Collective Effects I: Wakefield and Impedances

Gang Wang

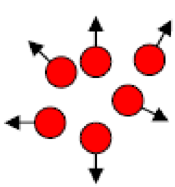
Outline

- Introduction
  - Collective effects, collective instabilities
- Wakefields and Impedances (Ultra-relativistic model)
  - Wake functions
  - Panofsky-Wenzel theorem
  - Cylindrical symmetric structure
  - Wake potentials, loss factor and kick factor
  - Impedances

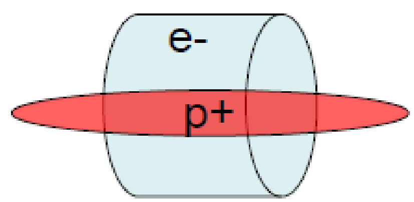
What are collective effects?

- In the single particle dynamics, the E&M fields due to the charged particle themselves are neglected when considering their motions.
- As the number of the particles increases, the particles' own fields (and fields induced by them) can start to affect their behavior, which is generally called the collective effects.

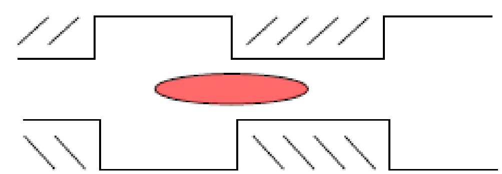
Beam interacts with itself: space charge,
IBS, Touschek effects



Beam interacts with localized electron cloud
or ion cloud

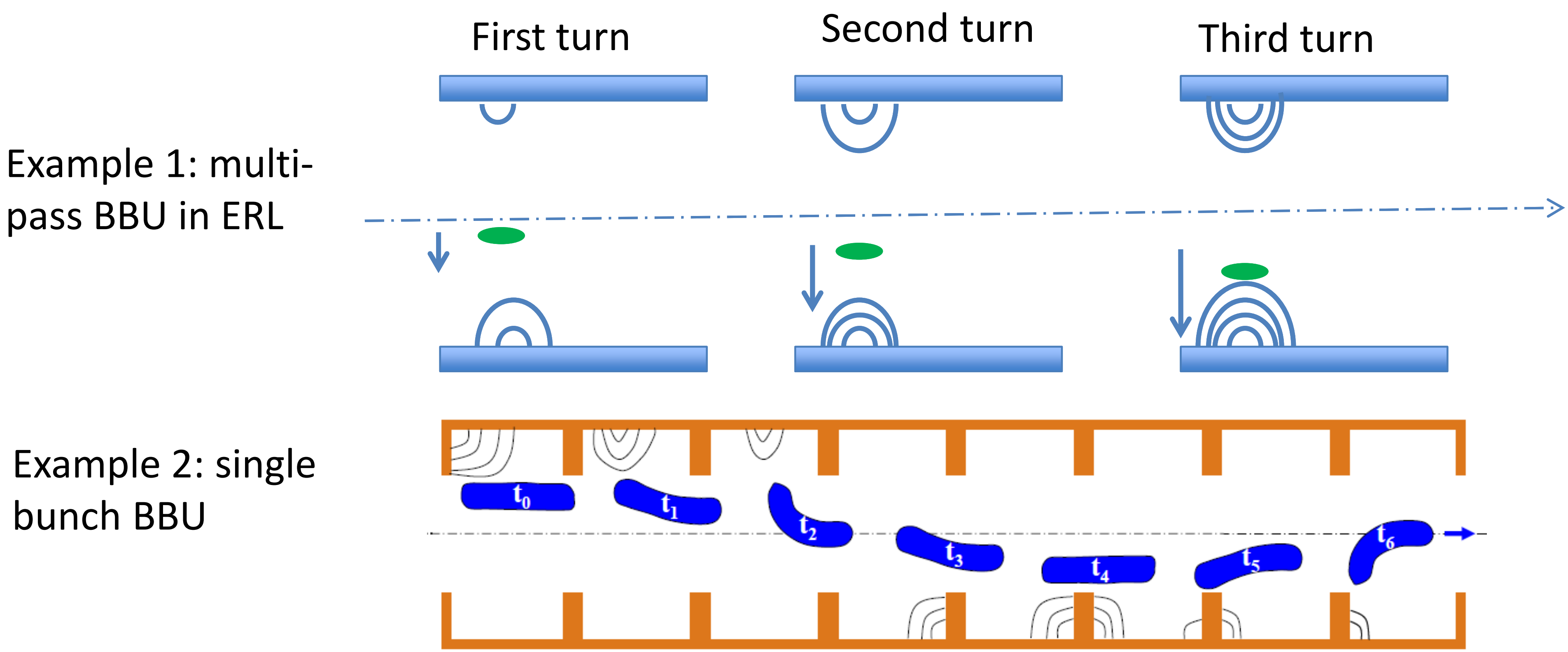


Beam interacts with machine: impedances-related instabilities.



Collective instabilities

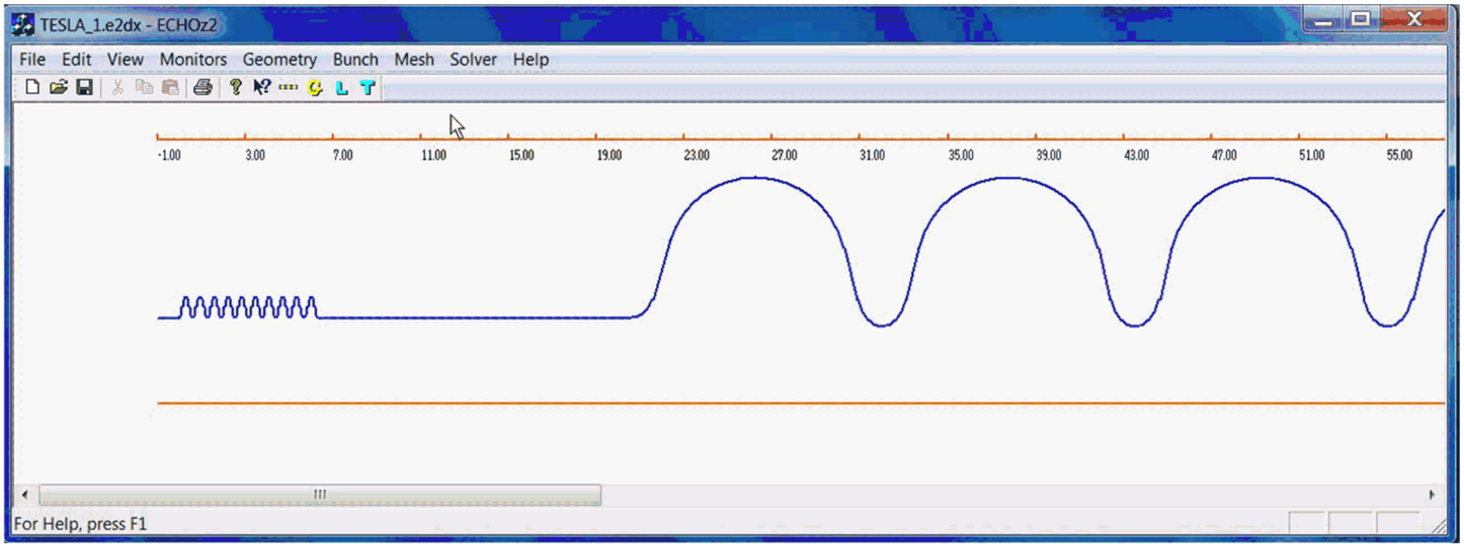
- The particle beam interacts with its surroundings to generate an electromagnetic field, known as wakefield. This field then acts back on the beam, perturbing its motion.
- Under unfavorable conditions, the perturbation on the beam are continuously enhanced by the wakefield, leading to the collective instabilities.



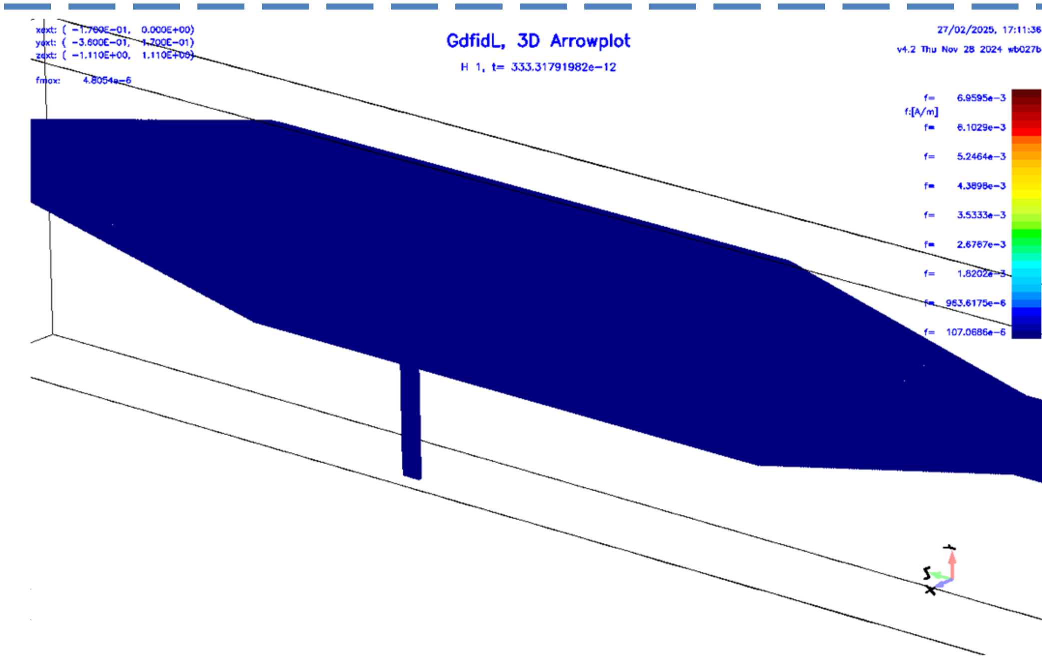
- **For the rest of the lecture, we will focus on a wakefield model developed for an ultra-relativistic beam,** $\gamma >> 1$

We are going to focus on Wakefield for today

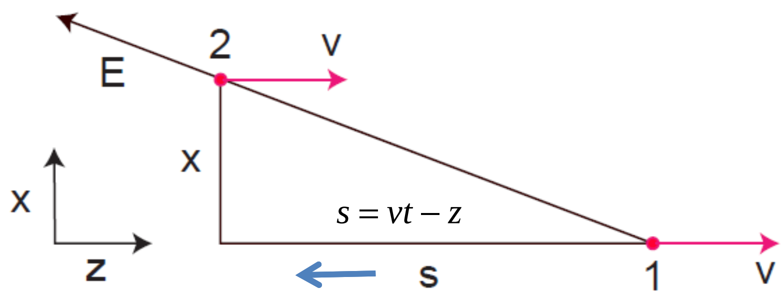
ECHO3D Simulation, taken from echo4d.de, Courtesy to I. Zagorodnov

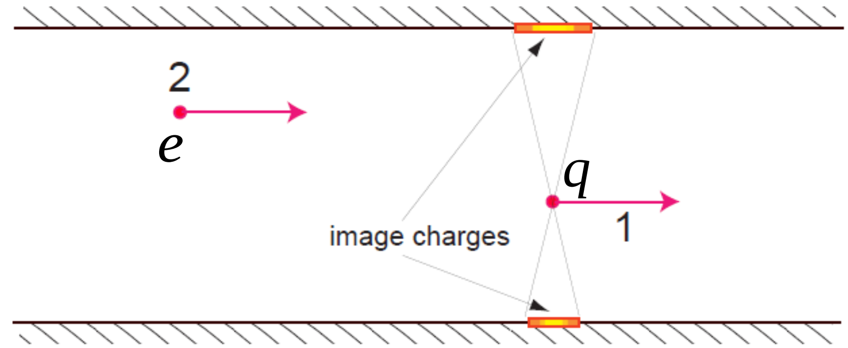


GdfidL Simulation,
EIC HSR polarimeter
Courtesy to Warner Bruns



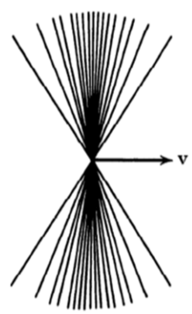
Ultra-relativistic beam and perfectly conducting beam pipe





$$\vec{E} = \frac{q\vec{R}}{4\pi\varepsilon_0\gamma^2 R^{*3}} \qquad \vec{B} = \frac{\vec{\beta}}{c}\times\vec{E}$$

$$R^{*2} = s^2 + x^2/\gamma^2$$



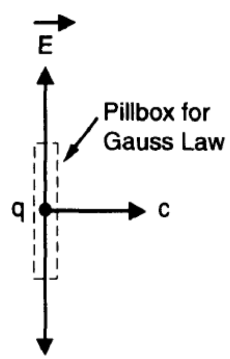
$$F_z = qE_z = -\frac{qes}{4\pi\varepsilon_0\gamma^2\left(s^2 + x^2/\gamma^2\right)^{3/2}}$$

$$F_x = q\left(E_x - c\beta B_y\right) = \frac{qex}{4\pi\varepsilon_0\gamma^4\left(s^2 + x^2/\gamma^2\right)^{3/2}}$$

At the limit of $\gamma \to \infty$

$$\vec{E} = \frac{q\hat{r}}{2\pi\varepsilon_0 r}\delta(z - ct)$$

$$\vec{B} = \frac{\hat{z}}{c}\times\vec{E}$$



For $\gamma \to \infty$, interaction among the particles and their images from the wall vanishes if

1. the wall is perfectly conducting, and
2. there are no discontinuities (cavities, bpms, bellows…).

(It is also assumed that particles go straight, i.e. no radiations from particles)

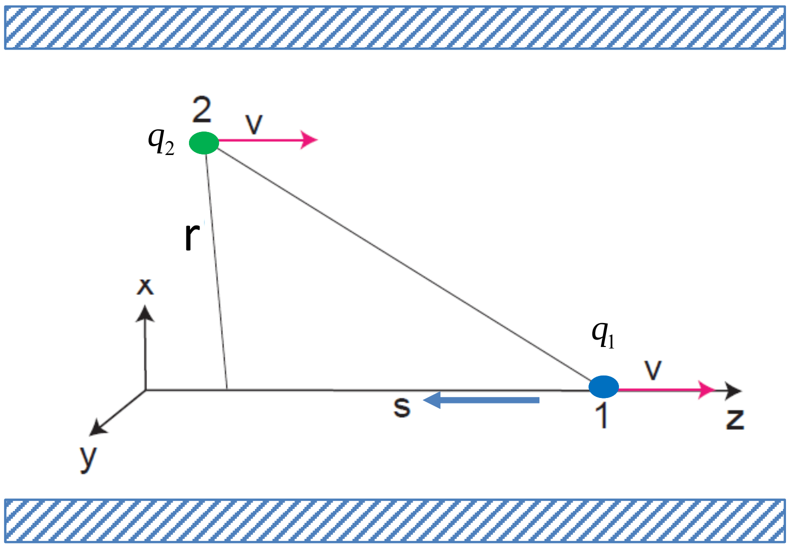
Wake Functions

- Rigid bunch approximation:

the motion of particles is not affected while passing through the structure

- Impulse approximation:

instead of the detailed E&M field in the structure, we care more about the total momentum change to the particles due to the wake field:



$$\Delta\vec{p}(x,y,s)=q_2\int_{-\infty}^{\infty}dt\left[\vec{E}(x,y,z,t)+c\hat{z}\times\vec{B}(x,y,z,t)\right]_{z=ct-s}$$

Longitudinal wake function*: $$w_l(x,y,s)=-\frac{c}{q_1q_2}\Delta p_z=-\frac{c}{q_1}\int_{-\infty}^{\infty}E_z(x,y,ct-s,t)\,dt\ \text{[V/C]}$$

Transverse wake function*: $$\vec{w}_t(x,y,s)=\frac{c}{q_1q_2}\Delta\vec{p}_\perp=\frac{c}{q_1}\int_{-\infty}^{\infty}\left[\vec{E}_\perp(x,y,z,t)+c\hat{z}\times\vec{B}(x,y,z,t)\right]_{z=ct-s}dt$$

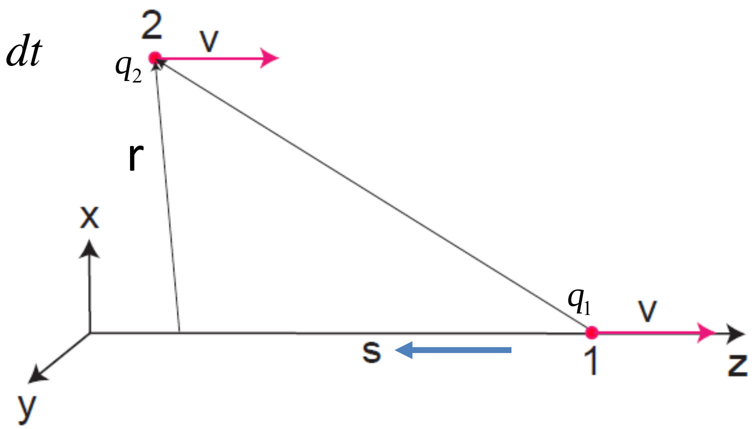
[V/C]

* These definition follow from ‘Impedances and Wakes in High-Energy Particle Accelerators’ by B. Zotter, which is different from those in ‘ Physics of Collective Beam Instabilities in High Energy Accelerators’ by A. Chao.

Panofsky-Wenzel Theorem

We want to find relation between longitudinal wake function and transverse wake function due to a structure (a piece of beam pipe, bpm, bellow, cavity....)

$$\nabla_s \times \Delta\vec{p}(x,y,s) = \nabla_s \times \int_{-\infty}^{\infty} \vec{F}(x,y,z,t)\Big|_{z=vt-s} dt = \int_{-\infty}^{\infty} \left[\nabla \times \vec{F}(x,y,z,t)\right]\Big|_{z=vt-s} dt$$



$$\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} - \hat{z}\frac{\partial}{\partial s} \qquad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{F} = q_2\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\nabla\times\vec{F} = q_2\nabla\times\vec{E} + q_2\nabla\times\left(\vec{v}\times\vec{B}\right)$$

$$= -q_2\frac{\partial\vec{B}}{\partial t} + q_2\vec{v}\left(\nabla\cdot\vec{B}\right) - q_2\left(\vec{v}\cdot\nabla\right)\vec{B}$$

$$= -q_2\frac{\partial\vec{B}}{\partial t} - q_2 v\frac{\partial}{\partial z}\vec{B}$$

$$\vec{v} = v\hat{z}$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

We assume the B field due to the structure has limited spatial range, i.e. it is localized.

$$\nabla_s\times\Delta\vec{p}(x,y,s) = -q_2\int_{-\infty}^{\infty}\left[\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial z}\right)\vec{B}(x,y,z,t)\right]_{z=vt-s} dt = -q_2\int_{-\infty}^{\infty}\frac{d}{dt}\vec{B}(x,y,vt-s,t)dt = 0$$

$$\frac{\partial}{\partial s}\Delta\vec{p}_\perp = -\vec{\nabla}_\perp\Delta p_z \quad \Longrightarrow \quad \boxed{\frac{\partial}{\partial s}\vec{w}_t(x,y,s) = \vec{\nabla}_\perp w_l(x,y,s)}$$

This is called Panofsky-Wenzel theorem

$$\left[\nabla_\perp\times\Delta\vec{p}_\perp(x,y,s)\right]\cdot\hat{z} = 0$$

*The derivation follows from USPAS note by K.Y. Ng.

Another Relation at $\beta \to 1$

$$\nabla_s \cdot \Delta\vec{p}(x,y,s) = \nabla_s \cdot \int_{-\infty}^{\infty} \vec{F}(x,y,z,t)\Big|_{z=vt-s} dt = \int_{-\infty}^{\infty} \left[\nabla \cdot \vec{F}(x,y,z,t)\right]\Big|_{z=vt-s} dt$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{j} = \rho v \hat{z}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 \frac{\partial}{\partial t} \vec{E} \right)$$

$$\Downarrow$$

$$\begin{aligned}
\nabla \cdot \vec{F} &= q_2 \nabla \cdot \vec{E} + q_2 \nabla \cdot \left(\vec{v} \times \vec{B}\right) \\
&= q_2 \frac{\rho}{\varepsilon_0} - q_2 \vec{v} \cdot \left(\nabla \times \vec{B}\right) \\
&= q_2 \frac{\rho}{\varepsilon_0} - q_2 \mu_0 \vec{v} \cdot \left(\rho v \hat{z} + \varepsilon_0 \frac{\partial}{\partial t} \vec{E} \right) \\
&= q_2 \frac{\rho}{\varepsilon_0 \gamma^2} - \frac{q_2}{c} \beta \frac{\partial}{\partial t} E_z \\
&\approx -\frac{q_2}{c} \frac{\partial}{\partial t} E_z
\end{aligned}$$

$$\begin{aligned}
\nabla_s \cdot \Delta\vec{p}(x,y,s) &= -\frac{q_2}{c} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t} E_z(x,y,z,t) \right]\Bigg|_{z=vt-s} dt \\
&= -\frac{q_2}{c} \int_{-\infty}^{\infty} \left\{ \frac{d}{dt} E_z(x,y,vt-s,t) - \left[v \frac{\partial}{\partial z} E_z(x,y,z,t) \right]\Bigg|_{z=vt-s} \right\} dt \\
&= \frac{q_2 v}{c} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z} E_z(x,y,z,t) \right]\Bigg|_{z=vt-s} dt \\
&= -\frac{q_2 v}{c} \frac{\partial}{\partial s} \int_{-\infty}^{\infty} E_z(x,y,vt-s,t)\, dt \\
&\approx -\frac{\partial}{\partial s} \Delta p_z(x,y,s)
\end{aligned}$$

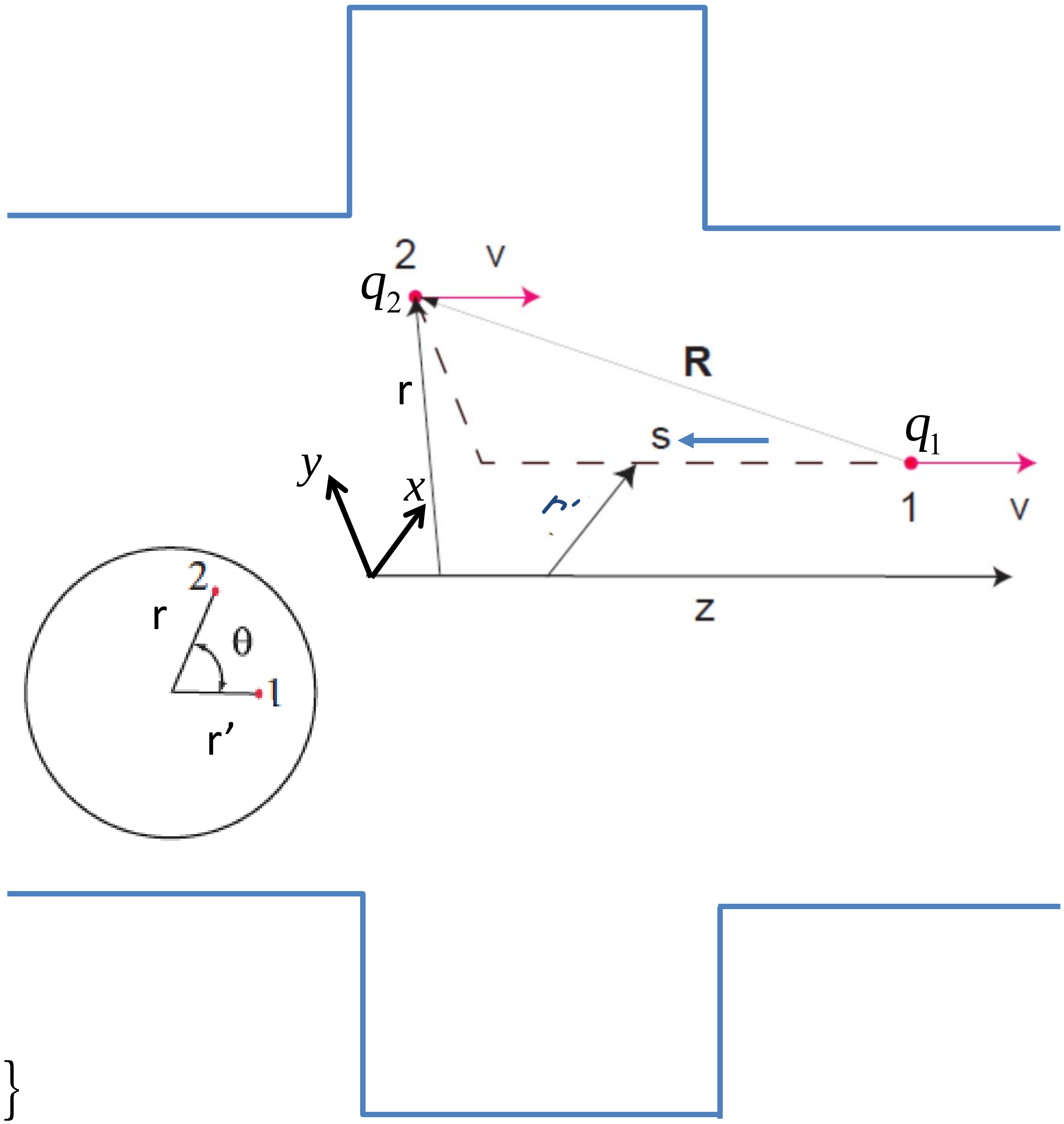
$$\boxed{\nabla_\perp \cdot \Delta\vec{p}(x,y,s) = 0}$$

Cylindrical symmetric structure I

$$\nabla_s \times \Delta\vec{p}(r',r,\theta,z) = 0$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial s}\Delta\vec{p}_\perp = -\vec{\nabla}_\perp \Delta p_z \\ \\ \left[\nabla_\perp \times \Delta\vec{p}_\perp\right]\cdot\hat{z} = 0 \end{cases}$$

$$\nabla_\perp \cdot \Delta\vec{p}(r',r,\theta,s) = 0$$



For a system with cylindrical symmetry, it is usually more convenient to decompose quantities into azimuthal modes:

$$\Delta\vec{p}(r',r,\theta,s) \sim \{\cos(m\theta), \sin(m\theta)\}$$

Cylindrical symmetric structure II

$$\frac{\partial}{\partial s}\Delta\vec{p}_\perp = -\vec{\nabla}_\perp \Delta p_z \quad \Rightarrow \begin{cases} \dfrac{\partial}{\partial s}\Delta p_\theta = -\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\Delta p_z \\ \dfrac{\partial}{\partial s}\Delta p_r = -\dfrac{\partial}{\partial r}\Delta p_z \end{cases}$$

$$\Delta p_r(r',r,\theta,s) = \sum_{m=0}^{\infty}\Delta\tilde{p}_{m,r}(r',r,s)\cos(m\theta)$$

$$\Longrightarrow \Delta p_z(r',r,\theta,s) = \sum_{m=0}^{\infty}\Delta\tilde{p}_{m,z}(r',r,s)\cos(m\theta)$$

$$\left[\nabla_\perp \times \Delta\vec{p}_\perp(r',r,\theta,s)\right]\cdot\hat{z} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial r}(r\Delta p_\theta) = \frac{\partial}{\partial\theta}\Delta p_r \quad \Longrightarrow \quad \Delta p_\theta(r',r,\theta,s) = \sum_{m=0}^{\infty}\Delta\tilde{p}_{m,\theta}(r',r,s)\sin(m\theta)$$

$$\vec{\nabla}_\perp \cdot \Delta\vec{p}(r',r,\theta,s) = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial r}(r\Delta p_r) = -\frac{\partial}{\partial\theta}\Delta p_\theta$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial s}\Delta\tilde{p}_{m,\theta} = \dfrac{m}{r}\Delta\tilde{p}_{m,z} & \dfrac{\partial}{\partial s}\Delta\tilde{p}_{m,r} = -\dfrac{\partial}{\partial r}\Delta\tilde{p}_{m,z} \\ \dfrac{\partial}{\partial r}(r\Delta\tilde{p}_{m,\theta}) = -m\Delta\tilde{p}_{m,r} & \dfrac{\partial}{\partial r}(r\Delta\tilde{p}_{m,r}) = -m\Delta\tilde{p}_{m,\theta} \end{cases}$$

$$\Longrightarrow \quad \frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(r\Delta\tilde{p}_{m,r}(r',r,s)\right)\right] - m^2\Delta\tilde{p}_{m,r}(r',r,s) = 0$$

$$\Delta\tilde{p}_{m,r}(r,s) = A_m(r',s)\,m r^{m-1} + \cancel{B_m(r',s)\,r^{-m-1}}$$

Cylindrical symmetric structure III

By analyzing the source term in the Maxwell equations, it can be shown that the driving term has an explicit dependence on r'

$$\rho = \sum_{m=0}^{\infty} \rho_m \qquad \vec{j} = \sum_{m=0}^{\infty} \vec{j}_m, \quad \vec{j}_m = c\rho_m \hat{s},$$

$$\rho_m = \frac{I_m}{\pi a^{m+1}(1+\delta_{m0})}\delta(s-ct)\delta(r-a)\cos m\theta,$$

$$a \Leftrightarrow r'$$

$$\vec{E}, \vec{B} \sim \boxed{I_m = qa^m}$$

$$A_m(r',s) = \frac{q_1 q_2}{v} r'^m W_m(s)$$

*Reference: A. Chao 'Physics of Collective Beam Instabilities in High Energy Accelerators', eq. (2.35)

$$\Delta\tilde{p}_{m,r}(r,s) = A_m(r',s)\, m r^{m-1}$$

$$\Delta\tilde{p}_{m,\theta}(r',r,s) = -A_m(r',s)\, m r^{m-1}$$

$$\Delta\tilde{p}_{m,z} = -A'_m(r',s)\, r^m$$

Analyzing the source term in the Maxwell equations

$$\Delta\tilde{p}_{m,r}(r,s) = \frac{q_1 q_2}{v} r'^m W_m(s)\, m r^{m-1}$$

$$\Delta\tilde{p}_{m,\theta}(r',r,s) = -\frac{q_1 q_2}{v} r'^m W_m(s)\, m r^{m-1}$$

$$\Delta\tilde{p}_{m,z}(r',r,s) = -\frac{q_1 q_2}{v} r'^m W'_m(s)\, r^m$$

Dependence of E&M field on the offset of leading particle

$$\tilde{E}_r = \begin{cases} -\frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(-\frac{imA}{k} + B - \frac{4I_m}{a^{2m}}\right)r^{m-1}, & \text{For } r<a \\ \frac{2I_m}{r^{m+1}} - \frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(-\frac{imA}{k} + B\right)r^{m-1}, & \text{For } a<r<b \end{cases}$$

$$\tilde{E}_\theta = \begin{cases} -\frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(\frac{imA}{k} - B + \frac{4I_m}{a^{2m}}\right)r^{m-1}, \\ \frac{2I_m}{r^{m+1}} - \frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(\frac{imA}{k} - B\right)r^{m-1}, \end{cases}$$

$$\tilde{B}_r = \begin{cases} \frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(\frac{imA}{k} + B - \frac{4I_m}{a^{2m}}\right)r^{m-1}, \\ -\frac{2I_m}{r^{m+1}} + \frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(\frac{imA}{k} + B\right)r^{m-1}, \end{cases}$$

$$\tilde{B}_\theta = \begin{cases} -\frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(\frac{imA}{k} + B - \frac{4I_m}{a^{2m}}\right)r^{m-1}, \\ \frac{2I_m}{r^{m+1}} - \frac{ikA}{2(m+1)}r^{m+1} + \frac{1}{2}\left(\frac{imA}{k} + B\right)r^{m-1}, \end{cases} \qquad (2.35)$$

$$\boxed{I_m = qa^m}$$

- In the region a<r<b, the constant A and B are determined by the boundary condition at the wall, hence, they are proportional to I_m .
- In the region r<a, the dependence may not be correct, for example, the space charge impedance (r<a)

$$W_m(z) = \frac{2L}{\gamma^2}\delta(z)\begin{cases} \ln\frac{b}{a} & \text{if } m=0, \\ \frac{1}{m}\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right) & \text{if } m>0, \end{cases} \qquad (2.55)$$

Cylindrical Symmetric Structure IV

$$w_l(r',r,\theta,s) = -\frac{c}{q_1 q_2}\Delta p_z(r',r,\theta,s) = \sum_{m=0}^{\infty} W'_m(s)\, r'^m\, r^m \cos(m\theta)$$

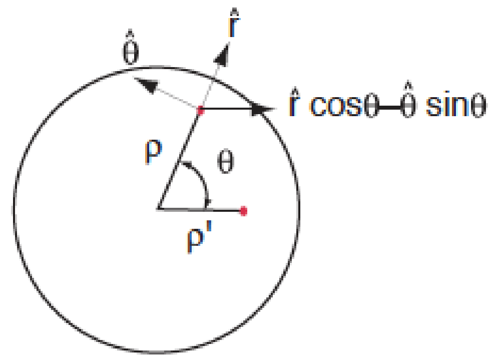
$$\vec{w}_t(r',r,\theta,s) = \frac{c}{q_1 q_2}\Delta \vec{p}_\perp(r',r,\theta,s) = \sum_{m=0}^{\infty} W_m(s)\, m r'^m\, r^{m-1}\left[\cos(m\theta)\hat{r} - \sin(m\theta)\hat{\theta}\right]$$

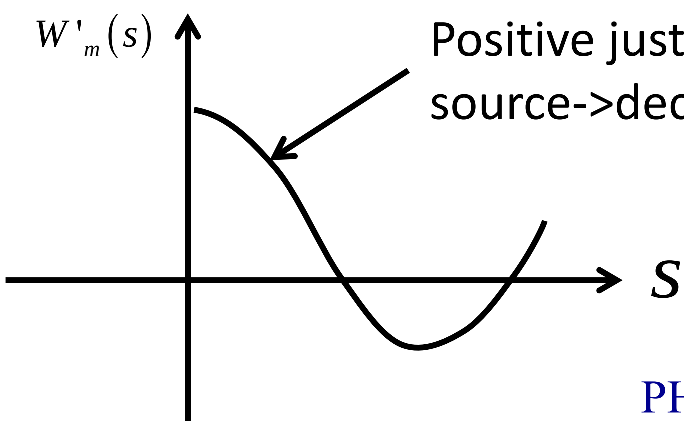
* In many references (by A. Chao, K.Y. Ng ...), $W_m(s)$ and $W'_m(s)$ are called wake functions.

$m=0$ $\quad w_l(r',r,\theta,s) = W'_0(s) \quad w_t(r',r,\theta,s) = 0$

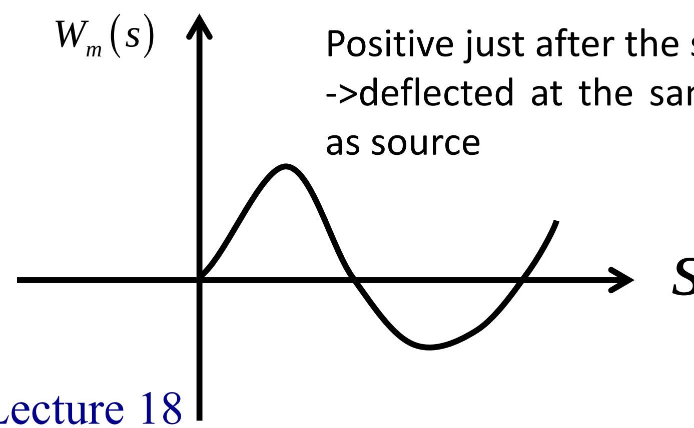
$m=1$ $\quad \vec{w}_t(r',r,\theta,s) = W_1(s)\, r'\left[\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right]$

$\quad w_l(r',r,\theta,s) = W'_1(s)\, r' r \cos(\theta)$





Positive just after the source->deceleration



Positive just after the source ->deflected at the same direction as source

Wake Potential

- In practice, usually only monopole mode (m=0) wake is considered for longitudinal wake field and only dipole mode (m=1) is considered for transverse mode.

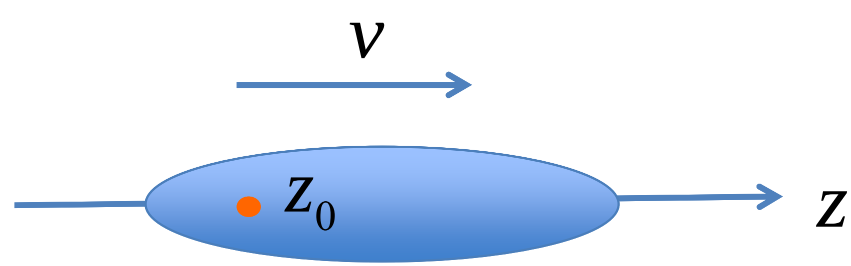
monopole longitudinal wake: $w_{//}(s) = W'_0(s)$ V/C

dipole transverse wake: $w_{\perp}(s) = W_1(s)$ V/(C*m)

- Wake potentials are defined to describe the momentum change induced by all particles in a bunch to a test unit charge:

$$V_{//}(z_0) = \frac{-c\Delta p_z(z_0)}{eQ_e} = \int_{z_0}^{\infty} \lambda(z_1) w_{//}(z_1 - z_0) dz_1 \quad \text{[V/C]}$$

$$\vec{V}_{\perp}(z_0) = \frac{c\Delta\vec{p}_{\perp}(z_0)}{eQ_e} = \int_{z_0}^{\infty} \langle \vec{x}_{\perp}(z_1) \rangle \lambda(z_1) w_{\perp}(z_1 - z_0) dz_1 \quad \text{[V/C]}$$



$\lambda(z)$ is line number density of a bunch

* If we observe at $z = z^*$ and use arriving time, $t = \frac{1}{c}(z^* - z)$ as longitudinal variables, above definition become

$$V_{//}(t_0) = \int_{-\infty}^{t_0} \lambda(t_1) w_{//}(t_0 - t_1) dt_1 \qquad \vec{V}_{\perp}(t_0) = \int_{-\infty}^{t_0} \langle \vec{x}_{\perp}(t_1) \rangle \lambda(t_1) w_{\perp}(t_0 - t_1) dt_1$$

Loss Factor and Kick Factor

- Once the longitudinal wake potential is known, the total energy change of a bunch to the wakefields is given by

$$\Delta U = -\int_{-\infty}^{\infty} \underbrace{\left[Q_e V_{//}(z)\right]}_{\text{Potential at slice (z,z+dz)}} \underbrace{\left[Q_e \lambda(z)\right]}_{\text{Charge in slice (z,z+dz)}} dz$$

Definition of Loss Factor: $\boxed{\kappa_{//} \equiv \frac{-\Delta U}{Q_e^2} = \int_{-\infty}^{\infty} V_{//}(z)\lambda(z)\,dz}$ [V/C]

- Similarly, the total transverse momentum change of a bunch to the wakefields is given by

$$\Delta \vec{P}_{\perp} = \int_{-\infty}^{\infty} \underbrace{\left[eQ_e \frac{\vec{V}_{\perp}(z)}{c}\right]}_{\text{Transverse momentum change of a particle at slice (z,z+dz).}} \underbrace{\left[\frac{Q_e}{e}\lambda(z)\right]}_{\text{Particle number in slice (z,z+dz)}} dz$$

Definition of Kick Factor: $\boxed{\vec{\kappa}_{\perp} = \frac{c\Delta \vec{P}_{\perp}}{Q_e^2} = \int_{-\infty}^{\infty} V_x(z)\lambda(z)\,dz}$ [V/C]

Impedances

- Although the time domain description of particle-environment interaction, the wake fields, contains all information, it is often more convenient to describe the interaction in frequency domain (convolution vs multiplication, calculate wakes in frequency domain can be easier sometimes, solving beam instability problems...), i.e. the impedances

$$Z_{//}(\omega) = \frac{1}{c}\int_0^\infty w_{//}(s) e^{i\omega s/c}\, ds \qquad [s*V/C]=[Ohm]$$

$$Z_{\perp}(\omega) = -\frac{i}{c}\int_0^\infty w_{\perp}(s) e^{i\omega s/c}\, ds \qquad [s*V/(C*m)]=[Ohm/m]$$

*The frequency ω is frequently allowed to have an imaginary part, in that case the transformation is actually Laplace transform, which is only defined for $\mathrm{Im}(\omega)\geq 0$

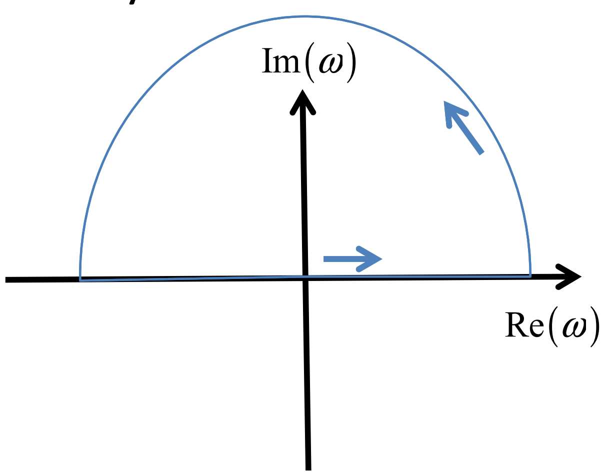
- The inverse transformations are

$$w_{//}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega) e^{-i\omega s/c}\, d\omega$$

$$w_{\perp}(s) = \frac{i}{2\pi}\int_{-\infty}^{\infty} Z_{\perp}(\omega) e^{-i\omega s/c}\, d\omega$$

* In complex ω plane, $Z_{//}(\omega)$ and $Z_{\perp}(\omega)$ should not have singularities in the upper half plane, i.e. $\mathrm{Im}(\omega)\geq 0$, in order to satisfy the causality condition:

$$w_{//}(s<0)=0 \qquad w_{\perp}(s<0)=0$$



Properties of Impedances

- Symmetry properties about positive and negative frequency

$$Z_{//}^*(\omega) = Z_{//}(-\omega) \quad \Rightarrow \begin{cases} \mathrm{Re}\left[Z_{//}(\omega)\right] = \mathrm{Re}\left[Z_{//}(-\omega)\right] \\ \mathrm{Im}\left[Z_{//}(\omega)\right] = -\mathrm{Im}\left[Z_{//}(-\omega)\right] \end{cases}$$

$$Z_{\perp}^*(\omega) = -Z_{\perp}(-\omega) \quad \Rightarrow \begin{cases} \mathrm{Re}\left[Z_{\perp}(\omega)\right] = -\mathrm{Re}\left[Z_{\perp}(-\omega)\right] \\ \mathrm{Im}\left[Z_{\perp}(\omega)\right] = \mathrm{Im}\left[Z_{\perp}(-\omega)\right] \end{cases}$$

- Relations between real part and imaginary part of impedances

$$w_{//}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega) e^{-i\omega s/c} d\omega \quad \Rightarrow w_{//}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\mathrm{Re}\left[Z_{//}(\omega)\right]\cos\left(\frac{\omega s}{c}\right) - \mathrm{Im}\left[Z_{//}(\omega)\right]\sin\left(\frac{\omega s}{c}\right)\right\} d\omega$$

$$w_l(s<0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\mathrm{Re}\left[Z_l(\omega)\right]\cos\left(\frac{\omega s}{c}\right) + \mathrm{Im}\left[Z_l(\omega)\right]\sin\left(\frac{\omega |s|}{c}\right)\right\} d\omega = 0 \quad \Rightarrow \int_{-\infty}^{\infty} \mathrm{Im}\left[Z_l(\omega)\right]\sin\left(\frac{\omega |s|}{c}\right) d\omega = -\int_{-\infty}^{\infty} \mathrm{Re}\left[Z_l(\omega)\right]\cos\left(\frac{\omega |s|}{c}\right) d\omega$$

$$\Rightarrow \quad w_l(s>0) = \frac{2}{\pi}\int_0^{\infty} \mathrm{Re}\left[Z_l(\omega)\right]\cos\left(\frac{\omega s}{c}\right) d\omega \qquad w_{\perp}(s>0) = \frac{2}{\pi}\int_0^{\infty} \mathrm{Re}\left[Z_{\perp}(\omega)\right]\sin\left(\frac{\omega s}{c}\right) d\omega$$

Kramers-Kronig relations:

$$Z_{//}(\omega) = -\frac{i}{\pi} P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega' - \omega} d\omega' \quad \Rightarrow \mathrm{Re}\left[Z_{//}(\omega)\right] = \frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[Z_{//}(\omega')\right]}{\omega' - \omega} d\omega' \qquad \mathrm{Im}\left[Z_{//}(\omega)\right] = -\frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left[Z_{//}(\omega')\right]}{\omega' - \omega} d\omega'$$

Many pictures and derivations used in the slides are taken from the following references:

[1] ‘Wake and Impedance’ by G.V. Stupakov, SLAC-PUB-8683;

[2] ‘Physics of Intensity Dependent Instabilities’ by K.Y. Ng, Lecture Notes in USPAS 2002;

[3] ‘Accelerator Physics’ by S.Y. Lee;

[4] ‘Physics of Collective Beam Instabilities in High Energy Accelerators’ by A. Chao;

[5] ‘Impedances and Wakes in High-Energy Particle Accelerators’ by B. Zotter and S. Kheifets.

What we learned today

- Apart from external fields (generated by magnets, cavities…), the motions of particles can be affected by fields induced by their own charge, either through direct Coulomb interactions (beam-beam, IBS…) or through their environment (wakefield). These effects are called collective effects, which can limit the performance of an accelerator.

- One of these adverse effects is called collective instabilities, which make particles significantly deviate from their designed trajectories (To be continued in the next class).

- For ultra relativistic particles, the collective instabilities are well described by a formalism using the quantities: **wakefields and impedances**. One of the advantages of using the formalism is that the effects are explicitly factorized into two parts: the beam (charge, distribution, bunch length…) and the environment (wakefields or impedances).