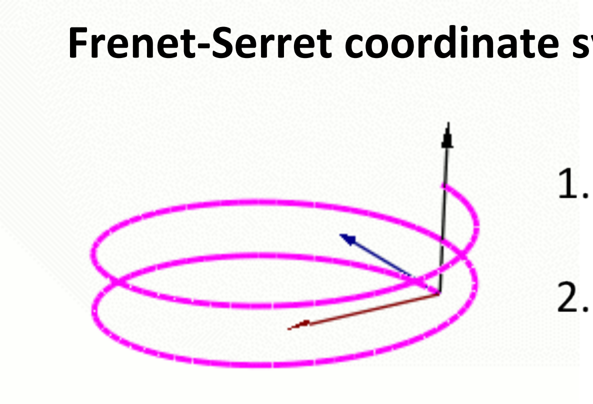
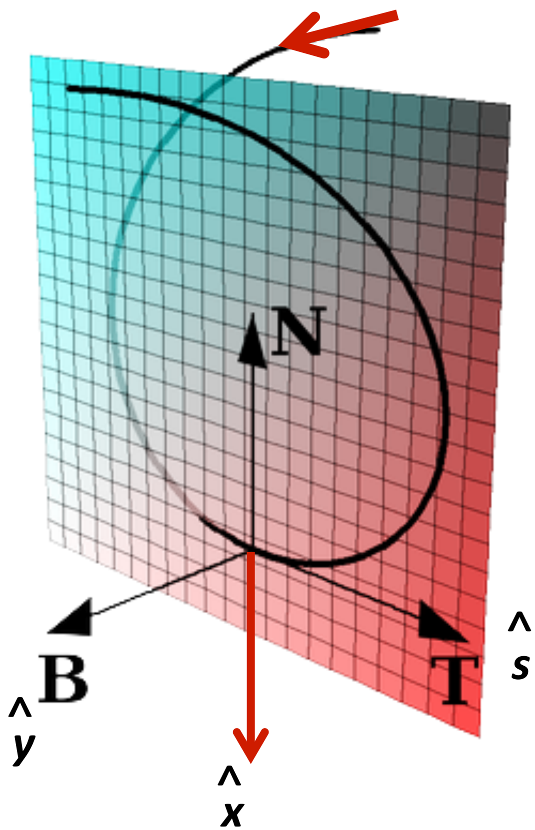
Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

Frenet-Serret coordinate system:

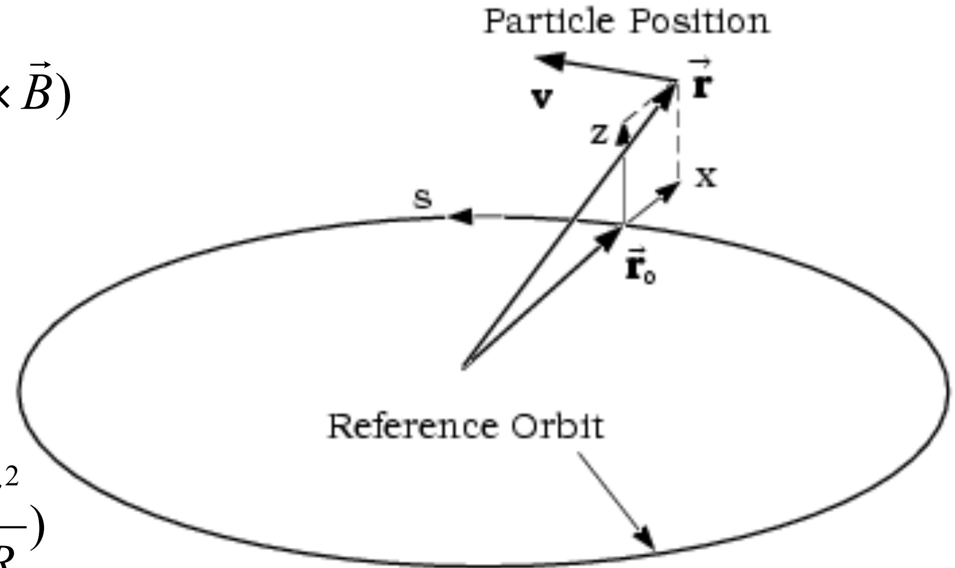


1. The tangential vector points toward the direction of beam motion
2. We define the normal vector points outward from the curve. Magnetic field direction y. So that $s = -x \times y$ (electron for this case)



http://en.wikipedia.org/wiki/Frenet-Serret

Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$



$$F_x = \frac{d}{dt}(\gamma m\dot{x}) = \frac{\gamma m\beta^2 c^2}{R} = -q\beta c B_y$$

$$F_y = \frac{d}{dt}(\gamma m\dot{y}) = q\beta c B_x$$

$$\frac{d}{dt}(\gamma m\dot{x}) \approx \gamma m(\ddot{x} - \frac{v^2}{R}) = \gamma m(\frac{\beta^2 c^2}{R^2}\frac{d^2 x}{d\theta^2} - \frac{v^2}{R})$$

How to transform from the original coordinate system into the Frenet-Serret coordinate system?

$$\theta = \frac{s}{R} = \frac{\beta c t}{R}$$

$$n = -\frac{\rho}{B_0}(\frac{\partial B_y}{\partial x})_{x=0}$$ → Constant when $B_y \sim x^0$ or x^1

$$\frac{d^2 x}{d\theta^2} + (1 + \frac{\rho}{B_0}\frac{\partial B_y}{\partial x})x = 0 \qquad \frac{d^2 y}{d\theta^2} + (-\frac{\rho}{B_0}\frac{\partial B_y}{\partial x})y(1 + 2\frac{x}{\rho}) = 0$$

$$\frac{d^2 x}{d\theta^2} + (1-n)x = 0 \qquad \frac{d^2 y}{d\theta^2} + ny = 0$$

$$\frac{dx}{d\theta} = \sqrt{1-n}(-A\sin\sqrt{1-n}\theta + B\cos\sqrt{1-n}\theta) \qquad \frac{dy}{d\theta} = \sqrt{n}(-C\sin\sqrt{n}\theta + D\cos\sqrt{n}\theta)$$

Number of transverse oscillation in one beam revolution ~ $\sqrt{1-n} / \sqrt{n}$, weak focusing!

For weak focusing, stable solution requires 0<n<1, which makes transverse (betatron) oscillation having a tune (number of circles per revolution) $Q_x, Q_y = \sqrt{1-n}, \sqrt{n}$ less than 1. The beam sizes in such machines scales with $C^{1/2}/(1-n)^{1/4}, C^{1/2}/n^{1/4}$

For a modern machine, especially light sources where smaller beam sizes are required to achieve higher beam brightness, strong focusing is required => external focusing magnets (quadrupoles) are often used.

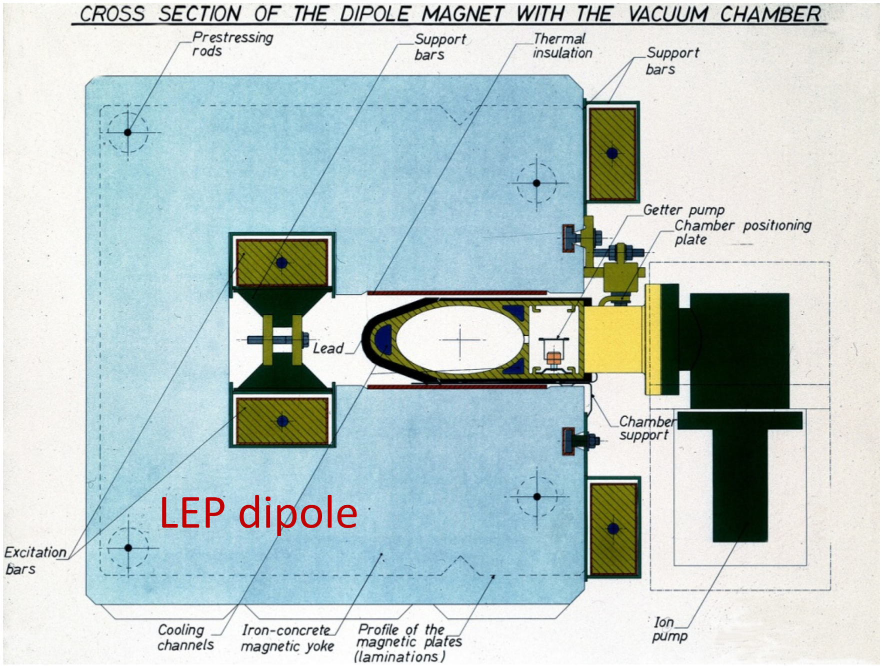
$$x'' + K_x(s)x = \pm \frac{\Delta B_z}{B\rho}, \quad y'' + K_y(s)y = \mp \frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm \frac{B_1}{B\rho}$$

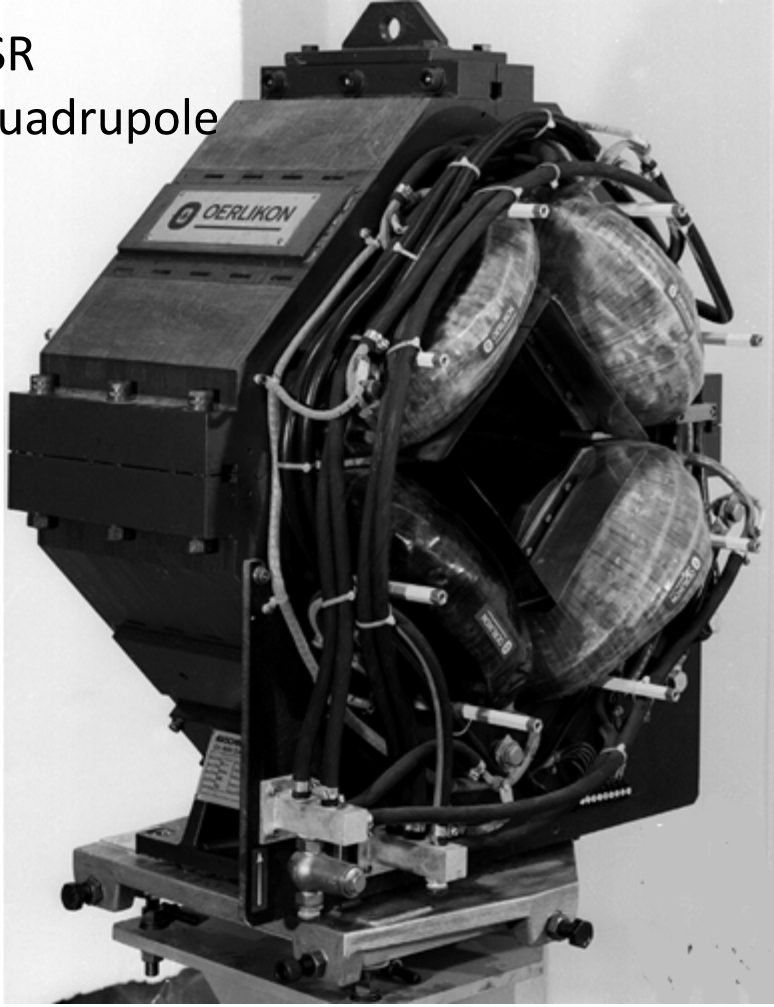
Higher order magnet, usually field errors

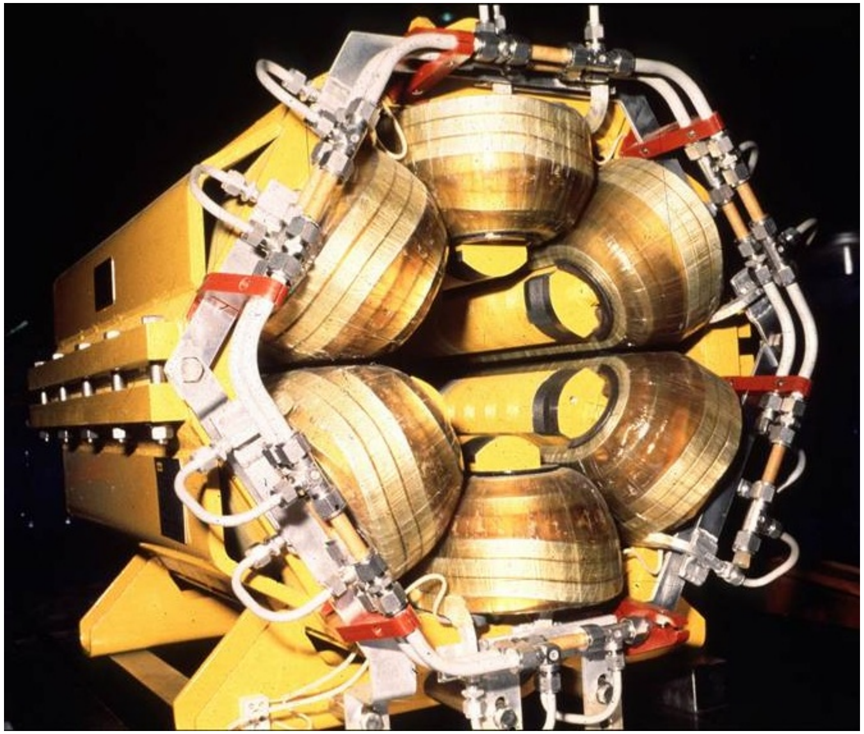
Natural focusing from dipoles

Focusing from quadrupoles



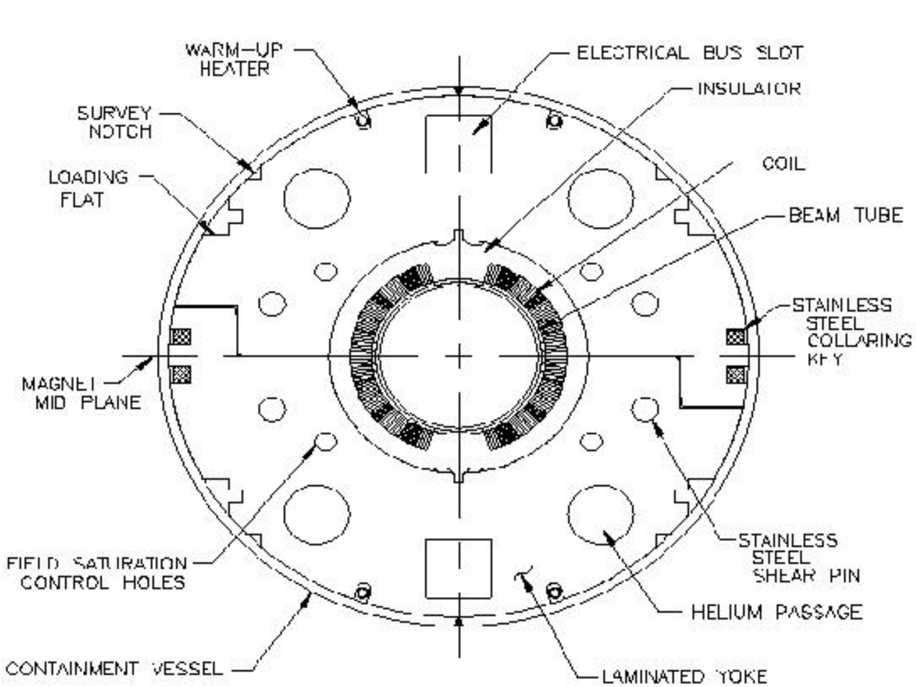
ISR quadrupole



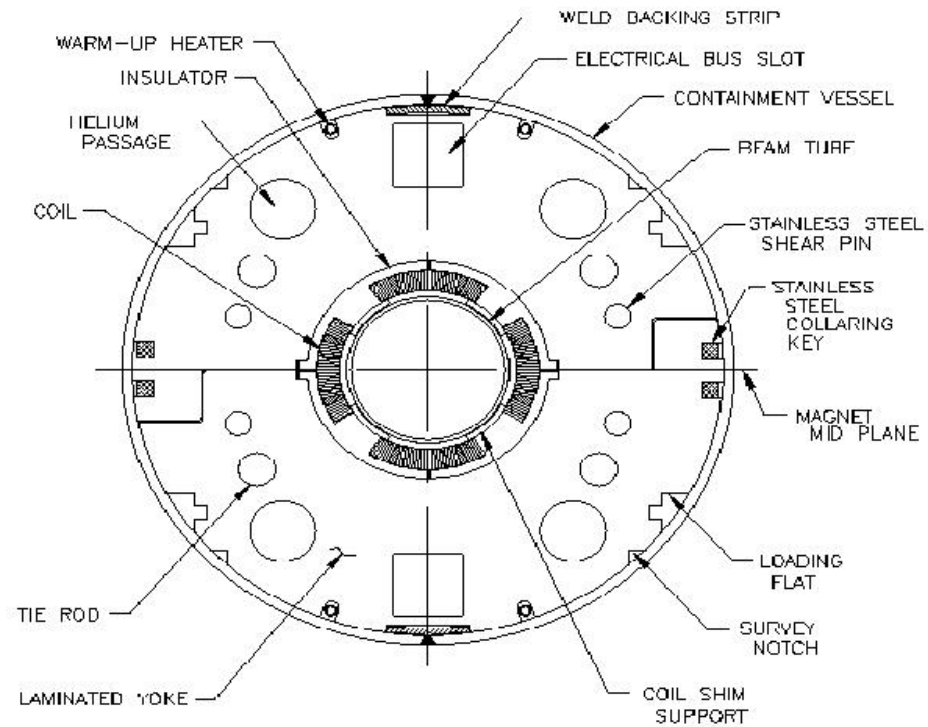


LEP Sextupole

RHIC ARC DIPOLE



RHIC ARC QUADRUPOLE



For two dimensional magnetic field, one can expand the magnetic field using **Beth representation**.

$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$

$$B_x = -\frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial y} = -\frac{1}{h_s}\frac{\partial A_s}{\partial y}, B_y = \frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial x} = \frac{1}{h_s}\frac{\partial A_s}{\partial x}$$

For h_s=1 or ρ=∞, one obtains the multipole expansion:

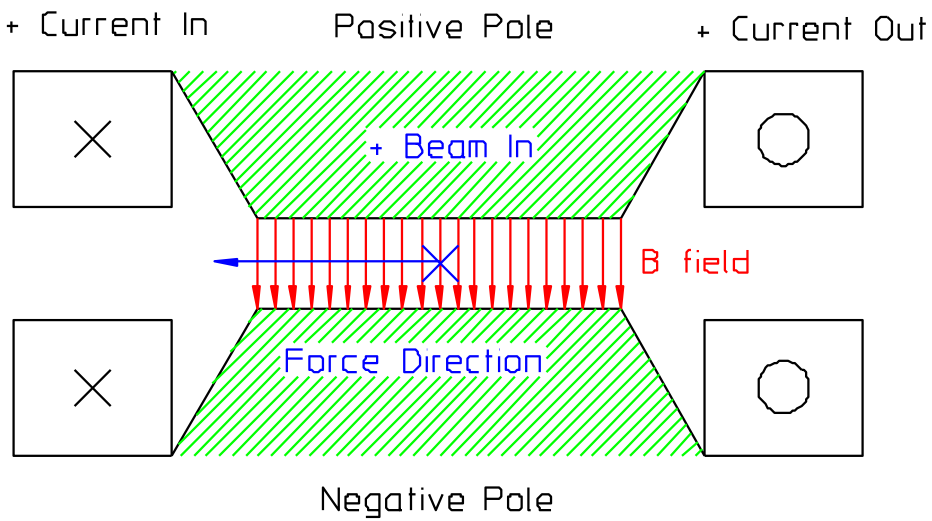
$$B_y + jB_x = B_0 \sum_n (b_n + ja_n)(x+jy)^n, \qquad A_s = \mathrm{Re}\left\{B_0 \sum_n \frac{1}{n+1}(b_n + ja_n)(x+jy)^{n+1}\right\}$$

b_0 : dipole, a_0 : skew (vertical) dipole; $B_y = B_0 b_0$, $B_x = B_0 a_0$,

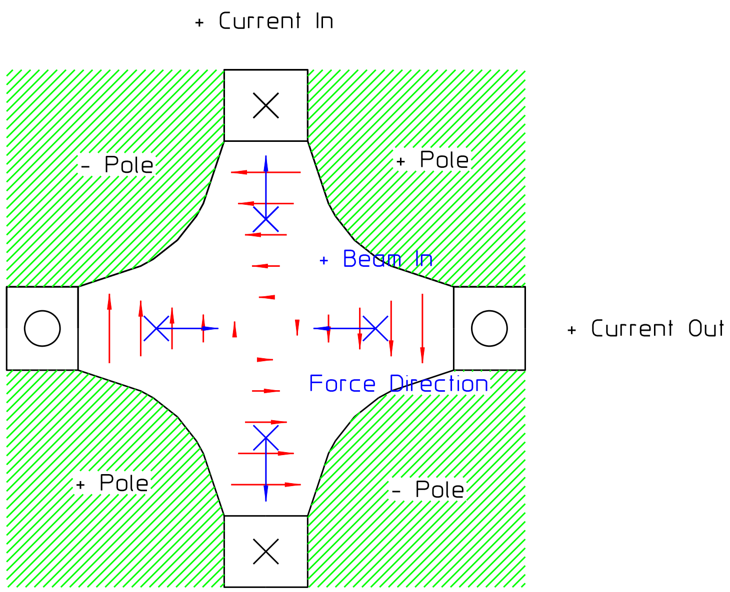
b_1 : quad, a_1 : skew quad; $B_y = B_0 b_1 x$, $B_x = B_0 b_1 y$, $B_y = -B_0 a_1 y$, $B_x = B_0 a_1 x$,

b_2 : sextupole, a_2 : skew sextupole;

$$\frac{1}{B\rho}(B_y + jB_x) = \mp\frac{1}{\rho}\sum_n (b_n + ja_n)(x+jy)^n$$

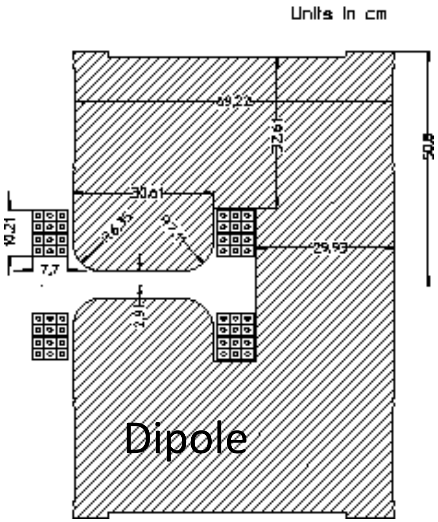


$$B_0 = \frac{\mu_0 N I}{g}, \quad A_s = B_0 x$$



$$B_1 = \frac{2\mu_0 N I}{a^2}, \quad A_s = \frac{B_1}{2}\left(x^2 - z^2\right)$$

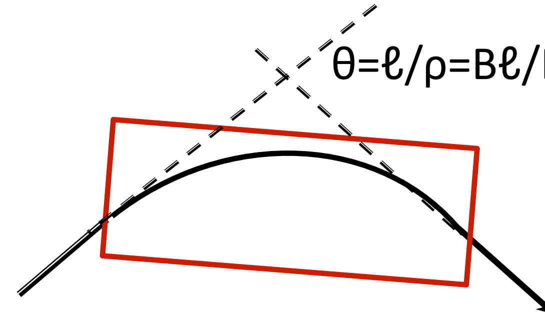
We will learn how to optimize the arrangement of beam control elements in achieving the wanted beam properties.



$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

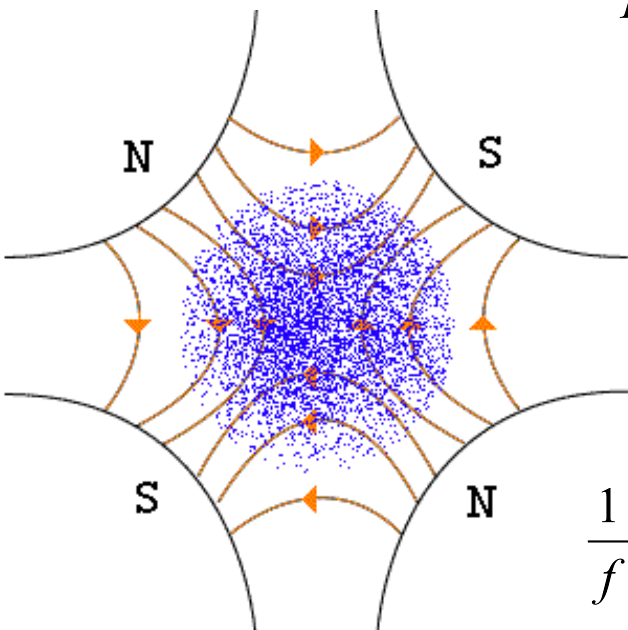
$$\gamma m \frac{\upsilon^2}{\rho} = q\upsilon B$$

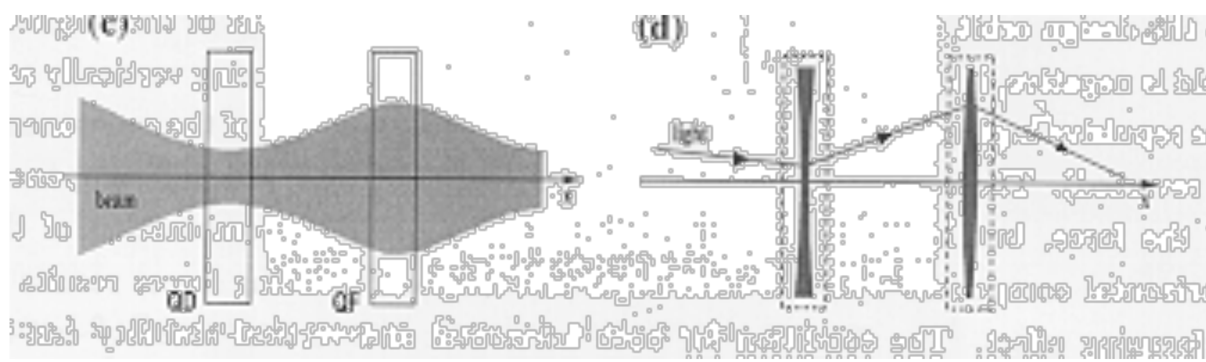
$$\rho = \frac{\gamma m v}{qB} = \frac{p}{qB}$$



$$B\rho[T-m] = \frac{p}{q} = \frac{A}{Z} \times 3.33564 \times p[GeV/c/u]$$

quadrupole

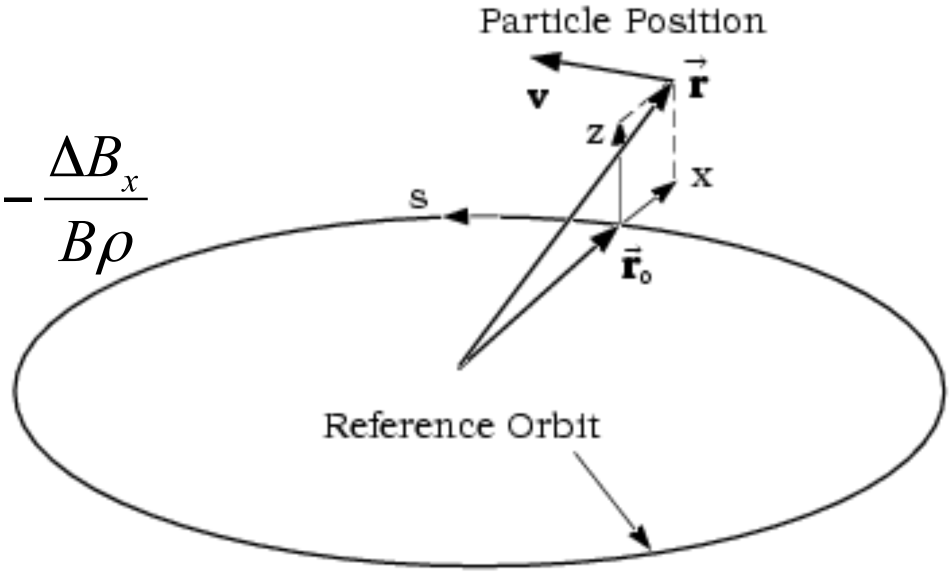




$$\frac{1}{f} = \mp \frac{B_1 \ell}{B\rho}$$

f>0, if focusing, f<0 if defocusing

How to solve the Hill's equation?



$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho}$$

Ideal Linear accelerator:

$$x'' + K_x(s)x = 0, \quad y'' + K_y(s)y = 0$$

Let X represent x or y:

$$X'' + K(s)X = 0,$$

$$X(s) = \begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s, s_0)\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

$M(s,s_0)$ is the betatron transfer matrix. For any two linearly independent solutions y_1, y_2 of Hill's equation, the Wronskian is independent of time

$$W(y_1, y_2, s) \equiv y_1 y_2' - y_1' y_2, \quad \frac{dW}{ds} = 0. \qquad W(s) = [\det M] W(s_0)$$

$$\mathrm{Det}(M(s_2,s_1))=1$$