Homework 7.

Problem 1. points. FODO cell.

Consider a general FODO cell comprised of two quadrupoles F and D separated by two drift sections, e.g. the structure below:

$$F:\ K_F = \frac{e}{pc}\frac{\partial B_y}{\partial x}, l_F;$$
$$O1:\ l_1$$
$$D:\ K_D = \frac{e}{pc}\frac{\partial B_y}{\partial x}, l_D;$$
$$O2:\ l_2$$

(a) write matrix (both x and y or 4x4) of general FODO cell (not assuming any limitations on K F,D).

(b) write stability criteria (for x and y) for periodic lattice built of this FOD cell. Hint – do not try to solve it!

(c,d) make transition to short lens approximation and assume equal strength of

$$l_F K_F = -K_D l_D = \frac{1}{f} = const, l_{F,D} \to 0$$
$$l = l_1 = l_2$$

and

(c) show that both x and y motion can be stable (e.g. prove so called strong focusing: combination of focusing and defocusing length can provide focusing in both directions);

(d) define (e.g solve) the stability criteria for such cell.

Solution

(a) We know already matricies of all these elements and need just multiply them in correct order

$$M_x = O_2 D_x O_1 F_x = \begin{bmatrix} 1 & l_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\varphi_D & \dfrac{\sinh\varphi_D}{\omega_D} \\ \omega_D \sinh\varphi_D & \cosh\varphi_D \end{bmatrix} \begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi_F & \dfrac{\sin\varphi_F}{\omega_F} \\ -\omega_F \sin\varphi_F & \cos\varphi_F \end{bmatrix}$$

$$M_y = O_2 D_y O_1 F_y = \begin{bmatrix} 1 & l_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi_D & \dfrac{\sin\varphi_D}{\omega_D} \\ -\omega_D \sin\varphi_D & \cos\varphi_D \end{bmatrix} \begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\varphi_F & \dfrac{\sinh\varphi_F}{\omega_F} \\ \omega_F \sinh\varphi_F & \cosh\varphi_F \end{bmatrix}$$

$$\omega_F = \sqrt{|K_F|}; \varphi_F = \omega_F l_F; \omega_D = \sqrt{|K_D|}; \varphi_D = \omega_D l_D$$

$$M_x = O_2 D_x O_1 F_x = \begin{bmatrix} \mathrm{ch}_D + l_2\omega_D \,\mathrm{sh}\varphi_D & \dfrac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D \\ \omega_D \,\mathrm{sh}_D & \mathrm{ch}_D \end{bmatrix} \begin{bmatrix} \mathrm{cs}_F - l_1\omega_F \,\mathrm{sn}_F & \dfrac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F \\ -\omega_F \,\mathrm{sn}_F & \mathrm{cs}_F \end{bmatrix}$$

$$= \begin{bmatrix} (\mathrm{ch}_D + l_2\omega_D\,\mathrm{sh}_D)(\mathrm{cs}_F - l_1\omega_F\,\mathrm{sn}_F) - \omega_F\,\mathrm{sn}_F\left(\dfrac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D\right) & (\mathrm{ch}_D + l_2\omega_D\,\mathrm{sh}\varphi_D)\left(\dfrac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F\right) + \mathrm{cs}_F\left(\dfrac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D\right) \\ \omega_D\,\mathrm{sh}_D(\mathrm{cs}_F - l_1\omega_F\,\mathrm{sn}_F) - \omega_F\,\mathrm{sn}_F\,\mathrm{ch}_D & \omega_D\,\mathrm{sh}_D\left(\dfrac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F\right) + \mathrm{cs}_F\,\mathrm{ch}_D \end{bmatrix}$$

and similarly ugly expression for vertical matrix,

$$M_y = O_2 D_y O_1 F_y =$$

$$= \begin{bmatrix} (\mathrm{cs}_D - l_2\omega_D\,\mathrm{sn}_D)(\mathrm{ch}_F + l_1\omega_F\,\mathrm{sh}_F) + \omega_F\,\mathrm{sh}_F\left(\dfrac{\mathrm{sn}_D}{\omega_D} + l_2\,\mathrm{cs}_D\right) & (\mathrm{cs}_D + l_2\omega_D\,\mathrm{sn}\varphi_D)\left(\dfrac{\mathrm{sh}_F}{\omega_F} + l_1\,\mathrm{ch}_F\right) + \mathrm{ch}_F\left(\dfrac{\mathrm{sn}_D}{\omega_D} + l_2\,\mathrm{cs}_D\right) \\ -\omega_D\,\mathrm{sn}_D(\mathrm{ch}_F + l_1\omega_F\,\mathrm{sh}_F) + \omega_F\,\mathrm{sh}_F\,\mathrm{cs}_D & -\omega_D\,\mathrm{sn}_D\left(\dfrac{\mathrm{sh}_F}{\omega_F} + l_1\,\mathrm{ch}_F\right) + \mathrm{ch}_F\,\mathrm{cs}_D \end{bmatrix}$$

(b) stability criteria are:

$$\left|Trace[M_{x,y}]\right| < 2$$

$$-2 < (\mathrm{ch}_D + l_2\omega_D\,\mathrm{sh}_D)(\mathrm{cs}_F - l_1\omega_F\,\mathrm{sn}_F) - \omega_F\,\mathrm{sn}_F\left(\frac{\mathrm{sh}_D}{\omega_D} + l_2\,\mathrm{ch}_D\right) + \omega_D\,\mathrm{sh}_D\left(\frac{\mathrm{sn}_F}{\omega_F} + l_1\,\mathrm{cs}_F\right) + \mathrm{cs}_F\,\mathrm{ch}_D < 2$$

$$-2 < (\mathrm{ch}_F + l_1\omega_F\,\mathrm{sh}_F)(\mathrm{cs}_D - l_2\omega_D\,\mathrm{sn}_D) - \omega_D\,\mathrm{sn}_D\left(\frac{\mathrm{sh}_F}{\omega_F} + l_1\,\mathrm{ch}_F\right) + \omega_F\,\mathrm{sh}_F\left(\frac{\mathrm{sn}_F}{\omega_F} + l_2\,\mathrm{cs}_D\right) + \mathrm{cs}_D\,\mathrm{ch}_F < 2$$

(c,d)

$$\begin{bmatrix} \cosh\varphi & \dfrac{\sinh\varphi}{\omega} \\ \omega\sinh\varphi & \cosh\varphi \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{bmatrix}; \begin{bmatrix} \cos\varphi & \dfrac{\mathrm{sih}\varphi}{\omega} \\ -\omega\sin\varphi & \cos\varphi \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix}$$

$$M_x = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{l}{f} & l \\ \dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 - \dfrac{l}{f} & l \\ -\dfrac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \left(\dfrac{l}{f}\right)^2 - \dfrac{l}{f} & 2l + \dfrac{l^2}{f} \\ -\dfrac{l}{f^2} & 1 + \dfrac{l}{f} \end{bmatrix}$$

$$M_y = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 - \left(\dfrac{l}{f}\right)^2 + \dfrac{l}{f} & 2l - \dfrac{l^2}{f} \\ -\dfrac{l}{f^2} & 1 - \dfrac{l}{f} \end{bmatrix}$$

Stability criteria is

$$\left|Trace[M_{x,y}]\right| < 2 \;\; \to \;\; -2 < 2 - \left(\frac{l}{f}\right)^2 < 2$$

$$\left(\frac{l}{f}\right)^2 < 4; \; \left|\frac{l}{f}\right| < 2$$

can be satisfied for both directions.

Problem 2. Matrix Gymnastics.

For a one-dimensional motion consider parameterization via the eigen vectors and their propagation along *s*:

$$Y(\mathrm{s}) = \begin{bmatrix} \mathrm{w(s)} \\ \mathrm{w}'(\mathrm{s}) + \dfrac{i}{\mathrm{w(s)}} \end{bmatrix}; Y^*(\mathrm{s}) = \begin{bmatrix} \mathrm{w(s)} \\ \mathrm{w}'(\mathrm{s}) - \dfrac{i}{\mathrm{w(s)}} \end{bmatrix};$$

$$\beta(\mathrm{s}) \equiv \mathrm{w}^2(\mathrm{s});\ \ \alpha(\mathrm{s}) = -\frac{\beta'(\mathrm{s})}{2} \equiv -\mathrm{w(s)w'(s)};\ \ \frac{d\psi}{ds} = \frac{1}{\mathrm{w}^2(\mathrm{s})} \equiv \frac{1}{\beta(\mathrm{s})};$$

$$Y(\mathrm{s}_2)\mathrm{e}^{i\Delta\psi} = \mathbf{M}(\mathrm{s}_1|\mathrm{s}_2)Y(\mathrm{s}_1) \Leftrightarrow Y^*(\mathrm{s}_2)\mathrm{e}^{-i\Delta\psi} = \mathbf{M}(\mathrm{s}_1|\mathrm{s}_2)Y^*(\mathrm{s}_1); \Delta\psi = \psi(\mathrm{s}_2) - \psi(\mathrm{s}_1).$$

or in other form as:

$$\left[Y(\mathrm{s}_2)\mathrm{e}^{i\Delta\psi}, Y^*(\mathrm{s}_2)\mathrm{e}^{-i\Delta\psi}\right] = \mathbf{M}(\mathrm{s}_1|\mathrm{s}_2)\left[Y(\mathrm{s}_1), Y^*(\mathrm{s}_1)\right];$$

$$\tilde{W}(\mathrm{s}) = \left[Y(\mathrm{s})\mathrm{e}^{i\psi(\mathrm{s})}, Y^*(\mathrm{s})\mathrm{e}^{-i\psi(\mathrm{s})}\right]; \tilde{W}(\mathrm{s}_2) = \mathbf{M}(\mathrm{s}_1|\mathrm{s}_2)\tilde{W}(\mathrm{s}_1);$$

Show that then matrix can be expressed through the values of envelopes **w**, and its derivatives **w'** and the "betatron" phase advance $\Delta\psi = \psi(\mathrm{s}_2) - \psi(\mathrm{s}_1)$ as:

$$\mathbf{M}(\mathrm{s}_1|\mathrm{s}_2) = \begin{bmatrix} \dfrac{\mathrm{w}_2}{\mathrm{w}_1}\cos\Delta\psi - \mathrm{w}_1'\mathrm{w}_2\sin\Delta\psi & \mathrm{w}_1\mathrm{w}_2\sin\Delta\psi \\ -\sin\Delta\psi\left(\dfrac{1}{\mathrm{w}_1\mathrm{w}_2} + \mathrm{w}_1'\mathrm{w}_2'\right) - \cos\Delta\psi\left(\dfrac{\mathrm{w}_1'}{\mathrm{w}_2} - \dfrac{\mathrm{w}_2'}{\mathrm{w}_1}\right) & \dfrac{\mathrm{w}_1}{\mathrm{w}_2}\cos\Delta\psi + \mathrm{w}_2'\mathrm{w}_1\sin\Delta\psi \end{bmatrix}$$

or in traditional terms:

$$\mathbf{M}(\mathrm{s}_1|\mathrm{s}_2) = \begin{bmatrix} \sqrt{\dfrac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1\sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\dfrac{\sin\Delta\psi\left(1 + \alpha_1\alpha_2\right) + \cos\Delta\psi\left(\alpha_2 - \alpha_1\right)}{\sqrt{\beta_1\beta_2}} & \sqrt{\dfrac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2\sin\Delta\psi\right) \end{bmatrix}$$

Hint: if you are using $\tilde{W}(\mathrm{s})$*, than use that* $\tilde{W}^T SW = -2iS$*. If you hate complex variables, than construct* $\tilde{U}(\mathrm{s}) = \left[\tilde{R}(\mathrm{s}), \tilde{Q}(\mathrm{s})\right]; \tilde{R}(\mathrm{s}) = \mathrm{Re}\,\tilde{Y}(\mathrm{s}); \tilde{Q}(\mathrm{s}) = \mathrm{Im}\,\tilde{Y}(\mathrm{s});$ *show that it is symplectic, and define how it transformed with s.*

Solution: While it is possible to do the exercise in a complex form like

$$\tilde{U}(s_2)=\mathbf{M}\tilde{U}(s_1) \rightarrow \mathbf{M}=\tilde{U}(s_1)\tilde{U}^{-1}(s_2)$$

$$\tilde{U}^{-1} = -\frac{1}{2i} S\tilde{U}^T S \rightarrow \mathbf{M}=-\frac{1}{2i}\tilde{U}(s_1) S\tilde{U}^T(s_2) S$$

$$\mathbf{M}=\frac{1}{2i}\tilde{U}(s_1) S\left(\tilde{U}^T(s_2) S\right)^T$$

$$\tilde{U}(s)=\begin{bmatrix} w(s) & w(s) \\ w'(s)+\dfrac{i}{w(s)} & w'(s)-\dfrac{i}{w(s)} \end{bmatrix}\begin{bmatrix} e^{i\psi(s)} & 0 \\ 0 & e^{-i\psi(s)} \end{bmatrix} \tag{1}$$

and just multiply the matrices. It is relatively easy but boring exercise to show that M in (1) is a real matrix. It takes few steps:

$$\tilde{U}^* = \tilde{U}\cdot\mathbf{P}; \mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \mathbf{PS} = -\mathbf{SP}; \mathbf{P}^T = \mathbf{P}; \tilde{U}^{*T} = \mathbf{P}\tilde{U}^*; \mathbf{P}^2 = \mathbf{I};$$

$$\mathbf{M}^* = \left(-\frac{1}{2i}\tilde{U}_1\mathbf{S}\tilde{U}_2^T\mathbf{S}\right)^* = \frac{1}{2i}\tilde{U}_1^*\mathbf{S}\tilde{U}_2^{*T}\mathbf{S} = \frac{1}{2i}\tilde{U}_1\cdot\mathbf{PSP}\cdot\tilde{U}_2^T\mathbf{S} = -\frac{1}{2i}\tilde{U}_1\mathbf{S}\tilde{U}_2^T\mathbf{S} = \mathbf{M}$$

It is interesting to use real form of the "eigen" vector relations:

$$Y_k = R_k + iQ_k; Y_k^* = R_k - iQ_k;$$

$$\mathbf{T}\cdot Y_k = e^{i\mu_k}Y_k; \mathbf{T}\cdot Y_k^* = e^{-i\mu_k}Y_k^* \rightarrow \mathbf{T}\cdot R_k = R_k\cos\mu_k - Q_k\sin\mu_k; \mathbf{T}\cdot Q_k = R_k\sin\mu_k + Q_k\cos\mu_k;$$

$$V^T S V = S;$$

$$V = \left[R_1, Q_1 ... R_n, Q_n\right]; \mathbf{T}\cdot V = V\cdot\Omega(C); \quad \Omega(C) = \begin{bmatrix} \Omega_1 & 0 & ... & 0 \\ 0 & \Omega_2 & ... & ... \\ .... & ... & ... & 0 \\ ... & ... & 0 & \Omega_n \end{bmatrix} \Omega_k(C) = \begin{bmatrix} \cos\mu_k & -\sin\mu_k \\ \sin\mu_k & \cos\mu_k \end{bmatrix}$$

$$\tilde{V}(s_2) = \mathbf{M}(\mathrm{s}_1|\mathrm{s}_2)\tilde{V}(s_1); \ \tilde{V}(s) = V(s)\Omega(s); \ V(s+C) = V(s); \Omega_k(s) = \begin{bmatrix} \cos\psi_k & -\sin\psi_k \\ \sin\psi_k & \cos\psi_k \end{bmatrix}$$

with obvious expression for matrix but already in real form:

$$\mathbf{M}(\mathrm{s}_1|\mathrm{s}_2) = \tilde{V}(s_1)\tilde{V}^{-1}(s_2) = V(s_1)\Omega(s_1)\Omega(s_2)V^{-1}(s_2)$$

$$V^{-1} = -SV^T S \rightarrow \mathbf{M}(\mathrm{s}_1|\mathrm{s}_2) = V(s_1)\Omega(s_1 - s_2) SV(s_2)^T S$$

In 1D case takes few multiplications

$$V(s)=\begin{bmatrix} w(s) & 0 \\ w'(s) & \dfrac{1}{w(s)} \end{bmatrix};\Omega(s)=\begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix};V^{-1}(s)=\begin{bmatrix} \dfrac{1}{w(s)} & 0 \\ -w'(s) & w(s) \end{bmatrix}$$

$$M(s_1|s_2)=\begin{bmatrix} w(s_1) & 0 \\ w'(s_1) & \dfrac{1}{w(s_1)} \end{bmatrix}\begin{bmatrix} \cos\Delta\psi & \sin\Delta\psi \\ -\sin\Delta\psi & \cos\Delta\psi \end{bmatrix}\begin{bmatrix} \dfrac{1}{w(s_2)} & 0 \\ -w'(s_2) & w(s_2) \end{bmatrix}=$$

$$\begin{bmatrix} w(s_1)\cos\Delta\psi & w(s_1)\sin\Delta\psi \\ w'(s_1)\cos\Delta\psi-\dfrac{\sin\Delta\psi}{w(s_1)} & w'(s_1)\sin\Delta\psi+\dfrac{\cos\Delta\psi}{w(s_1)} \end{bmatrix}\begin{bmatrix} \dfrac{1}{w(s_2)} & 0 \\ -w'(s_2) & w(s_2) \end{bmatrix}=$$

$$\begin{bmatrix} \dfrac{w(s_1)\cos\Delta\psi}{w(s_2)}-w'(s_2)w(s_1)\sin\Delta\psi & w(s_2)w(s_1)\sin\Delta\psi \\ \dfrac{ad-1}{c} & \dfrac{w(s_2)\cos\Delta\psi}{w(s_1)}-w'(s_1)w(s_2)\sin\Delta\psi \end{bmatrix}$$

Substitution to α,β is trivial. #