize that our solution (172) is nothing other than the transport matrix eq. (169-1) with matrix **M(s)** being a simple combination of 2n columns $M_k(s)$:

$$\mathbf{M}(s)=\left[M_1(s),M_2(s),...,M_{2n}(s)\right].$$

Eq. (171) then makes it equivalent to eqs. (169) and (170). Finally, we use notion $\mathbf{M}(s_o|s)$ to clearly demonstrate that $\mathbf{M}(s_o)=\mathbf{I}$ at azimuth $\boldsymbol{s_o}$

In differential calculus, the solution is defined as

$$\mathbf{M}(s_o|s)=\exp\left[\int_{s_o}^{s}\mathbf{D}(s)ds\right]=\lim_{N\to\infty}\prod_{k=1}^{N}(\mathbf{I}+\mathbf{D}(s_k))\Delta s; \tag{174}$$

$$\Delta s=(s-s_o)/N;\ s_k\in\{s_o+(k-1)\cdot\Delta s, s_o+k\cdot\Delta s\}$$

The fact that the transport matrix for a linear Hamiltonian system has unit determinant (i.e., the absence of dissipation!)

$$\det\mathbf{M}=\exp\left[\int_{s_o}^{s}Trace(\mathbf{D}(s))ds\right]=1. \tag{175}$$

is the first indicator of the advantages that follow.

Let us consider the invariants of motion characteristic of linear Hamiltonian systems, i.e., invariants of the symplectic phase space. Starting from the bilinear form of two independent solutions of eq. (165), $X_1(s)$ and $X_2(s)$, *(it is obvious that* $X^T\mathbf{S}X=0$*)* we show that

$$X_2^T(s)\cdot\mathbf{S}\cdot X_1(s)=X_2^T(s_o)\cdot\mathbf{S}\cdot X_1(s_o)=inv\,. \tag{176}$$

The proof is straightforward

$$\frac{d}{ds}\left(X_2^T\cdot\mathbf{S}\cdot X_1\right)=X_2^{T\prime}\cdot\mathbf{S}\cdot X_1+X_2^T\cdot\mathbf{S}\cdot X_1{}'=X_2^T\cdot\left((\mathbf{SD})^{\mathbf{T}}\mathbf{S}+\mathbf{SSD}\right)\cdot X_1{}'\equiv 0\,.$$

Proving that transport matrices for Hamiltonian system are symplectic is very similar:

$$\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}=\mathbf{S}. \tag{177}$$

Beginning from the simple fact that the unit matrix is symplectic: $\mathbf{I}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{I}=\mathbf{S}$, i.e. $\mathbf{M}\left(s_o|s_o\right)$ is symplectic, and following with the proof that $\mathbf{M}^{\mathbf{T}}\left(s_o|s\right)\cdot\mathbf{S}\cdot\mathbf{M}\left(s_o|s\right)=\mathbf{M}^{\mathbf{T}}\left(s_o|s_o\right)\cdot\mathbf{S}\cdot\mathbf{M}\left(s_o|s_o\right)=\mathbf{S}$:

$$\frac{d}{ds}\left(\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}\right)=\mathbf{M}^{\mathbf{T}\prime}\cdot\mathbf{S}\cdot\mathbf{M}+\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}'=\mathbf{M}^{\mathbf{T}}\cdot\left((\mathbf{SD})^{\mathbf{T}}\mathbf{S}+\mathbf{SSD}\right)\cdot\mathbf{M}\equiv\mathbf{0}\ \#$$

Symplectic square matrices of dimensions *2n x 2n*, which include unit matrix **I**, create a symplectic group, where the product of symplectic matrices also is a symplectic matrix.. The symplectic condition (177) is very powerful and should not be underappreciated. Before going further, we should ask ourselves several questions: How can the inverse matrix of **M** be found? Are there invariants of motion to hold-on to? Can something specific be said about a real accelerator wherein there are small but all-important perturbations beyond the linear equation of motions?

As you probably surmised, the Hamiltonian method yield many answers, and is why it is so vital to research.

We can count them: The general transport matrix **M** (solution of $\mathbf{M}' = \mathbf{D}(s) \cdot \mathbf{M}$ with arbitrary **D**) has $(2n)^2$ independent elements. Because the symplectic condition $\mathbf{M}^{\mathrm{T}} \cdot \mathbf{S} \cdot \mathbf{M} - \mathbf{S} = \mathbf{0}$ represents an asymmetric matrix with n-diagonal elements equivalently being zeros, and the conditions above and below the diagonal are identical – then only the *n(2n-1)* condition remains and only the *n(2n+1)* elements are independent. For *n=1 (1D)* there is only one condition, for *n=2* there are 6 conditions, and n=3 (3D) there are 15 conditions. Are these facts of any use in furthering this exploration?

Group G is defined as a set of elements, with a definition of a product of any two elements of the group; $P = A \bullet B \in G$; $A, B \in G$. The product must satisfy the associative law : $A \bullet (B \bullet C) = (A \bullet B) \bullet C$; there is an unit element in the group $I \in G; I \bullet A = A \bullet I = A : \forall A \in G$; and inverse elements: $\forall A \in G; \exists B(called\, A^{-1}) \in G : A^{-1}A = AA^{-1} = I.$

As you probably surmised, the Hamiltonian method yield many answers, and is why it is so vital to research.

We can count them: The general transport matrix **M** (solution of $\mathbf{M}' = \mathbf{D}(s)\cdot\mathbf{M}$ with arbitrary **D**) has *(2n)*^{2} independent elements. Because the symplectic condition $\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}-\mathbf{S}=\mathbf{0}$ represents an asymmetric matrix with n-diagonal elements equivalently being zeros, and the conditions above and below the diagonal are identical – then only the *n(2n-1)* condition remains and only the *n(2n+1)* elements are independent. For *n=1 (1D)* there is only one condition, for *n=2* there are 6 conditions, and n=3 (3D) there are 15 conditions. Are these facts of any use in furthering this exploration?

First, symplecticity makes the matrix determinant to be unit:

$$\det\left[\mathbf{M}^{\mathbf{T}}(s)\cdot\mathbf{S}\cdot\mathbf{M}(s)\right]=\det\mathbf{S}\ \rightarrow\left(\det\mathbf{M}(s)\right)^2=1\rightarrow\det\mathbf{M}=\pm1;\quad \det\mathbf{M}(0)=1\rightarrow\det\mathbf{M}=1\ \#$$

i.e., it preserves the 2n-D phase space volume occupied by the ensemble of particles (system):

$$\int\prod_{i=1}^{n}dq_i dP^i = inv \tag{178}$$

The other invariants preserved by symplectic transformations are called Poincaré invariants and are the sum of projections onto the appropriate over- manifold in two, four…. (2n-2) dimensions:

$$\sum_{i=1}^{n}\iint dq_i dP^i = inv;\ \sum_{i\neq j}\iint\ \iint dq_i dP^i dq_j dP^j = inv\ldots\ldots \tag{179}$$

1 Look at a simple n=1 case with 2x2 matrices to verify that the symplectic product is reduced to determine

$$\mathbf{M}_{2x2}=\begin{vmatrix} a & b \\ c & d \end{vmatrix};\ \mathbf{S}_{2x2}=\sigma;\Rightarrow\mathbf{M}^{\mathbf{T}}\cdot\sigma\cdot\mathbf{M}=\det\mathbf{M}\cdot\sigma \tag{Note-4}$$

For example, matrix **M** can be represented as n^2 combinations of 2x2 matrices M_{ij}:

$$\mathbf{M} = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nn} \end{bmatrix}. \tag{180}$$

Using equation (Note-4), we easily demonstrate the requirement for the symplectic condition (177) is that the sum of determinants in each row of these 2x2 matrices is equal to one; the same is true for the columns:

$$\sum_{i=1}^{n} \det\left[M_{ij}\right] = \sum_{j=1}^{n} \det\left[M_{ij}\right] = 1 \tag{181}$$

with a specific prediction for decoupled matrices, which are block diagonal:

$$\mathbf{M} = \begin{bmatrix} M_{11} & 0\dots & 0 \\ 0 & \dots & 0 \\ 0 & 0\dots & M_{nn} \end{bmatrix}; \quad \det\left[M_{ii}\right] = 1. \tag{182}$$

Other trivial and useful features are: for the columns

$$\mathbf{M} = \begin{bmatrix} C_1 & C_2 & .... & C_{2n-1} & C_{2n} \end{bmatrix} \Rightarrow$$

$$C_{2k+1}^T S C_{2k} = -C_{2k}^T S C_{2k+1} = 1, \tag{183}$$

$$others\ are\ 0$$

or lines of the symplectic matrix:

$$\mathbf{M} = \begin{bmatrix} L_1 \\ L_2 \\ .... \\ L_{2n-1} \\ L_{2n} \end{bmatrix} \Rightarrow -L_{2k+1}^T S L_{2k} = L_{2k}^T S L_{2k+1} = 1, others\ are\ 0 \tag{184}$$

We could go further, but we will stop here by showing the most incredible feature of symplectic matrices, viz., that it is easy to find their inverse (recall there is no general rule for inverting a 2n x 2n matrix!) Thus, multiplying eq. (177) from left by –**S** we get

$$-\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}=\mathbf{I}\ \Rightarrow\ \mathbf{M}^{-1}=-\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}. \tag{185}$$

As an easy exercise for 2x2 symplectic (i.e. with unit determinant – see note below) matrices, you can show that $\mathbf{M}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (183) gives $\mathbf{M}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is a much less trivial task to invert 6x6 matrix; hence, the power of symplecticity allows us enact many theoretical manipulations that otherwise would be impossible. Obviously, and easy to prove, transposed symplectic and inverse symplectic matrices also are also symplectic:

$$\mathbf{M}^{-1\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}^{-1}=\mathbf{S}; \qquad \mathbf{M}\cdot\mathbf{S}\cdot\mathbf{M}^{T}=\mathbf{S}. \tag{186}$$

Now you know as much as any physicist need to know about symplicticity