Home Work PHY 554 #7.

HW 1 (5 points): RF cavity beam loading/unloading.

A short ultra-relativistic (1-v/c <<1) bunch with charge of 5 nC is passing through a 0.3 meter long 500 MHz pillbox accelerating cavity operating at the fundamental TM_{010} with peak accelerating field of 5 MV/m.

(1) Find the change of the cavity voltage ΔV/V (accelerating field) after the beam passes through it as function of the phase of the beam passing the cavity. What are the maximum and minimum ΔV/V?

(2) How the beam loading ΔV/V depends on the accelerating field? At what level of accelerating it reaches ΔV/V 1%?

(a) Assume that beam does not change velocity in the cavity;
(b) Hint – use energy conservation law
(c) Assume that relative change of the voltage ΔV/V is small, e.g. the beam loading can be treated as a perturbation.

Solution:

First, we need to find the energy gain by each electron in the cavity operating at E=5 MV/m using RF phase in the center as the reference:

$$dz \cong ct;$$

$$\Delta E = ec\mathbf{E}\int_{-L/2c}^{L/2c}\cos(\omega t+\varphi)dt = eL\mathbf{E}\frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}}\cos\varphi = eV_{RF}\cos\varphi$$

$$FF = \frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}} = 0.636179.. \qquad V_{RF} = L\mathbf{E}\cdot FF = 0.9543\ MV$$

It means that that energy take/given by the beam is

$$\Delta U = qV_{RF}\cos\varphi = \Delta U_o\cos\varphi$$
$$q = 5nC = 5\cdot 10^{-9}C; V_{RF} = 0.9543\cdot 10^{6}V$$
$$\Delta U_o = \text{sign}[q]\cdot 4.77\cdot 10^{-3}J$$

Naturally, when energy is taken by electron beam, $q\cos\varphi>0$, RF voltage in the cavity drops (it is called beam loading) and with beam loses energy, $q\cos\varphi<0$, RF voltage increases. To know the voltage change we need to know what EM energy is stored in the RF cavity. We should use the your favorite units system (SI)

$$W = \int \left(\varepsilon_o \frac{\vec{\mathbf{E}}^2}{2} + \mu_o \frac{\vec{\mathbf{H}}^2}{2} \right) dV = \frac{\varepsilon_o}{2} \int \vec{\mathbf{E}}_o^{\,2}\, dV$$

or GSG

$$W = \frac{1}{8\pi} \int \left(\vec{\mathbf{E}}^2 + \vec{\mathbf{H}}^2 \right) dV = \frac{1}{8\pi} \int \vec{\mathbf{E}}_o^{\,2}\, dV$$

and the field pattern we derived for TM_{010} mode

$$\mathbf{E}_o = \hat{z} \cdot E_o \cdot J_o \left(2.405 \frac{r}{a} \right);\ E_o = 5 \cdot 10^6 \frac{V}{m} \equiv 166.7\ Gs$$

where radius of the cavity, *a*, is defined by its frequency:

$$J_o(ka) \equiv J_o\left(\frac{\omega}{c} a \right) = 0 : TM_{010} \rightarrow \frac{\omega}{c} a = 2.405$$

$$a = \frac{2.405c}{\omega} = \frac{2.405}{2\pi} \frac{c}{f} = 0.2295\, m$$

and then integrate Bessel function over the radius of the cavity

$$\int J_o^{\,2} r\, dr d\theta dz = 2\pi L \int_o^a J_o^{\,2}(kr) r dr = \frac{2\pi L}{k^2} \int_o^{x_o} J_o^{\,2}(x) x dx;$$

$$x_o = 2.404825557695773..:\quad \int_o^{x_o} J_o^{\,2}(x) x dx = 0.779325$$

$$\frac{2\pi L}{k^2} \int_o^{x_o} J_o^{\,2}(x) x dx = 1.337 \cdot 10^{-2} m^3 = 1.337 \cdot 10^4 cm^3$$

Then using your favorite units, we got identical

$$W = 1.48\, J = 1.48 \cdot 10^7\, erg$$

Side note: *A smart RF engineer would use known value of R_{sh}/Q*

$$\frac{R_{sh}}{Q_0} = \frac{V_{RF}^2}{\omega_0 W} \rightarrow W = \frac{V_{RF}^2}{\omega_0} / \frac{R_{sh}}{Q_0}$$

for pillbox cavity of 196 Ohm (Slide 10, Lecture 11) to get the same:

$$\frac{R_{sh}}{Q_0} = 196; V_{RF} = 9.54E5\ V; \omega_0 = 3.141593E9\ Hz \Rightarrow W = 1.48 J$$

Finally we should notice that

$$V_{RF} \sim \sqrt{W} \rightarrow \frac{\Delta V_{RF}}{V_{RF}} = \frac{1}{2} \frac{\Delta W}{W}; \Delta W = -\Delta U$$

and maximum voltage drop in our case is

$$\Delta V_{RF} = -V_{RF} \frac{1}{2} \frac{\Delta U}{W} = -1.54\ kV;\ \frac{\Delta V_{RF}}{V_{RF}} = -1.6 \cdot 10^{-3} = 0.16\%$$

Beam loading dependence on the accelerating field (RF voltage) is very simple to find from following

$$\Delta U = qV_{RF} \sim E_o; W \sim V_{RF}{}^2 \sim E_o{}^2 \Rightarrow \frac{\Delta V_{RF}}{V_{RF}} \sim \frac{1}{V_{RF}} \sim \frac{1}{E_o}$$

e.g. beam loading is inverse proportional to the accelerating field. Thus, to increase beam loading from 0.16% to 1% we should make the accelerating voltage to be 0.16 Eo = 0.8 MV/m.

HW 2 (3 points): Cavities filled with ferrite material are used for RF system requiring large frequency tuning range. The frequency is controlled by applying external magnetic field, B_{ext}, to the ferrite material and by doing so to change it magenta permeability $\mu(B_{ext})$. A 300 m in circumference AGS synchrotron accelerates polarized protons from total energy of 2.5 GeV to 25 GeV.

(a) Calculate the range of the beam revolution frequency in AGS;
(b) Assuming 100% filling by ferrite, what should be ratio of μ_{max} to μ_{min}. Where μ should have maximum value?

Note: RF systems operate on a fixed integer harmonic of the revolution frequency.

Solution:

(a) Rest energy of a proton is 0.9837 GeV. It means that Lorentz factor changes from 2.66 to 26.6 and v/c changes from 0.9269 to 0.9993, e.g. revolution frequency increases 1.0781 fold during acceleration from 927.5 kHz to 999.987 kHz (e.g.1 MHz!).

(b) Since the frequency of an RF cavity scales the same as speed of light in the media:

$$\omega_{res} = \frac{\omega_o}{\sqrt{\varepsilon\mu}} \rightarrow \mu \propto \omega_{res}^{-2}$$

one should reduce μ 1.162-fold to accommodate necessary change in resonant frequency.

HW 3 (2 points): In RF cavity operating at 500 MHz, amplitude of the magnetic field at the part surface is 500 Gs or 500 Oe. Find power losses per square meter of the surface for:

(a) Cu cavity*
(b) SRF cavity with surface resistance, $R_s = 5\ 10^{-9}$ Ohm.

How much water you can heat from 20 C° to 40 C° in one hour (3,600 second) by cooling such Cu cavity?

*Hint: you may use the conductivity of Cu or scale R_s from results shown in Lecture 10. Thermal capacitance of water is 4,179 J/kg/ C°.

Solution:

We should use formula for surface losses for a good conductor (it is in SI units):

$$\frac{P_{loss}}{A} = \frac{1}{2} R_s \left|\vec{\mathbf{H}}_{//}\right|^2$$

The most confusing is to transfer H from CGS (Gs = Oe) units to SI (A/m) with coefficient 1000/4π : H=3.98 10^4 A/m: With A=1 m^2 power lost is simply

$$P_{loss} = \frac{1}{2} R_s \left|\vec{\mathbf{H}}_{//}\right|^2 A = \frac{1}{2} R_s \left|\vec{\mathbf{H}}_{//}\right|^2$$

and for SRF cavity we would have 3.96 W losses per one square meter of the surface. For Cu surface impedance scales with the frequency

$$R_s = \sqrt{\frac{\omega\mu}{\sigma}} [\Omega]$$

In slide 8, lecture 11 we shown that for Cu R_s=10 mOhm at frequency of 1.5 GHz, which is 3 time higher than in our case. Thus

$$R_s(Cu, 500 Mhz) = \frac{10 m\Omega}{\sqrt{3}} \approx 5.8 m\Omega$$

and power loss density is 4.57 MW per m^2. In one hour the EM field generates 1.65 10^{10} J in 1 m^2 of the Cu surface. Heating one kg (e.g. one liter) of water by 20K requires 83.6 kJ: hence this power will heat from 20 C° to 40 C° 197 tons of water! e.g. a cube approximately 6m x 6 m x 6m.