PHY 564
Advanced Accelerator Physics
Lecture 11
Full 3D linearized motion in Accelerators

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Lecture 11. Full 3D matrices. Chromatic effects. Longitudinal (energy and time, synchrotron) oscillations in storage rings.

While we considered in many details 2D (transverse) matrices, we left aside a more complicated (and heavier) full 3D linear matrices and stability in periodic systems. Here is a brief recollection of what can be done (usually by computers) for linearized motion in arbitrary accelerating system.

First, let's remember equation of motion for the reference particle, e.g. that whose trajectory

$$\begin{gathered} \vec{r}(s)=\vec{r}_o(s);\ f' \equiv \frac{df}{ds};\ \ \vec{\tau}=\vec{r}_o{}'(s);\ \vec{\tau}'=-|\vec{\tau}'|\vec{n};\vec{b}=[\vec{n}\times\vec{\tau}] \\ \vec{r}=\vec{r}_o(s)+x\cdot\vec{n}(s)+y\cdot\vec{b}(s); \\ \frac{d\vec{\tau}}{ds}=-K(s)\cdot\vec{n};\ \frac{d\vec{n}}{ds}=K(s)\cdot\vec{\tau}-\kappa(s)\cdot\vec{b};\ \frac{d\vec{b}}{ds}=\kappa(s)\cdot\vec{n}; \\ P_1=P_x;P_2=(1+Kx)P_s+\kappa\left(P_x y-P_y x\right);\ P_3=P_y; \end{gathered} \tag{11-1}$$

we use as a reference (*x=y=0, Px=Py=0*), whose arrival schedule

$$t(s)=t_o(s) \tag{11-2}$$

and energy (momentum)

$$cp_o(s)=\sqrt{E_o(s)^2-m^2c^4} \tag{11-3}$$

we follow. We should note that the Frenet-Serret coordinate system is uniquely defined when curvature of the reference trajectory:

$$K(s)\equiv\frac{1}{\rho(s)}=\left|\frac{d^2\vec{r}_o}{ds^2}\right|\equiv|\vec{\tau}'|\,. \tag{11-4}$$

is non zero. It also defines the direction of the normal and by-normal vectors $\vec{n},\vec{b}$:

$$\vec{n}=-\frac{\vec{r}_o{}''}{\left|\vec{r}_o{}''\right|};\vec{b}=\frac{1}{\left|\vec{r}_o{}''\right|}\left[\vec{r}_o{}'\times\vec{r}_o{}''\right]; \tag{11-5}$$

Note that curvature in (11-4) is positively defined. It means that even for a wiggly planar trajectories one either should use torsion (local rotation) or alternating sign of the curvature – we are using the later.

When curvature is zero (a straight line piece of the reference trajectory), both $\vec{n},\vec{b}$ are not uniquely defined, e.g. we can use torsion (rotation in $\vec{n},\vec{b}$ plane) as an instrument. We already used it for calculating matrices of solenoid and SQ-quadrupole. The only one important condition remains: you start from fully defined the $\vec{n},\vec{b}$ at the end the curved section, turn them around as many times and in any direction your want, but that you put the vectors $\vec{n},\vec{b}$ into the required directions at beginning of next curved section.

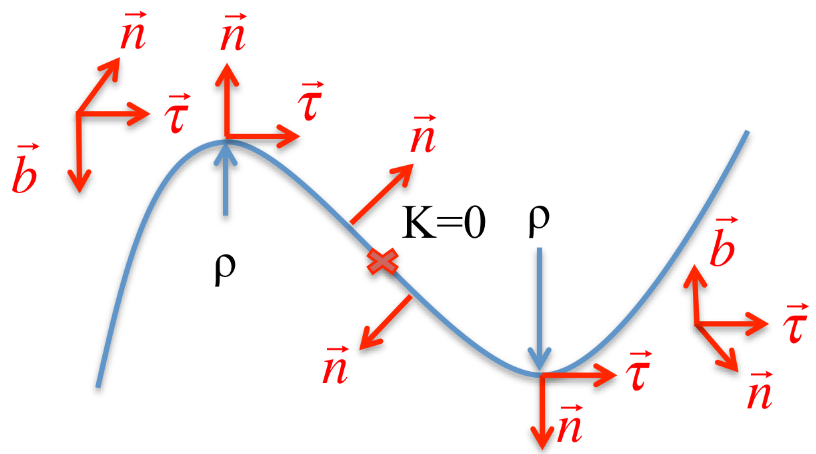


Fig.11-1. Even for a plane trajectory, the curvature direction $\vec{r}_o''$ can change, at some point (or at the straight line) the direction of the normal and bi-normal vectors has to flip (180-degrees rotation about $\vec{\tau}$) Hence, for such reference trajectories there must be either non-zero torsion (rotation) or provision for alternating sign of curvature. For planar trajectory defining direction of the bi-normal vector is sufficient to determine the sign of the curvature – usually it is selected to aim from the plain to the viewer.

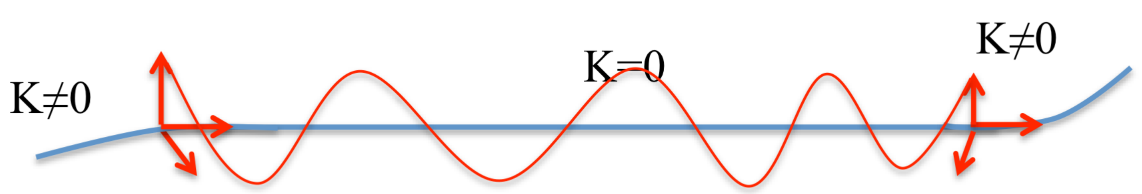


Fig. 11-2. Allowed manipulations with $\vec{n},\vec{b}$ in a straight section: it can be any helix with variable thread, but with two unit vectors $\vec{n},\vec{b}$ perpendicular to each other $\vec{b} = [\vec{n} \times \vec{\tau}]$.

We derived the conditions for the reference particle with

$$\vec{r}=\vec{r}_o(s);\ t=t_o(s);\ \ H=H_o(s)=E_o(s)+\varphi_o(s,t_o(s)) \tag{11-6}$$

as:

$$\begin{gathered}
K(s)=-\frac{e}{p_o(s)c}\left(B_y\left(0,0,s,t_o(s)\right)+\frac{c}{\mathrm{v}_o}E_x\left(0,0,s,t_o(s)\right)\right);\\
B_x\left(0,0,s,t_o(s)\right)=\frac{c}{\mathrm{v}_o}E_y\left(0,0,s,t_o(s)\right);\\
\frac{dt_o(s)}{ds}=\frac{1}{\mathrm{v}_o(s)}=\frac{H_o\left(0,0,s,t_o(s)\right)-e\varphi_o\left(0,0,s,t_o(s)\right)}{p_o(s)c^2}\equiv\frac{E_o(s)}{p_o(s)c^2};\\
\frac{dE_o(s)}{ds}\equiv eE_s\left(0,0,s,t_o(s)\right).
\end{gathered} \tag{11-7}$$

We also introduced longitudinal Canonical pair, which has zero values for reference particle:

$$\left\{\tau=-c(t-t_o(s)),\ \delta=\left(H-E_o(s)-e\varphi_o(s,t)\right)/c\right\} \tag{11-8}$$

Finally, we had derived the complete expression for Hamiltonian, and also the linearized Hamiltonian for an arbitrary accelerator expanding about the point $f\left(x=0,y=0,s,t_o(s)\right)\to f|_{ref}$ in time and space. In normalized Canonical coordinates with $p_{norm}=mc$:

$$\begin{gathered}
\pi_x=\frac{P_1}{mc};\ \pi_y=\frac{P_3}{mc};\ \pi_\tau=\frac{\delta}{mc};\\
\mathcal{H}_L=\frac{mc}{p_o}\cdot\frac{\pi_x^{\ 2}+\pi_y^{\ 2}}{2}+\left(\frac{mc}{p_o}\right)^3\cdot\frac{\pi_\tau^{\ 2}}{2}+\\
L\left(x\pi_y-y\pi_x\right)+g_x x\pi_\tau+g_y y\pi_\tau+\\
+\frac{F}{mc}\frac{x^2}{2}+\frac{N}{mc}xy+\frac{G}{mc}\frac{y^2}{2}+\frac{U}{mc}\frac{\tau^2}{2}+\frac{F_x}{mc}x\tau+\frac{F_y}{mc}y\tau;
\end{gathered} \tag{11-9}$$

where we separated parts of the Hamiltonian into three lines: quadratic form of momenta (“kinetic”), products of momenta and coordinates (mixed) and quadratic form of coordinates (“potential”).

The coefficients of the Hamiltonian are:

$$K = -\frac{e}{p_o c}\left(B_y + \frac{c}{\mathrm{v}_o}E_x\right);\ c_p = \frac{mc}{p_o} = \frac{1}{\beta_o\gamma_o};$$

$$f = \frac{F}{mc} = -K\cdot\frac{e}{mc^2}\left(B_y + \frac{2c}{\mathrm{v}_o}E_x\right) - \frac{e}{mc^2}\left(\frac{\partial B_y}{\partial x} + \frac{\partial E_x}{\partial x}\right) + \frac{mc}{p_o}\left(\frac{eB_s}{2mc^2}\right)^2 + \frac{mc}{p_o}\left(\frac{eE_x}{p_o c}\right)^2;$$

$$g = \frac{G}{mc} = \frac{e}{mc^2}\left(\frac{\partial B_x}{\partial y} - \frac{c}{\mathrm{v}_o}\frac{\partial E_y}{\partial y}\right) + \frac{mc}{p_o}\left(\frac{eB_s}{2mc^2}\right)^2 + \frac{mc}{p_o}\left(\frac{eE_z}{p_o c}\right)^2; \qquad ;\ (11\text{-}10)$$

$$2n = \frac{2N}{mc} = \frac{e}{mc^2}\left(\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right) - \frac{c}{\mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right)\right) + K\cdot\frac{e}{mc^2}\left(B_x - \frac{2c}{\mathrm{v}_o}E_y\right) + \frac{mc}{p_o}\left(\frac{eE_z}{p_o c}\right)\left(\frac{meE_x}{p_o c}\right);$$

$$L = \kappa + \frac{e}{2p_o c}B_s;\quad u = \frac{U}{mc} = \frac{e}{mc\mathrm{v}_o}\frac{\partial E_s}{\partial ct};\ g_x = \left(\frac{mc}{p_o}\right)^2\frac{eE_x}{p_o c} - \frac{mc^2}{p_o \mathrm{v}_o}K;\ g_y = \left(\frac{mc}{p_o}\right)^2\frac{eE_y}{p_o c};$$

$$f_x = \frac{F_x}{mc} = \frac{e}{mc^2}\frac{\partial B_y}{\partial ct} + \frac{e}{mc\mathrm{v}_o}\frac{\partial E_x}{\partial ct};\ f_y = \frac{F_y}{mc} = -\frac{e}{mc^2}\frac{\partial B_x}{\partial ct} + \frac{e}{mc\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

Note, that the Hamiltonian (11-9) is dimensionless and its coefficients are either dimensionless or have dimension of 1/L or $1/L^2$. Not all coefficients are important in all case. For example, in ultra-relativistic case, high powers of $mc/p_o = 1/\beta_o\gamma_o$ can be neglected. First will disappear terms in red, then in blue terms could become weak (but not always negligible – beware of this!)

$$\pi_x = \frac{P_1}{mc}; \quad \pi_y = \frac{P_3}{mc}; \quad \pi_\tau = \frac{\delta}{mc};$$

$$\begin{aligned} \mathcal{H}_L = {} & \frac{mc}{p_o} \cdot \frac{\pi_x^{\,2} + \pi_y^{\,2}}{2} + \left(\frac{mc}{p_o}\right)^3 \cdot \frac{\pi_\tau^{\,2}}{2} + \\ & L\left(x\pi_y - y\pi_x\right) + g_x x\pi_\tau + g_y y\pi_\tau + \\ & + \frac{F}{mc}\frac{x^2}{2} + \frac{N}{mc}xy + \frac{G}{mc}\frac{y^2}{2} + \frac{U}{mc}\frac{\tau^2}{2} + \frac{F_x}{mc}x\tau + \frac{F_y}{mc}y\tau; \end{aligned} \tag{11-9}$$

$$K = -\frac{e}{p_o c}\left(B_y + \frac{c}{\mathrm{v}_o}E_x\right); \; c_p = \frac{mc}{p_o} = \frac{1}{\beta_o \gamma_o};$$

$$\begin{aligned} f &= \frac{F}{mc} = -K \cdot \frac{e}{mc^2}\left(B_y + \frac{2c}{\mathrm{v}_o}E_x\right) - \frac{e}{mc^2}\left(\frac{\partial B_y}{\partial x} + \frac{\partial E_x}{\partial x}\right) + \frac{mc}{p_o}\left(\frac{eB_s}{2mc^2}\right)^2 + \frac{mc}{p_o}\left(\frac{eE_x}{p_o c}\right)^2; \\ g &= \frac{G}{mc} = \frac{e}{mc^2}\left(\frac{\partial B_x}{\partial y} - \frac{c}{\mathrm{v}_o}\frac{\partial E_y}{\partial y}\right) + \frac{mc}{p_o}\left(\frac{eB_s}{2mc^2}\right)^2 + \frac{mc}{p_o}\left(\frac{eE_z}{p_o c}\right)^2; \\ 2n &= \frac{2N}{mc} = \frac{e}{mc^2}\left(\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right) - \frac{c}{\mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right)\right) + K \cdot \frac{e}{mc^2}\left(B_x - \frac{2c}{\mathrm{v}_o}E_y\right) + \frac{mc}{p_o}\left(\frac{eE_z}{p_o c}\right)\left(\frac{meE_x}{p_o c}\right); \\ L &= \kappa + \frac{e}{2p_o c}B_s; \quad u = \frac{U}{mc} = \frac{e}{mc\mathrm{v}_o}\frac{\partial E_s}{\partial ct}; \; g_x = \left(\frac{mc}{p_o}\right)^2 \frac{eE_x}{p_o c} - \frac{mc^2}{p_o \mathrm{v}_o}K; \; g_y = \left(\frac{mc}{p_o}\right)^2 \frac{eE_y}{p_o c}; \\ f_x &= \frac{F_x}{mc} = \frac{e}{mc^2}\frac{\partial B_y}{\partial ct} + \frac{e}{mc\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; \; f_y = \frac{F_y}{mc} = -\frac{e}{mc^2}\frac{\partial B_x}{\partial ct} + \frac{e}{mc\mathrm{v}_o}\frac{\partial E_y}{\partial ct}. \end{aligned} \tag{11-10}$$

We can easily write matrix form of the Hamiltonian and **D**-matrix and to derive cubic equation for its eigen values. We discussed that eigen values of **D**-matrix are coming in pairs with of eigen values with opposite sign, with reduces equation to a bi-quadratic of power n:

$$p(\lambda) = \det[\mathbf{D} - \lambda\mathbf{I}] = \prod_{k=1}^{3}(\lambda - \lambda_k)(\lambda + \lambda_k) = \prod_{k=1}^{3}(\lambda^2 - \lambda_k{}^2) = \lambda^6 + a_4\lambda^4 + a_2\lambda^2 + a_o = 0 \qquad (11\text{-}11)$$

In detail:

$$\mathbf{H} = \begin{pmatrix} f & 0 & n & L & f_x & g_x \\ 0 & \frac{mc}{p_o} & -L & 0 & 0 & 0 \\ n & -L & g & 0 & f_y & g_y \\ L & 0 & 0 & \frac{mc}{p_o} & 0 & 0 \\ f_x & 0 & f_y & 0 & u & 0 \\ g_x & 0 & g_y & 0 & 0 & \left(\frac{mc}{p_o}\right)^3 \end{pmatrix}; \quad \mathbf{D} = \begin{pmatrix} 0 & \frac{mc}{p_o} & -L & 0 & 0 & 0 \\ -f & 0 & -n & -L & -f_x & -g_x \\ L & 0 & 0 & \frac{mc}{p_o} & 0 & 0 \\ -n & L & -g & 0 & -f_y & -g_y \\ g_x & 0 & g_y & 0 & 0 & \left(\frac{mc}{p_o}\right)^3 \\ -f_x & 0 & -f_y & 0 & -u & 0 \end{pmatrix}. \qquad (11\text{-}12)$$

One better use Mathematica to get expressions for coefficients: some are relatively short…

$$a_4 = \sum_{k=1}^{3} \lambda_k^2 = 2L^2 + c_p\left(f + g + \left(\frac{mc}{p_o}\right)^2 u\right); c_p \equiv \frac{mc}{p_o};$$

$$a_2 = -\left(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2\right) = L^4 - c_p{}^4\left(f_x{}^2 + f_y{}^2\right) + 2c_p{}^3 L^2 u + c_p{}^2\left(f \cdot g - n^2\right) + \tag{11-13}$$

$$+c_p\left(4L\left(f_x g_y - f_y g_x\right) - L^2\left(f + g\right) + u\left(f + g - g_x{}^2 - g_y{}^2\right)\right)$$

but the main coefficient is rather long and ugly:

$$a_o = Det[\mathbf{D}] = \prod_{k=1}^{3}\left(-\lambda_k^2\right) = -\lambda_1^2\lambda_2^2\lambda_3^2 =$$

$$c_p{}^5\left(f \cdot g \cdot u + 2nf_x f_y - f \cdot f_y{}^2 - g \cdot f_x{}^2 - n^2 u\right) + c_p{}^4 L^2\left(f_x{}^2 + f_y{}^2 + f \cdot u + g \cdot u\right) \tag{11-14}$$

$$+c_p{}^3 L^4 u + c_p{}^2\left(\left(f_y g_x - f_x g_y\right)^2 + u\left(2g_x g_y n - g \cdot g_x{}^2 - f \cdot g_y{}^2\right)\right) + c_p L^2 u\left(g_x{}^2 + g_y{}^2\right).$$

Naturally, the cubic equation can be solved analytically:

$$\lambda_1^2 = -\frac{1}{6}\left(2a_4 + \frac{2^{4/3}\left(3a_2 - a_4{}^2\right)}{q} - 2^{2/3} q\right);$$

$$\lambda_{2,3}^2 = -\frac{1}{6}\left(2a_4 - \frac{2^{1/3}\left(1 \pm \mathrm{i}\sqrt{3}\right)\left(3a_2 - a_4{}^2\right)}{q} \mp \mathrm{i} \cdot 2^{-1/3}\left(\mathrm{i} + \sqrt{3}\right) q\right); \tag{11-15}$$

$$q = \left(3\sqrt{3}\sqrt{27a_0{}^2 + 4a_2{}^3 - 18a_0 a_2 a_4 - a_2{}^2 a_4{}^2 + 4a_0 a_4{}^3} - 27a_0 + 9a_2 a_4 - 2a_4{}^3\right)^{1/3}.$$

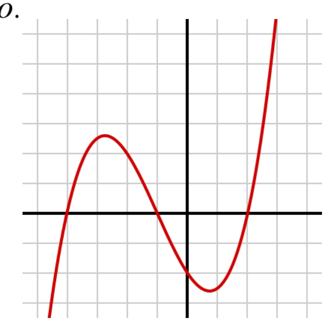
Number or real and complex λ_k^2 is determine by the discriminant of the cubic equation (11-11)

$$\Delta = 18a_o a_2 a_4 - 4a_o a_4{}^3 + a_2{}^2 a_4{}^2 - 4a_2{}^3 - 27a_o{}^2$$

Two other combinations play important role in defining branches of roots of cubic equation:

$$\Delta_o = a_4{}^2 - 3a_2;\ \Delta_1 = 2a_4{}^3 - 9a_4 a_2 + 27a_0.$$

If Δ > 0, then the cubit equation has three distinct real roots:



This corresponds to six distinct eigen values comprised of pairs $\lambda_k, -\lambda_k$ *when none of* λ_k^2 *is zero.* λ_k *is real or purely imaginary depending on the sign of* λ_k^2.

If Δ = 0 then the cubic equation has a multiple root and all of its roots are real.

If $\Delta_o = 0$ than all λ_k^2 *are identical*

$$\lambda_k{}^2 = -\frac{a_4}{3}, k = 1,2,3$$

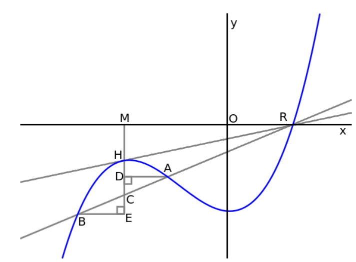
with sign of a_4 *defining if* λ_k *is real or purely imaginary.*

When $a_4 \neq 0$ *the level of degeneration (maximum height of the eigen vectors) is 3.*

When $a_4 = 0$, *all eigen vectors are zero and level of denervation can be 6. But requirement of* $a_4 = 0$, $\Delta_o = 0$ *and* $\Delta = 0$ *mean that* $a_2 = 0$ *and* $a_0 = 0$, *e.g. the characteristic equation is*

$$p(\lambda) = \det[\mathbf{D} - \lambda \mathbf{I}] = \lambda^6; \rightarrow p(\mathbf{D}) = \mathbf{D}^6 = 0 \rightarrow \exp(\mathbf{D}s) = \mathbf{I} + \sum_{n=1}^{5} \frac{\mathbf{D}^n s^n}{n!}$$

If $\Delta_o \neq 0$ than all λ_k^2 *are identical there is a double root*



$$\lambda_k^2 = \frac{9a_o - a_2 a_4}{2\Delta_o}; k = 1,2$$

and one unique

$$\lambda_3^2 = \frac{4a_2 a_4 - 9a_o - a_4{}^2}{\Delta_o}$$

and we have degeneration of at least second of second order.

If Δ < 0, then the cubic equation has one real root and two non-real complex conjugate roots. Generally this would correspond to a non-degenerated case.

https://en.wikipedia.org/wiki/Cubic_function

Even though, all of these expressions are explicit and analytical, it still preferable to give a computer to crack the numbers… and to use them to calculate step by step matrices using the most general Sylvester formula. It is especially true for case when we have time-dependent or accelerating fields and can not no longer rely on piece-wise constancy of the Hamiltonian matrix. Meanwhile, computes still can split the steps in sufficiently short steps in s and calculate the transport matrix (using Sylvester formula for exact symplecticity!) to an arbitrary good accuracy.

For periodic system (such as a storage ring), one than can solve cubic equation on $\left(\lambda+\lambda^{-1}\right)$ of its eigen values

$$\det\left[T(s)-\lambda I\right]=0 \rightarrow \left(\lambda+\lambda^{-1}\right)^3+b_2\left(\lambda+\lambda^{-1}\right)^2+b_1\left(\lambda+\lambda^{-1}\right)+b_o=0;$$

$$\mathbf{T}=\left[\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix}\right] \qquad (11\text{-}16)$$

$$p_6(\lambda)=\prod_{k=1}^{3}\left(\lambda^{-1}{}_k-\lambda\right)\left(\lambda_k-\lambda\right)=0;\quad \left(\lambda^{-1}{}_k-\lambda\right)\left(\frac{\lambda_k}{\lambda}\right)=-\left(\lambda_k-\lambda^{-1}\right);$$

$$\left(\lambda_k-\lambda^{-1}\right)\left(\lambda_k-\lambda\right)=\left(\lambda^2{}_k-\lambda_k\left(\lambda+\lambda^{-1}\right)+1\right)=-\lambda_k\left(\left(\lambda+\lambda^{-1}\right)-\left(\lambda_k+\lambda_k{}^{-1}\right)\right);$$

$$\frac{1}{\lambda^3}p_6(\lambda)=\prod_{k=1}^{3}\left(\left(\lambda+\lambda^{-1}\right)-\left(\lambda_k+\lambda_k{}^{-1}\right)\right)=\tilde{p}_3\left(\lambda+\lambda^{-1}\right)\rightarrow \tilde{p}_3\left(\lambda+\lambda^{-1}\right)=0.$$

$$\lambda^6+\alpha_5\lambda^5+\alpha_4\lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+1=0$$

$$\lambda^{-6}+\alpha_5\lambda^{-5}+\alpha_4\lambda^{-4}+\alpha_3\lambda^{-3}+\alpha_2\lambda^{-2}+\alpha_1\lambda^{-1}+1=0$$

$$1+\alpha_5\lambda+\alpha_4\lambda^2+\alpha_3\lambda^3+\alpha_2\lambda^4+\alpha_1\lambda^5+\lambda^6=0 \Rightarrow$$

$$\alpha_1=\alpha_5=-Trace[\mathbf{T}];$$

$$\alpha_2=\alpha_4=\det A+\det E+\det J+Tr(A)\cdot Tr(E)+Tr(A)\cdot Tr(J)+Tr(J)\cdot Tr(E)$$
$$-Tr(BD)-Tr(CG)-Tr(FH)$$

....

and check that the 3D motion is stable

$$\left|\lambda_k\right| = 1;\ \lambda_k = e^{i\mu_k};\ \mu_k = 2\pi Q_k;\ k = 1,2,3 \tag{11-17}$$

and define three eigen vectors and their complex conjugates:

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i\dfrac{q_{kx}}{\mathrm{w}_{kx}}\right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i\dfrac{q_{ky}}{\mathrm{w}_{ky}}\right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}}\right) e^{i\chi_{k\tau}} \end{bmatrix}; Y_k(s+C) = Y_k(s); T(s) Y_k(s) = e^{i\mu_k} Y_k(s); k = 1,2,3 \tag{2-18}$$

with the symplectic orthogonality relations that we already discussed:

$$Y_k^{\ T} S Y_j = 0;\ \ Y^{*}{}_j^{\ T} S Y_k = 2i\delta_{kj}; \tag{2-19}$$

which will apply multiple (15 to be exact!) relations on the component of the eigen vectors, with the simples being:

$$q_{kx} + q_{ky} + q_{k\tau} = 1;\ k = 1,2,3 \tag{2-20}$$

Frequently there are a lot simpler cases, some of which we going to consider.

Accelerator with constant energy – closed orbit.

One of the most used approximations (and simplification) is coming from the fact that in the most of the accelerators (especially in storage ring) longitudinal (or so called synchrotron) oscillations are very slow, when compared with transverse (or so called betatron) oscillations. Specifically, in most of typical storage rings it takes from few hundreds to few thousands of turns to complete one oscillation. Furthermore, in hadron storage ring, where losses on synchrotron are practically absent, one can operate beam in so-called coasting mode – e.g. without any AC fields. Thus, let's consider such an accelerator and study how particles motion depends on their energy (momentum p_o) and explicitly no time dependence.

$$\mathcal{H}_L = \frac{mc}{p_o}\cdot\frac{\pi_x^{\ 2}+\pi_y^{\ 2}}{2} + \left(\frac{mc}{p_o}\right)^3\cdot\frac{\pi_\tau^{\ 2}}{2} +$$

$$L\left(x\pi_y - y\pi_x\right) + g_x x\pi_\tau + g_y y\pi_\tau + \frac{F}{mc}\frac{x^2}{2} + \frac{N}{mc}xy + \frac{G}{mc}\frac{y^2}{2}; \qquad (11\text{-}21)$$

$$\pi_x = \frac{P_1}{mc};\ \ \pi_y = \frac{P_3}{mc};\ \ \pi_\tau = \frac{\delta}{mc};$$

Since the energy of the particle is constant but time is slipping:

$$\frac{d}{ds}\pi_\tau = -\frac{\partial H}{\partial \tau} = 0 \to \pi_\tau = const;$$

$$\frac{d}{ds}\tau_\tau = \frac{\partial H}{\partial \pi_\tau} = g_x x + g_y y + \left(\frac{mc}{p_o}\right)^3\cdot\pi_\tau \qquad (11\text{-}22)$$

we can simplify the equations of motion for 2D case plus energy dependence and time slippage:

$$\mathcal{H}_L = \mathcal{H}_\beta + \mathcal{H}_\delta; \; Z^T = \left(x, \pi_x, y, \pi_y\right);$$

$$\mathcal{H}_\beta = \frac{mc}{p_o} \cdot \frac{\pi_x{}^2 + \pi_y{}^2}{2} + \frac{F}{mc}\frac{x^2}{2} + \frac{N}{mc}xy + \frac{G}{mc}\frac{y^2}{2} + L\left(x\pi_y - y\pi_x\right); \tag{11-23}$$

$$\mathcal{H}_\delta = \left(\frac{mc}{p_o}\right)^3 \cdot \frac{\pi_\tau{}^2}{2} + g_x x \pi_\tau + g_y y \pi_\tau;$$

$$\frac{d}{ds}Z = D_\beta \cdot Z + \pi_\tau \cdot F_\delta; \;\; \pi_\tau \cdot C = S\frac{\partial}{\partial Z}H_\delta; \; F_\delta{}^T = C \cdot \begin{bmatrix} 0 & -g_x & 0 & -g_y \end{bmatrix};$$

or in explicit matrix form:

$$\frac{dZ}{ds} = D \cdot Z + \pi_\tau \cdot C; \quad D = \begin{bmatrix} 0 & 1 & -L & 0 \\ -f & 0 & -n & -L \\ L & 0 & 0 & 1 \\ -n & L & -g & 0 \end{bmatrix}; C = \begin{bmatrix} 0 \\ -g_x \\ 0 \\ -g_y \end{bmatrix}. \tag{11-24}$$

$$\frac{d\tau}{ds} = g_x x + g_y y + \left(\frac{mc}{p_o}\right)^3 \pi_\tau; \;\; g_x = \left(\frac{mc}{p_o}\right)^2 \frac{eE_x}{p_o c} - \frac{mc^2}{p_o \mathrm{v}_o}K; \; g_y = \left(\frac{mc}{p_o}\right)^2 \frac{eE_y}{p_o c}$$

We shall note that for ultra-relativistic particles (or in the absence of the electric fields!) only the curvature K of the trajectory remains as the driving term g_x for transverse motion. Solution for of the in-homogeneous equation for Z can be trivially expressed using 4x4 transport matrix:

$$Z = Z_\beta + \pi_\tau \cdot R;\ \frac{dZ_\beta}{ds} = D \cdot Z_\beta; \frac{dR}{ds} = D \cdot R + C;$$

$$\mathbf{M} \equiv \mathbf{M}_{4x4}; \frac{d\mathbf{M}(s)}{ds} = D \cdot \mathbf{M}(s);\quad Z_\beta(s) = \mathbf{M}(s) Z_{\beta o};$$

$$R(s) = \mathbf{M}(s) A(s);\quad \mathbf{M}(s)\frac{dA}{ds} = \pi_\tau \cdot C \Rightarrow \frac{dA}{ds} = \mathbf{M}^{-1}(s) C(s); \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S}; \tag{11-25}$$

$$\Rightarrow A(s) = \int_o^s \mathbf{M}^{-1}(\xi) C(\xi) d\xi;\ R(s) = \mathbf{M}(s)\left(A_o + \int_o^s \mathbf{M}^{-1}(\xi) C(\xi) d\xi \right).$$

$$R(s) = \int_o^s \mathbf{M}(\xi|s) C(\xi) d\xi; \eta$$

For periodic system we can find "periodic transverse orbit" for an off-momentum particle:

$$\eta(s+C) = \int_o^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi = \mathbf{T}(s)\eta(s) + \int_s^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi$$

$$\mathbf{M}(\xi|s+C) = \mathbf{T}(s)\mathbf{M}(\xi|s); \mathbf{T}(s) \equiv \mathbf{M}(s|s+C); \tag{11-26}$$

$$\eta(s+C) = \eta(s) \Rightarrow (\mathbf{I} - \mathbf{T})\eta(s) = \int_s^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi;$$

$$\eta(s) = (\mathbf{I} - \mathbf{T}(s))^{-1} \int_s^{s+C} \mathbf{M}(\xi|s+C) C(\xi) d\xi;\quad Z = Z_\beta + \pi_\tau \cdot \eta(s).$$

We will find expression for such closed periodical orbit expressed via eigen vectors – naturally the results would be identical. The $\eta = \begin{bmatrix} \eta_x & \eta_{px} & \eta_y & \eta_{py} \end{bmatrix}$- function is called transverse dispersion (picking analogy from optics). Unfortunately in accelerator physics terminology there is a number of confusions… and frequently the dispersion is represented by $D = \begin{bmatrix} D_x & D_{px} & D_y & D_{py} \end{bmatrix}$. Read the context to be sure…

Next natural step is to look onto the slippage of the particle in time for a particle without betatron oscillations $Z_\beta = 0$ (we will add them later):

$$Z = \pi_\tau \cdot \eta(s); \frac{d\tau}{ds} = \left(g_x\eta_x + g_y\eta_y + \left(\frac{mc}{p_o}\right)^3 \right)\pi_\tau;$$
$$\tau(s) = f_\tau(s)\pi_\tau; \quad f_\tau(s) = f_\tau(0) + \left(\frac{mc}{p_o}\right)^3 \cdot s + \int_0^s \left(g_x(\xi)\eta_x(\xi) + g_y(\xi)\eta_y(\xi)\right)d\xi. \tag{11-27}$$

First (red) term corresponds to the velocity dependence on the particles energy – it is weak for ultra-relativistic particles moving very-very close to the speed of the light, but it is important for hadron accelerators. Hence, we will keep it. Again, for a periodic system we

$$f_\tau(s+C) = f_\tau(s) + \left(\frac{mc}{p_o}\right)^3 \cdot C + \int_s^{s+C} \left(g_x(\xi)\eta_x(\xi) + g_y(\xi)\eta_y(\xi)\right)d\xi = \eta_\tau \cdot C;$$
$$\eta_\tau = \frac{1}{C}\int_0^C \left(g_x\eta_x + g_y\eta_y\right)ds + \left(\frac{mc}{p_o}\right)^3; \tag{11-28}$$

It worth expressing it for a simple case when electric field is zero

$$\eta_\delta = \frac{p_o}{mc}\eta_\tau = \left(\frac{mc}{p_o}\right)^2 - \frac{1}{C}\frac{c}{\mathrm{v}_o}\int_0^C K(s)\eta_x(s)ds = \frac{1}{\beta_o{}^2\gamma_o{}^2} - \frac{1}{\beta_o}\langle K\eta_x\rangle \tag{11-28}$$

e.g. the dependence of the travel time around the storage ring on particles momentum as two components: one corresponds to increase in velocity (kinematic) and the other (geometrical) to - typically - elongation of the trajectory in bending magnets – particles with higher energy travel at larger radius. In general, η_τ can be either negative or positive. When two terms cancel each other, travel time around storage ring is energy independent – this energy is called critical. If the geometrical term $\left\langle g_x\eta_x + g_y\eta_y \right\rangle$ in the accelerator is positive, the accelerator does not have critical energy. Such conditions do not come naturally and require a special, so call negative compaction factor lattice:

$$\alpha_c = \left\langle g_x\eta_x + g_y\eta_y \right\rangle > 0 \,.$$

Synchrotron oscillations – first look.

Here we will assume that longitudinal oscillations (if stable) are slow. Let's initially introduce longitudinal field

$$u = -\frac{e}{p_o c^2}\frac{\partial E_{rf}}{\partial t}$$

and see how it affects betatron oscillations.

What is interesting, that we can formally separate energy dependence on the energy from betatron oscillations using a Canonical transformation:

$$\begin{aligned}\tilde{H}\left(X_\beta\right) = H\left(\tilde{X}+X_\delta\right) - \frac{\partial F}{\partial s} = H\left(\tilde{X}+X_\delta\right) + \\ +\eta'_x\tilde{\pi}_\tau\left(\tilde{\pi}_x+\eta_{px}\tilde{\pi}_\tau\right) - \eta'_{px}\tilde{\pi}_\tau\tilde{x} + \eta'_y\tilde{\pi}_\tau\left(\tilde{\pi}_y+\eta_{py}\tilde{\pi}_\tau\right) - \eta'_{py}\tilde{\pi}_\tau\tilde{y}\end{aligned} \qquad (11\text{-}30)$$

while we can prove that matrix of such transformation is symplectic, it is also very easy to do using a generation function noticing that $\tilde{\pi}_\tau = \pi_\tau$ is not changing during the transformation

$$\begin{aligned}F\left(X,\tilde{P}\right) &= \left(x-\eta_x\tilde{\pi}_\tau\right)\left(\tilde{\pi}_x+\eta_{px}\tilde{\pi}_\tau\right) + \left(y-\eta_y\tilde{\pi}_\tau\right)\left(\tilde{\pi}_y+\eta_{py}\tilde{\pi}_\tau\right) + \tau\tilde{\pi}_\tau \\ \pi_\tau &= \frac{\partial F}{\partial \tau} = \tilde{\pi}_\tau;\ \ \tilde{\tau} = \frac{\partial F}{\partial \tilde{\pi}_\tau} = \tau - \eta_x\tilde{\pi}_x + \eta_{px}x - \eta_y\tilde{\pi}_y + \eta_{py}y; \\ x_\beta &= \tilde{x} = \frac{\partial F}{\partial \tilde{\pi}_x} = x - \eta_x\tilde{\pi}_\tau; y_\beta = \tilde{y} = \frac{\partial F}{\partial \tilde{\pi}_y} = y - \eta_y\tilde{\pi}_\tau; \\ \pi_x &= \frac{\partial F}{\partial x} = \tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau;\ \ \pi_y = \frac{\partial F}{\partial y} = \tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau.\end{aligned} \qquad (11\text{-}31)$$

In new coordinates (we will drop the $_\beta$ index for compactness. Let's consider that there is no betatron oscillations. Brining all terms together, we would arrive a longitudinal Hamiltonian

$$\mathcal{H}_\tau = \left(\left(\frac{mc}{p_o} \right)^3 + g_x \eta_x + g_y \eta_y \right) \frac{\pi^2{}_\tau}{2} + \frac{U}{mc} \frac{\tau^2}{2}; \tag{11-32}$$

which can be solved in a traditional manner. But since the oscillations assumed to be very slow, we can average the Hamiltonian to

$$\langle \mathcal{H}_\tau \rangle = \eta_\tau \frac{\pi^2{}_\tau}{2} + \frac{\langle U \rangle}{mc} \frac{\tau^2}{2}; \tag{11-33}$$

with stable solution when $\eta_\tau \frac{\langle U \rangle}{mc} > 0$ and simple oscillator solution:

$$\begin{gathered} \Omega_s = \sqrt{\eta_\tau \frac{\langle U \rangle}{mc}};\ Q_s = \frac{\mu_s}{2\pi} = \frac{\Omega_s C}{2\pi}; \\ \tau = a \cdot \cos(\Omega_s s + \varphi_s); \pi_\tau = -\frac{a\Omega_s}{\eta_\tau} \sin(\Omega_s s + \varphi_s); \end{gathered} \tag{11-34}$$

This solution is an approximation – beware of this.

What we learned today

- We can continue with exact description and parameterization of linear motion in accelerator to 3D
- While equations become more evolved, we still have analytical expressions for the eigen values and can use Sylvester formulae for calculating matrices of any element in accelerator
- Since expressions are becoming cumbersome, letting computers to do accurate step-by-step job is a good idea
- Stability of the periodic system or one turn matrix in accelerator has the same appearance, but with less obvious connections to the matrix elements – expressions are just too long
- We made a first look into an approximate description – typical for accelerator books – of slow synchrotron oscillations in storage rings. We will continue this in next class..