Derivation of FEL Hamiltonian in Saldin's Textbook

Gang Wang

We start from the following general Hamiltonian for an electron in a wiggler (eq. (2.1) from the textbook):

$$H\left(p_z,z,t\right)=\left[\left(p_z c+eA_z\right)^2+e^2\left(A_\perp+A_w\right)^2+m_e^2c^4\right]^{1/2}-e\phi\,. \tag{1}$$

For the extended canonical transformation (see APPENDIX 1: eq. (A.1.8) and eq. (A.1.10)), we transform from the old variables

$$x_0=t\,, \tag{2}$$

and

$$x_1=z\,, \tag{3}$$

to the new variables

$$\bar{x}_0=z\,, \tag{4}$$

and

$$\bar{x}_1=\psi=\left(k+k_w\right)z-\omega t\,. \tag{5}$$

In order to use eq. (A.1.10) to find the new momentum, $\bar{p}_0$, which is the new Hamiltonian with a negative sign according to equation 4 lines above the end of APPENDIX 1 in the textbook, we express the old variables in terms of the new variables as:

$$x_0=t=\frac{k+k_w}{\omega}z-\frac{\bar{x}_1}{\omega}=\frac{k+k_w}{\omega}\bar{x}_0-\frac{\bar{x}_1}{\omega}\,, \tag{6}$$

and

$$x_1=\bar{x}_0\,. \tag{7}$$

Inserting eq. (6), (7) and the old canonical momentum:

$$p_0=-H\left(z,p_z,t\right), \tag{8}$$

and

$$p_1=p_z\,, \tag{9}$$

into eq. (A.1.10) of the textbook yields

$$\bar{p}_0=p_0\frac{\partial x_0}{\partial \bar{x}_0}+p_1\frac{\partial x_1}{\partial \bar{x}_0}=-\frac{k+k_w}{\omega}H+p_z\,, \tag{10}$$

and

$$\bar{p}_1=p_0\frac{\partial x_0}{\partial \bar{x}_1}+p_1\frac{\partial x_1}{\partial \bar{x}_1}=\frac{1}{\omega}H\,. \tag{11}$$

From the equation 4 lines above the end of APPENDIX 1 in the textbook, the new Hamiltonian is thus

$$\bar{H}=-\bar{p}_0=\frac{k+k_w}{\omega}E-p_z\left(\psi,E,z\right), \tag{12}$$

where we used the fact that $H=E$ is the energy of the electron. Solving eq. (1) for p_z produces

$$p_z = \frac{1}{c}\sqrt{(E+e\phi)^2 - m_e^2c^4 - e^2\left(A_\perp + A_w\right)^2} - \frac{e}{c}A_z, \tag{13}$$

and hence the new Hamiltonian is

$$\bar{H} = \frac{k+k_w}{\omega}E + \frac{e}{c}A_z - \frac{1}{c}\sqrt{(E+e\phi)^2 - m_e^2c^4 - e^2\left(A_\perp + A_w\right)^2} \ . \tag{14}$$

The Gauge transformation:

$$\phi(z,t) \to \phi'(z,t) = \phi(z,t) - \frac{1}{c}\frac{\partial\chi(z,t)}{\partial t} \ , \tag{15}$$

and

$$A_z(z,t) \to A_z'(z,t) = A_z(z,t) + \frac{\partial\chi(z,t)}{\partial z}, \tag{16}$$

does not change the magnetic field $\vec{B}$ since

$$\vec{B}' = \nabla\times\vec{A}' = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z + \chi(z,t) \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{B}, \tag{17}$$

and (see eq. (6.9) of Jackson 3rd edition)

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \vec{E} \ . \tag{18}$$

Taking

$$\chi(z,t) = c\int_{t_0}^{t}\phi(z,\tau)d\tau - \int_{z_0}^{z}A_z(z_1,t_0)dz_1 \ , \tag{19}$$

eq. (15) and (16) becomes

$$\phi'(z,t) = \phi(z,t) - \frac{1}{c}\frac{\partial\chi(z,t)}{\partial t} = 0 \tag{20}$$

and

$$A_z'(z,t) = A_z(z,\tau) + c\frac{\partial}{\partial z}\int_{t_0}^{t}\phi(z,\tau)d\tau - \frac{\partial}{\partial z}\int_{z_0}^{z}A_z(z_1,t_0)dz_1 = -c\int_{t_0}^{t}E_z(z,\tau)d\tau \ . \tag{21}$$

Hence the Hamiltonian after the Gauge transformation becomes

$$\tilde{\bar{H}} = \frac{k+k_w}{\omega}E - e\int_{t_0}^{t}E_z(z,\tau)d\tau - \frac{1}{c}\sqrt{E^2 - m_e^2c^4 - e^2\left(A_\perp + A_w\right)^2} \ , \tag{22}$$

which can be written as

$$\tilde{\bar{H}} = \frac{k+k_w}{\omega}E + \frac{e}{\omega}\int_{\psi_0(z)}^{\psi}E_z(z,\psi_1)d\psi_1 - \frac{1}{c}\sqrt{E^2 - m_e^2c^4 - e^2\left(A_\perp + A_w\right)^2} \ . \tag{23}$$

Assuming the radiation filed is much weaker than the wiggler field, we take the following approximation for the last term of eq. (23):

$$\sqrt{E^2 - m_e^2c^4 - e^2\left(\vec{A}_\perp + \vec{A}_w\right)^2} \approx \sqrt{(E_0+P)^2 - e^2A_w^2 - m_e^2c^4} - \frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{\sqrt{(E_0+P)^2 - e^2A_w^2 - m_e^2c^4}} \ ,$$

(24)

with

$$P = E - E_0 \tag{25}$$

being the energy deviation of an electron and E_0 being the designed energy of the FEL. The wiggler potential is related to wiggler field through

$$\nabla \times \vec{A}_w = \vec{B}_w \ , \tag{26}$$

which is

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_{w,x} & A_{w,y} & A_{w,z} \end{vmatrix} = -\frac{\partial}{\partial z} A_{w,y} i + \frac{\partial}{\partial z} A_{w,x} j = B_{w,x} i + B_{w,y} j + B_{w,z} k \tag{27}$$

Hence

$$A_{w,y} = -\int_{z_0}^{z} B_{w,x}(z_1) dz_1 = -\frac{B_w}{k_w} \left[\sin(k_w z) - \sin(k_w z_0) \right] \ , \tag{28}$$

and

$$A_{w,x} = \int_{\tilde{z}_0}^{z} B_{w,y}(z_1) dz_1 = \frac{B_w}{k_w} \int_{\tilde{z}_0}^{z} d\cos(k_w z_1) = \frac{B_w}{k_w} \left[\cos(k_w z) - \cos(k_w \tilde{z}_0) \right] \ . \tag{29}$$

As a constant term in magnetic potential will not change the electric and magnetic field, we choose z_0 and $\tilde{z}_0$ such that they disappear from the potential, i.e.

$$A_{w,y} = -\frac{B_w}{k_w} \sin(k_w z), \tag{30}$$

and

$$A_{w,x} = \frac{B_w}{k_w} \cos(k_w z). \tag{31}$$

From eq. (30) and eq. (31), we find

$$e^2 A_w^2 + m_e^2 c^4 = \frac{e^2 B_w^2}{k_w^2} + m_e^2 c^4 = \left(\frac{e^2 B_w^2}{\gamma_0^2 m_e^2 c^4 k_w^2} + \frac{1}{\gamma_0^2} \right) \gamma_0^2 m_e^2 c^4 = \left(\frac{1 + K^2}{\gamma_0^2} \right) E_0^2 = \frac{E_0^2}{\gamma_z^2} \ . \tag{32}$$

Using eq. (32) and expanding to the second order of P / E_0 , the square root factor in the R.H.S. of eq. (24) becomes

$$\sqrt{(E_0 + P)^2 - e^2 A_w^2 - m_e^2 c^4} \approx E_0 \beta_z \left\{ 1 + \frac{P}{E_0 \beta_z^2} - \frac{1}{2} \frac{1}{\beta_z^2 \gamma_z^2} \left(\frac{P}{E_0 \beta_z} \right)^2 + O\left(\frac{P^3}{E_0^3} \right) \right\}, \tag{33}$$

and

$$\frac{1}{\sqrt{(E_0 + P)^2 - e^2 A_w^2 - m_e^2 c^4}} \approx \frac{1}{E_0 \beta_z} \left\{ 1 - \frac{P}{E_0 \beta_z^2} + \frac{1}{2} \left(\frac{P}{E_0 \beta_z} \right)^2 \left(\frac{3}{\beta_z^2} - 1 \right) + O\left(\frac{P^3}{E_0^3} \right) \right\}. \tag{34}$$

Inserting eq. (33) and eq. (34) into eq. (24) yields

$$\sqrt{E^2 - m_e^2c^4 - e^2\left(\vec{A}_\perp + \vec{A}_w\right)^2}$$
$$\approx E_0\beta_z + \frac{P}{\beta_z} - \frac{P^2}{2\beta_z^3\gamma_z^2E_0} - \frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{E_0\beta_z}\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2} - 1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right]. \tag{35}$$

Inserting eq. (25) and eq. (35) into eq. (23) leads to

$$\tilde{\tilde{H}} = \left(\frac{k+k_w}{\omega} - \frac{\beta_z}{c}\right)E_0 + \left(\frac{k+k_w}{\omega} - \frac{1}{\beta_z c}\right)P + \frac{P^2}{2c\beta_z^3\gamma_z^2E_0}$$
$$+\frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{E_0\beta_z c}\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2} - 1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right] + \frac{e}{\omega}\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1. \tag{36}$$

Doing a scale transformation (it is always canonical as discussed in eq. (9.9) of 'Classical Mechanics' by Goldstein) for the generalized coordinates and momentum defined in eq. (11) and (5) as follows

$$\begin{aligned} \bar{p}_1 &\to \tilde{p}_1 = \omega\bar{p}_1 = H = E_0 + P \\ \bar{x}_1 &\to \tilde{x}_1 = \bar{x}_1 = \psi \\ \tilde{\tilde{H}} &\to \hat{\tilde{H}} = \omega\tilde{\tilde{H}} \end{aligned} \tag{37}$$

yields

$$\begin{aligned} \hat{\tilde{H}} &= \omega\tilde{\tilde{H}} \\ &= \left(k + k_w - \frac{\omega}{c}\beta_z\right)E_0 + \left(k + k_w - \frac{\omega}{\beta_z c}\right)P + \frac{\omega P^2}{2c\beta_z^3\gamma_z^2E_0} \\ &\quad+\frac{e^2\omega\vec{A}_\perp\cdot\vec{A}_w}{E_0\beta_z c}\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2} - 1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right] + e\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1 \end{aligned}. \tag{38}$$

Transformation of subtracting constants from the Hamiltonian, generalized coordinates and generalized momentum is also canonical and will not change the equations of motion as suggested by the Hamiltonian equation and the direct conditions for a canonical transformation (see page 391 of Goldstein). Hence we subtract the first term from eq. (38) and E_0 from $\tilde{p}_1$. The resulting Hamiltonian becomes

$$\begin{gathered} \tilde{H}(\psi,P,z) = CP + \frac{\omega P^2}{2c\beta_z^3\gamma_z^2E_0} + \\ \frac{e^2\omega\vec{A}_\perp\cdot\vec{A}_w}{E_0\beta_z c}\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2} - 1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right] + e\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1 \end{gathered}, \tag{39}$$

with a newly defined variable

$$C = k + k_w - \frac{\omega}{\beta_z c}. \tag{40}$$

Now we will try to calculate $\vec{A}_\perp \cdot \vec{A}_w$. From eq. (18), the radiation field is related to its magnetic potential by

$$\vec{E} = -\vec{\nabla}\phi(z,t) - \frac{1}{c}\frac{\partial \vec{A}(z,t)}{\partial t}. \tag{41}$$

The transverse component of eq. (41) reads

$$E_x(z,t) = -\frac{1}{c}\frac{\partial A_x(z,t)}{\partial t}, \tag{42}$$

and

$$E_y(z,t) = -\frac{1}{c}\frac{\partial A_y(z,t)}{\partial t}. \tag{43}$$

Assuming the radiation field satisfy (the third equation in chaper 2.1.1)

$$E_x(z,t) + iE_y(z,t) = \tilde{E}(z)e^{i(kz-\omega t)}, \tag{44}$$

and inserting it into eq. (42) and (43) produces

$$A_x(z,t) + iA_y(z,t) = -c\int \tilde{E}(z)e^{i(kz-\omega t)}\,dt = -\frac{ic}{\omega}\tilde{E}(z)e^{i(kz-\omega t)}. \tag{45}$$

Using eq. (30), eq. (31) and eq. (45), we obtain

$$\begin{aligned}\vec{A}_\perp \cdot \vec{A}_w &= A_x A_{w,x} + A_y A_{w,y} \\ &= \mathrm{Re}\left[\left(A_x + iA_y\right)\cdot\left(A_{w,x} - iA_{w,y}\right)\right] \\ &= \frac{1}{2}\left(A_x + iA_y\right)\cdot\left(A_{w,x} - iA_{w,y}\right) + c.c. \\ &= -\frac{ic}{2\omega}\tilde{E}(z)e^{i(kz-\omega t)}\frac{B_w}{k_w}\left[\cos(k_w z) + i\sin(k_w z)\right] + c.c. \\ &== -\frac{ic}{2\omega}\frac{B_w}{k_w}\tilde{E}(z)e^{i(kz+k_w z-\omega t)} + c.c.\end{aligned}. \tag{46}$$

Inserting eq. (46) into eq. (39) yields

$$\begin{aligned}\tilde{H}(\psi,P,z) &= CP + \frac{\omega P^2}{2c\beta_z^3\gamma_z^2 E_0} + e\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1 \\ &\quad + \left[\frac{1}{2i}\frac{e^2}{E_0\beta_z}\frac{B_w}{k_w}\tilde{E}(z)e^{i(kz+k_w z-\omega t)} + c.c.\right]\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2} - 1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right] \\ &= CP + \frac{\omega P^2}{2c\beta_z^3\gamma_z^2 E_0} + e\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1 \\ &\quad - \left[Ue^{i\psi} + U^* e^{-i\psi}\right]\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2} - 1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right]\end{aligned}, \tag{47}$$

where we defined the complex radiation potential as

$$U(z) \equiv -\frac{1}{2i}\frac{e^2}{E_0\beta_z}\frac{B_w}{k_w}\tilde{E}(z) = -\frac{eK}{2i\gamma_0}\tilde{E}(z) = -\frac{e\theta_s}{2i}\tilde{E}(z)\ . \tag{48}$$

Taking the ultra-relativistic limit of $\beta_z \approx 1$, eq. (47) becomes

$$\tilde{H}(\psi,P,z) = CP + \frac{\omega P^2}{2c\gamma_z^2 E_0} - \left(Ue^{i\psi} + U^* e^{-i\psi}\right)\cdot\left[1 - \frac{P}{E_0} + \left(\frac{P}{E_0}\right)^2\right] + e\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1\,. \tag{49}$$

APPENDIX:

$$\begin{aligned}
\tilde{H} &= \frac{k+k_w}{\omega}(E_0+P) + \frac{e}{\omega}\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1 \\
&\quad -\frac{1}{c}\left\{E_0\beta_z + \frac{P}{\beta_z} - \frac{P^2}{2\beta_z^3\gamma_z^2E_0} - \frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{E_0\beta_z}\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2}-1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right]\right\} \\
&= \left(\frac{k+k_w}{\omega} - \frac{\beta_z}{c}\right)E_0 + \left(\frac{k+k_w}{\omega} - \frac{1}{\beta_z c}\right)P + \frac{P^2}{2c\beta_z^3\gamma_z^2E_0} \\
&\quad + \frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{E_0\beta_z c}\left[1 - \frac{P}{E_0\beta_z^2} + \frac{1}{2}\left(\frac{3}{\beta_z^2}-1\right)\left(\frac{P}{E_0\beta_z}\right)^2\right] + \frac{e}{\omega}\int_{\psi_0(z)}^{\psi} E_z(z,\psi_1)d\psi_1
\end{aligned}$$

$$\sqrt{\left(E_0+P\right)^2-e^2A_w^2-m_e^2c^4}$$

$$=E_0\sqrt{1-\frac{1}{\gamma_z^2}+2\frac{P}{E_0}+\left(\frac{P}{E_0}\right)^2}$$

$$=E_0\sqrt{\beta_z^2+2\frac{P}{E_0}+\left(\frac{P}{E_0}\right)^2}$$

$$=E_0\beta_z\sqrt{1+2\frac{P}{E_0\beta_z^2}+\left(\frac{P}{E_0\beta_z}\right)^2}$$

$$=E_0\beta_z\left\{1+\frac{1}{2}\left[2\frac{P}{E_0\beta_z^2}+\left(\frac{P}{E_0\beta_z}\right)^2\right]-\frac{1}{8}\left[2\frac{P}{E_0\beta_z^2}+\left(\frac{P}{E_0\beta_z}\right)^2\right]^2+..\right\}$$

$$\approx E_0\beta_z\left\{1+\frac{P}{E_0\beta_z^2}-\frac{1}{2}\frac{1}{\beta_z^2\gamma_z^2}\left(\frac{P}{E_0\beta_z}\right)^2+O\left(\frac{P^3}{E_0^3}\right)\right\}$$

$$\frac{1}{\sqrt{\left(E_0+P\right)^2-e^2A_w^2-m_e^2c^4}}$$

$$=\frac{1}{E_0\sqrt{1-\frac{1}{\gamma_z^2}+2\frac{P}{E_0}+\left(\frac{P}{E_0}\right)^2}}$$

$$=\frac{1}{E_0\sqrt{\beta_z^2+2\frac{P}{E_0}+\left(\frac{P}{E_0}\right)^2}}$$

$$=\frac{1}{E_0\beta_z\sqrt{1+2\frac{P}{E_0\beta_z^2}+\left(\frac{P}{E_0\beta_z}\right)^2}}$$

$$=\frac{1}{E_0\beta_z}\left\{1-\frac{1}{2}\left[2\frac{P}{E_0\beta_z^2}+\left(\frac{P}{E_0\beta_z}\right)^2\right]+\frac{3}{8}\left[2\frac{P}{E_0\beta_z^2}+\left(\frac{P}{E_0\beta_z}\right)^2\right]^2+..\right\}$$

$$\approx\frac{1}{E_0\beta_z}\left\{1-\frac{P}{E_0\beta_z^2}+\frac{1}{2}\left(\frac{P}{E_0\beta_z}\right)^2\left(\frac{3}{\beta_z^2}-1\right)+O\left(\frac{P^3}{E_0^3}\right)\right\}$$

$$
\begin{aligned}
A_{w,y} &= -\int_{z_0}^{z} B_{w,x}\left(z_1\right)dz_1 \\
&= -\int_{z_0}^{z} B_w \cos\left(k_w z_1\right)dz_1 \\
&= -\frac{B_w}{k_w}\int_{z_0}^{z} d\sin\left(k_w z_1\right) \\
&= -\frac{B_w}{k_w}\left[\sin\left(k_w z\right)-\sin\left(k_w z_0\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{E^2 - m_e^2c^4 - e^2\left(\vec{A}_\perp + \vec{A}_w\right)^2} &\approx \sqrt{E^2 - m_e^2c^4 - e^2\left(2\vec{A}_\perp\cdot\vec{A}_w + A_w^2\right)} \\
&= \sqrt{\left(E^2 - e^2A_w^2 - m_e^2c^4\right)\left(1-\frac{2e^2\vec{A}_\perp\cdot\vec{A}_w}{E^2 - e^2A_w^2 - m_e^2c^4}\right)} \\
&= \sqrt{\left(E^2 - e^2A_w^2 - m_e^2c^4\right)}\left(1-\frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{E^2 - e^2A_w^2 - m_e^2c^4}\right)
\end{aligned}
$$

$$
\bar{H} = -\bar{p}_0 = \frac{k+k_w}{\omega}E - p_z\left(\psi, E, z\right)
$$

$$
\bar{H} = -\bar{p}_0 = \frac{k+k_w}{\omega}E - p_z\left(\psi, E, z\right)
$$

$$
p_z \approx \frac{1}{c}\left(\sqrt{\left(E+e\phi\right)^2 - m_e^2c^4 - e^2\vec{A}_w^2} - \frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{\sqrt{\left(E+e\phi\right)^2 - m_e^2c^4 - e^2\vec{A}_w^2}}\right) - \frac{e}{c}A_z \qquad (50)
$$

$$
\begin{aligned}
p_z &= \frac{1}{c}\sqrt{\left(E+e\phi\right)^2 - m_e^2 c^4 - e^2\left(A_\perp + A_w\right)^2} - \frac{e}{c}A_z \\
&\approx \frac{1}{c}\sqrt{\left(E+e\phi\right)^2 - m_e^2 c^4 - e^2\vec{A}_w^2 - 2e^2\vec{A}_\perp\cdot\vec{A}_w} - \frac{e}{c}A_z \\
&= \frac{1}{c}\sqrt{\left[\left(E+e\phi\right)^2 - m_e^2 c^4 - e^2\vec{A}_w^2\right]\left(1 - \frac{2e^2\vec{A}_\perp\cdot\vec{A}_w}{\left(E+e\phi\right)^2 - m_e^2 c^4 - e^2\vec{A}_w^2}\right)} - \frac{e}{c}A_z \\
&\approx \frac{1}{c}\left(\sqrt{\left(E+e\phi\right)^2 - m_e^2 c^4 - e^2\vec{A}_w^2} - \frac{e^2\vec{A}_\perp\cdot\vec{A}_w}{\sqrt{\left(E+e\phi\right)^2 - m_e^2 c^4 - e^2\vec{A}_w^2}}\right) - \frac{e}{c}A_z
\end{aligned}
$$