Homework 19. Due November 18
Problem 1. 20 points. 1D emittance
For an ensemble or a distribution function of particles 1D geometrical emittance is defined as

$$\varepsilon_y^{\ 2} = \langle y^2\rangle\langle y'^2\rangle - \langle yy'\rangle^2;$$

$$\langle g(y,y')\rangle = \frac{\sum_{n=1}^{N_p} g(y_n, y'_n)}{N_p} = \int f(y,y')g(y,y')\,dy\,dy';$$

1. Show that the emittance is invariant to a Canonical linear (symplectic matrix) transformation of

$$\begin{bmatrix} y \\ y' \end{bmatrix} = M \begin{bmatrix} y \\ y' \end{bmatrix}$$

Note: use the fact that $\varepsilon_y^{\ 2} = \det\Sigma; \Sigma = \begin{bmatrix} \langle y^2\rangle & \langle yy'\rangle \\ \langle yy'\rangle & \langle y'^2\rangle \end{bmatrix}$; and find transformation rule for the Σ matrix.

2. For one-dimensional betatron (y) distribution find components of eigen vector $\mathbf{w}_\mathbf{y}$ and $\mathbf{w'}_\mathbf{y}$ generating a given (positively defined)

$$\Sigma = \begin{bmatrix} \langle y^2\rangle & \langle yy'\rangle \\ \langle yy'\rangle & \langle y'^2\rangle \end{bmatrix};$$

This operation is called matching the beam into the beam-line optics.

Solution.
Problem 1. (1) Let's prove that

$$\begin{bmatrix} \tilde{y} \\ \tilde{y}' \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} = M \begin{bmatrix} y \\ y' \end{bmatrix} = M \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \rightarrow \tilde{\Sigma} = M\Sigma M^T$$

by observing that

$$\Sigma_{ij} = \langle X_i X_j \rangle;$$

$$\tilde{\Sigma}_{ij} = \tilde{\Sigma}_{ji} = \langle \tilde{X}_i \tilde{X}_j \rangle = \langle M_{ik} X_k X_n M_{nj} \rangle =$$
$$= M_{ik} \langle X_k X_n \rangle M_{nj} = M_{ik}\Sigma_{ij} M_{nj}$$

(where we use the fact that one can extract constants from the averaging brakets) which in matrix form is equivalent to

$$\tilde{\Sigma} = M\Sigma M^T$$

The rest is easy since $\det M = 1$:

$$\det\tilde{\Sigma} = \det M \det\Sigma \det M^T = \det\Sigma$$

Shorter proof: $\Sigma = X \otimes X^T \rightarrow \tilde{\Sigma} = \tilde{X} \otimes \tilde{X}^T = M \cdot X \otimes X^T \cdot M^T = M \cdot \Sigma \cdot M^T$ #

(2) Let's remember that

$$y = a\mathrm{w}_y \cos\psi_y;\; y' = a\left(\mathrm{w}'_y \cos\psi_y - \frac{1}{\mathrm{w}_y}\sin\psi_y \right)$$

and calculate averages using randomness of particles' phases

$$\left\langle \cos^2\psi_y \right\rangle = \frac{1}{2}; \left\langle \cos\psi_y \sin\psi_y \right\rangle = 0; \left\langle \sin^2\psi_y \right\rangle = \frac{1}{2}; \frac{\left\langle a^2 \right\rangle}{2} = \varepsilon_y$$

$$\left\langle y^2 \right\rangle = \left\langle a^2 \mathrm{w}_y{}^2 \cos^2\psi_y \right\rangle = \mathrm{w}_y{}^2 \frac{\left\langle a^2 \right\rangle}{2} = \beta_y \frac{\left\langle a^2 \right\rangle}{2};$$

$$\left\langle yy' \right\rangle = \left\langle a^2 \mathrm{w}_y \cos\psi_y \left(\mathrm{w}'_y \cos\psi_y - \frac{1}{\mathrm{w}_y}\sin\psi_y \right) \right\rangle = \mathrm{w}_y \mathrm{w}'_y \frac{\left\langle a^2 \right\rangle}{2} = -\alpha_y \frac{\left\langle a^2 \right\rangle}{2};$$

$$\left\langle y'^2 \right\rangle = \left\langle a^2 \left(\mathrm{w}'_y \cos\psi_y - \frac{1}{\mathrm{w}_y}\sin\psi_y \right)^2 \right\rangle = \frac{1 + \left(\mathrm{w}'_y \mathrm{w}_y\right)^2}{\mathrm{w}_y{}^2} \frac{\left\langle a^2 \right\rangle}{2} = \frac{1+\alpha_y{}^2}{\beta_y}.$$

$$\Sigma = \begin{bmatrix} \left\langle y^2 \right\rangle & \left\langle yy' \right\rangle \\ \left\langle yy' \right\rangle & \left\langle y'^2 \right\rangle \end{bmatrix} = \varepsilon_y \begin{bmatrix} \beta_y & -\alpha_y \\ -\alpha_y & \dfrac{1+\alpha_y{}^2}{\beta_y} \end{bmatrix}.$$

Thus, for 1D case it one can use this relation to design matched lattice for a given Σ matrix of the beam – for example at injection point into a storage ring. This matching minimizes RMS amplitudes of particles oscillation in the storage ring.