PHY 564
Advanced Accelerator Physics
Lecture 13

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Lecture 13. Linear Hamiltonian system and their transport matrices

Compact form

We went through following discussion during previous lectures:

1. A linear Hamiltonian n-dimensional system with s as independent variable and n canonical pairs of variables

$$X^T = \{q_1, P_1, ..., q_n, P_n\} \Leftrightarrow \{x_1, ..., x_{2n}\} \\ x_{2k-1} = q_k; x_{2k} = P_k; k = 1,..,n \tag{1}$$

is fully described by its Hamiltonian

$$H(X,s) = \frac{1}{2} X^T \mathbf{H}(s) X; \; \mathbf{H}^T(s) = \mathbf{H}(s) \tag{2}$$

where $\mathbf{H}(s)$ is $2n$ x 2n symmetric matrix with coefficients, in general, depending on (e.g. being functions of) s:

$$[\mathbf{H}]_{ij} = h_{ij}(s). \tag{3}$$

Equations of motions can be written in a compact matrix form

$$\frac{dq_i}{ds} = \frac{\partial H}{\partial P_i} = \sum_{j=1}^{2n} h_{ij}(s) x_j; \frac{dP_i}{ds} = -\frac{\partial H}{\partial q_i} = -\sum_{j=1}^{2n} h_{ij}(s) x_j$$

$$\frac{dX}{ds} = \mathbf{S} \cdot \mathbf{H} \cdot X; \quad \mathbf{S} = \begin{bmatrix} \sigma & 0 & .. & 0 \\ 0 & \sigma & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \sigma \end{bmatrix}; \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4}$$

$$\mathbf{S} = [s_{ij}]; s_{2k-1,2k} = -s_{2k,2k-1} = 1; \; \textit{othewise}\ 0$$

or even more compact form

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X; \quad \mathbf{D}(s) = \mathbf{S} \cdot \mathbf{H}(s). \tag{5}$$

An arbitrary solution of eq. (5) with initial conditions X_o at s_o can be written as

$$\begin{gathered} X(s) = \mathbf{M}\left(s_o|s\right)X_o; \quad X\left(s_o\right) = X_o; \\ \frac{d\mathbf{M}}{ds} = \mathbf{D}(s)\cdot\mathbf{M};\ \ \mathbf{M}\left(s_o|s_o\right) = \mathbf{I};\ \left[\mathbf{I}\right]_{ik} = \delta_{ik}. \end{gathered}. \tag{6}$$

where $\mathbf{I}$ is an unit 2nx2n matrix $\mathbf{A}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{A} = \mathbf{A},\ \forall\mathbf{A}$. We proved that since $trace\left[\mathbf{D}\right] = 0$ than $\det\mathbf{M} = 1$ and furthermore matrix is $\mathbf{M}$ symplectic:

$$\begin{gathered} \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S} \Leftrightarrow \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S} \Leftrightarrow \mathbf{M}\mathbf{S}\mathbf{M}^T = \mathbf{S}; \quad \mathbf{S}^T = -\mathbf{S};\ \ \mathbf{H}^T = \mathbf{H}; \\ \mathbf{I}^T\mathbf{S}\mathbf{I} \equiv \mathbf{S};\ \ \frac{d\left(\mathbf{M}^T\mathbf{S}\mathbf{M}\right)}{ds} = \left(\frac{d\mathbf{M}^T}{ds}\mathbf{S}\mathbf{M} + \mathbf{M}^T\mathbf{S}\frac{d\mathbf{M}}{ds}\right) = \mathbf{M}^T\left(\mathbf{H}^T\mathbf{S}^T\mathbf{S} + \mathbf{S}\mathbf{S}\mathbf{H}\right)\mathbf{M} = \mathbf{0}\,\# \end{gathered}. \tag{7}$$

In general case of arbitrary s-depended Hamiltonian

$$\begin{gathered} \mathbf{M}\left(s_o|s\right) = \prod_{i=0}^{K-1}\left(\mathbf{I} + \mathbf{SH}\left(s_i^*\right)\cdot\left(s_{i+1} - s_i\right)\right); s_i^* \in \left\{s_i, s_{i+1}\right\}; K \to \infty, s_{i+1} - s_i \to 0 \\ Ordered: \prod_{i=0}^{N-1}\left(\mathbf{I} + \mathbf{SH}\left(s_i^*\right)\cdot\left(s_{i+1} - s_i\right)\right) \equiv \ldots\left(\mathbf{I} + \mathbf{SH}\left(s_{i+1}^*\right)\cdot\left(s_{i+2} - s_{i+1}\right)\right)\left(\mathbf{I} + \mathbf{SH}\left(s_i^*\right)\cdot\left(s_{i+1} - s_i\right)\right)\ldots. \end{gathered}. \tag{8}$$

one needs a computer to calculate the transport matrix. Please note that since, in general, matrices do not commute, and the order of multiplication MUST be respected. What is important for us, that it exists.

In practice, one can design a step-wise Hamiltonian, whose coefficient stay constant at some interval $\{s_i, s_{i+1}\}$. Than we can find solution of (6) is a very simple form:

$$\frac{d\mathbf{M}_i}{ds} = \mathbf{D}_i \cdot \mathbf{M}_i \rightarrow \mathbf{M}_i(s) = e^{\mathbf{D}_i(s-s_i)}; e^{\mathbf{A}}\Big|_{def} \equiv \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{A}^k}{k!}. \tag{9}$$

We call $\mathbf{M}(s_o|s)$ a transport matrix from s_o to s. Hence, we can rewrite (6) as

$$X(s_1) = \mathbf{M}_1 \cdot X_o;\ X(s_2) = \mathbf{M}_1 \cdot X(s_1); ....X(s) = \mathbf{M}_{k+1}(s_k|s) \cdot X(s_k); s \in \{s_k, s_{k+1}\}. \tag{10}$$

If $s \in \{s_k, s_{k+1}\}$, we can apply consequent multiplication of matrices of “elements”:

$$\begin{gathered} \mathbf{M}_k(s_k|s) = \mathbf{M}_{k+1}(s_k|s)\mathbf{M}_k ....\mathbf{M}_3 \cdot \mathbf{M}_2 \cdot \mathbf{M}_1; \\ \mathbf{M}_i = \exp\left[\mathbf{D}_i(s_i - s_{i-1})\right]; i = 1,2...,k;\ \ \mathbf{M}_{k+1}(s_k|s) = \exp\left[\mathbf{D}_{k+1}(s - s_k)\right] \end{gathered}. \tag{11}$$

Please note that since, in general, matrices do not commute, the order of multiplication **MUST be respected**: matrix of the first “element” is the right, matrix of the last “element” – the one on the left! We used here complete matrices of elements which “particle” passed completely (1…,k) and use *s*-dependent matrix for k+1 element where “particle” is located.

The remaining task for us is to calculate matrices of various elements. The method of calculation depends on the structure of the matrix **D** and its eigen values, which satisfy characteristic (polynomial of power 2n) equation:

$$\det[\mathbf{D} - \lambda\mathbf{I}] = p(\lambda) = \prod_{k=1}^{2n}(\lambda_k - \lambda);\ \ p(\lambda_k) = 0 \tag{12}$$

Since we are interested in Hamiltonian linear system, it has an additional nice feature that each eigen value pairs with one having the same value but an opposite sign:

$$p(\lambda) = \det[\mathbf{SH} - \lambda\mathbf{I}] = \det[\mathbf{SH} - \lambda\mathbf{I}]^T = \det\left[\mathbf{H}^T\mathbf{S}^T - \lambda\mathbf{I}\right] = \left(-1^{2n}\right)\det\left[\mathbf{SH}^T + \lambda\mathbf{I}\right] = p(-\lambda)$$

$$\det[\mathbf{D} - \lambda\mathbf{I}] = p(\lambda) = \prod_{k=1}^{n}(\lambda - \lambda_k)(\lambda + \lambda_k) = \prod_{k=1}^{n}\left(\lambda^2 - \lambda_k^2\right)$$

which reduces the order of polynomial we have to solve to n and eigen values to pairs: $\{\lambda_k, -\lambda_k\}$. This allows us to evaluate all eigen values analytically for 3 dimensional linear Hamiltonian motion! And making them really trivial for 1D case: $\lambda^2 - \lambda_1^2 = 0$.

There is a famous Sylvester formula, which allow to calculate function of matrices. When matrix **D** has *2n* unique (non-equal) eigen values, it can be diagonalized and Sylvester formula for exponent is a "piece of cake": sum of 2n polynomials of power 2n-1 of matrix **D** (remember $\mathbf{D}^0 \equiv \mathbf{I}$):

$$\exp[\mathbf{D}s] = \sum_{k=1}^{2n} e^{\lambda_k s} \prod_{\lambda_j \neq \lambda_k} \frac{\mathbf{D} - \lambda_j\mathbf{I}}{\lambda_k - \lambda_j} \tag{13}$$

Since $\mathbf{D},\mathbf{I}$ commute, the order of multiplication is arbitrary. Formula (13) can be further simplified because eigen values come in pairs: $\{\lambda_k, -\lambda_k\}$.

$$\exp[\mathbf{D}s] = \sum_{k=1}^{n}\left(\frac{e^{\lambda_k s} + e^{-\lambda_k s}}{2}\mathbf{I} + \frac{e^{\lambda_k s} - e^{-\lambda_k s}}{2\lambda_k}\mathbf{D}\right)\prod_{j\neq k}\left(\frac{\mathbf{D}^2 - \lambda_j^2\mathbf{I}}{\lambda_k^2 - \lambda_j^2}\right) \tag{14}$$

In general case, some eigen values can have multiplicity in the characteristic equation:

$$\det\left[\mathbf{D}-\lambda\mathbf{I}\right]=p(\lambda)=\prod_{k=1}^{m}\left(\lambda_k-\lambda\right)^{n_k} \tag{15}$$

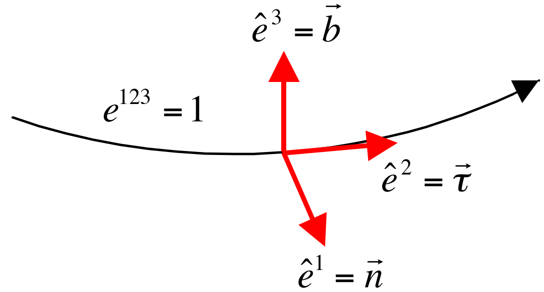
It amy indicated that matrix **D** can be only reduced to a Jordan form… Than Sylvester formula becomes a bit more involved

$$\exp[\mathbf{D}s]=\sum_{k=1}^{m}\left\lfloor e^{\lambda_k s}\prod_{i\neq k}\left\{\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_k-\lambda_i}\sum_{j=0}^{n_k-1}\left(\frac{\mathbf{D}-\lambda_k\mathbf{I}}{\lambda_i-\lambda_k}\right)^j\right\}^{n_i}\sum_{p=0}^{n_k-1}\frac{s^p}{p!}\left(\mathbf{D}-\lambda_k\mathbf{I}\right)^p\right\rfloor \tag{16}$$

where $n_k < 2n$ is multiplicity of eigen value λ_k. We will need this form only in very few occasions. If computer does it for you, you can easily put n_k -> 2n and use (16) – all extra terms will be automatically zero! It is important – use this formula without fear.

Accelerator Hamiltonian

Before going to a general accelerator elements with constant parameters, let's remind that we are dealing with curvilinear coordinate system:



$$\vec{\tau} = \frac{d\vec{r}_o(s)}{ds} = \vec{r}_o{}'$$

$$\vec{n} = -\vec{r}_o{}'' \Big/ \left|\vec{r}_o{}''\right|$$

$$\vec{b} = \left[\vec{n} \times \vec{\tau}\right]$$

$$\vec{r} = \vec{r}_o(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s). \tag{17}$$

with contra-variant coordinate components:

$$q^1 = x;\ q^2 = s,\ q^3 = y. \tag{18}$$

and covariant component of Canonically conjugate generalized momenta

$$P_1 = P_x; P_2 = (1 + Kx)P_s + \kappa\left(P_x y - P_y x\right);\ P_3 = P_y;\ \ \vec{P} = \vec{p} + \frac{e}{c}\vec{A}; \tag{19}$$

with regular (time *t* as independent variable) Hamiltonian of

$$H = c\sqrt{\begin{array}{l}(1+Kx)^{-2}\left(\left(P_2 - \frac{e}{c}A_2\right) + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)\right)^2 \\ + \left(P_1 - \frac{e}{c}A_1\right)^2 + \left(P_3 - \frac{e}{c}A_3\right)^2 + m^2c^2\end{array}} + e\varphi \tag{20}$$

By choosing s as independent variables, we change the set of canonical pair to

$$\{x,P_1\},\{y,P_3\}\{-ct,H/c\} \tag{21}$$

and corresponding exact Hamiltonian in accelerator physics (s-dependent) form:

$$\begin{aligned} h^* = -(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2} \\ + \frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) \end{aligned} \tag{22}$$

Then we considered motion of particles close to the design trajectory. While by setting the vector potential to be zero at the design orbit, we made $\{x,P_1\},\{y,P_3\}$ to be small (if necessary infinitesimally small) values near the ideal trajectory. It allows to think about expanding the Hamiltonian (22) near the reference orbit.

Meanwhile, H,t could be an arbitrary large numbers. Hence, we introduced a reference particle traveling at design orbit with design energy $H_o(s)$ and at designed time $t_o(s)$. Then we introduced new Canonical variables, naturally using a Canonical transformation, which can be infinitesimally small for particle with small deviations fro the reference particles:

$$\left\{\tau = -c(t - t_o(s)), \delta = \left(H - E_o(s) - e\varphi_o(s,t)\right)/c\right\} \tag{23}$$

and three canonical pairs become:

$$\{x,P_1\},\{y,P_3\}\{\tau,\delta\} \Rightarrow X^T = \{x,P_1,y,P_3,\tau,\delta\} \equiv \{x,P_x,y,P_y,\tau,\delta\} \equiv \{q_1,P_1,q_2,P_2,q_3,P_3\} \tag{24}$$

The nice feature of variables (24) that they are real Canonical pairs and when we use them, our equations of motion remain Hamiltonian – with all benefits it brings. We also made all coordinates to have dimension of length, and all Canonical momenta the dimension of mechanical momentum.

We also discussed that we can scale momenta by a constant and make it dimensionless:

$$\pi_1 \equiv \pi_x = \frac{P_1}{p}; \pi_3 \equiv \pi_y = \frac{P_3}{p}; \ \pi_o \equiv \pi_s = \frac{\delta}{p}. \qquad (25)$$

Selecting $p = mc$ makes the equations dimensionless, but always Hamiltonian. It is frequently refereed as normalized coordinated.

If energy of the particle is constant and $p = p_o$ is it full momentum, for infinitesimally small values the values become close to angles of trajectories. Unfortunately this is true only if transvers components of vector potential are equal to zero. When this is not a case and transverse component of vector potential are not zero:

$$x' = \frac{dx}{ds} = \frac{P_x - \frac{e}{c}A_x}{p_o}; y' = \frac{dy}{ds} = \frac{P_y - \frac{e}{c}A_y}{p_o}. \qquad (26)$$

pairs $(x, x'), (y, y')$ are not Canonical and differential equations connecting them are not Hamiltonian. Naturally, if p_o is a function of s, the transport matrix in the scale coordinates would not have an unit determinant. It is easy to see from a following: let's consider an infinitesimally small change of p_o in a drift-space (no focusing!)

$$p_{o1} = p_o + g\delta s; \ \ x_1 = x_o + x'^{*}_o \delta s; \quad x'_o \geq x'^{*}_o \geq x'_o \left(1 - \frac{g\delta s}{p_o + g\delta s}\right); \qquad (27)$$
$$x_1' = x'_o \frac{p_o}{p_o + g\delta s} x'_o \left(1 - \frac{g\delta s}{p_o + g\delta s}\right).$$

Jacobean in (x,x') coordinate system is definitely less then one… this, beware of using (x,x') and (y,y') when energy of particle is changing or if the transverse component of vector potential are not zero (for example in a solenoid!). Otherwise, it is very convenient and frequently used in accelerator physics books and papers.

One more time, when all component of the vector bellow are small

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \equiv \begin{bmatrix} x \\ P_1 \\ y \\ P_3 \\ \tau \\ \delta \end{bmatrix} \equiv \begin{bmatrix} x \\ P_x \\ y \\ P_y \\ z \\ P_z \end{bmatrix}; \tag{28}$$

we can expand Hamiltonian to the form in eq. (2) – all linear terms are killed by assumption that reference particles has the designed trajectory:

$$\begin{array}{l} \tilde{h} = \dfrac{P_1^2 + P_3^2}{2p_o} + F\dfrac{x^2}{2} + Nxy + G\dfrac{y^2}{2} + L\left(xP_3 - yP_1\right) + \\ \dfrac{\delta^2}{2p_o}\cdot\dfrac{m^2c^2}{p_o^2} + U\dfrac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau \end{array}; \tag{29}$$

with

$$\begin{aligned} &\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_oc}B_y - \frac{e}{p_oc}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_oc}\right)^2\right] - \frac{e}{p_o\mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o\mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2; \\ &\frac{G}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_oc}\right)^2\right] - \frac{e}{p_o\mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2; \qquad ;(30) \\ &\frac{2N}{p_o} = \left[\frac{e}{p_oc}\frac{\partial B_x}{\partial x} - \frac{e}{p_oc}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_oc}B_x - \frac{e}{p_o\mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o\mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right) \\ &L = \kappa + \frac{e}{2p_oc}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3}; \\ &F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}. \end{aligned}$$

$$\mathbf{H}=\begin{bmatrix} F & 0 & N & L & F_x & g_x \\ 0 & 1/p_o & -L & 0 & 0 & 0 \\ N & -L & G & 0 & F_y & g_y \\ L & 0 & 0 & 1/p_o & 0 & 0 \\ F_x & 0 & F_y & 0 & U & 0 \\ g_x & 0 & g_y & 0 & 0 & \dfrac{m^2c^2}{p_o^3} \end{bmatrix};\mathbf{D}=\begin{bmatrix} 0 & 1/p_o & -L & 0 & F_x & 0 \\ -F & 0 & -N & -L & 0 & -g_x \\ L & 0 & 0 & 1/p_o & F_y & 0 \\ -N & L & -G & 0 & 0 & -g_y \\ g_x & 0 & g_y & 0 & 0 & \dfrac{m^2c^2}{p_o^3} \\ -F_x & 0 & -F_y & 0 & -U & 0 \end{bmatrix} \quad (31)$$

For normalized coordinates it is

$$\mathbf{H}=\begin{bmatrix} \frac{F}{mc} & 0 & \frac{N}{mc} & L & \frac{F_x}{mc} & g_x \\ 0 & \frac{mc}{p_o} & -L & 0 & 0 & 0 \\ \frac{N}{mc} & -L & G & 0 & \frac{F_y}{mc} & g_y \\ L & 0 & 0 & \frac{mc}{p_o} & 0 & 0 \\ \frac{F_x}{mc} & 0 & \frac{F_y}{mc} & 0 & \frac{U}{mc} & 0 \\ g_x & 0 & g_y & 0 & 0 & \left(\frac{mc}{p_o}\right)^3 \end{bmatrix};\mathbf{D}=\begin{bmatrix} 0 & \frac{mc}{p_o} & -L & 0 & \frac{F_x}{mc} & 0 \\ -\frac{F}{mc} & 0 & -\frac{N}{mc} & -L & 0 & -g_x \\ L & 0 & 0 & \frac{mc}{p_o} & \frac{F_y}{mc} & 0 \\ -\frac{N}{mc} & L & -G & 0 & 0 & -g_y \\ g_x & 0 & g_y & 0 & 0 & \left(\frac{mc}{p_o}\right)^3 \\ -\frac{F_x}{mc} & 0 & -\frac{F_y}{mc} & 0 & -\frac{U}{mc} & 0 \end{bmatrix} \quad (32)$$

and the case of $p = p_o = const$

$$\mathbf{H} = \begin{bmatrix} \frac{F}{p_o} & 0 & \frac{N}{p_o} & L & \frac{F_x}{p_o} & g_x \\ 0 & 1 & -L & 0 & 0 & 0 \\ \frac{N}{p_o} & -L & G & 0 & \frac{F_y}{p_o} & g_y \\ L & 0 & 0 & 1 & 0 & 0 \\ \frac{F_x}{p_o} & 0 & \frac{F_y}{p_o} & 0 & \frac{U}{p_o} & 0 \\ g_x & 0 & g_y & 0 & 0 & \left(\frac{mc}{p_o}\right)^2 \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & 1 & -L & 0 & \frac{F_x}{p_o} & 0 \\ -\frac{F}{p_o} & 0 & -\frac{N}{p_o} & -L & 0 & -g_x \\ L & 0 & 0 & 1 & \frac{F_y}{p_o} & 0 \\ -\frac{N}{p_o} & L & -G & 0 & 0 & -g_y \\ g_x & 0 & g_y & 0 & 0 & \left(\frac{mc}{p_o}\right)^2 \\ -\frac{F_x}{p_o} & 0 & -\frac{F_y}{p_o} & 0 & -\frac{U}{p_o} & 0 \end{bmatrix} \quad (33)$$

While different in the appearance all three matrices **D** in (31-33) have identical eigen values and therefore the same Sylvester form of exp[**D**s].

We write all possible form of 6x6 matrices and one can explore in detail all possible 6x6 transport matrices generated by (31-33).

To limit the pain, we will explore matrices of elements with DC fields. While they most natural and common elements we use in accelerators, they also allow complete evaluation of the matrices. For example, it makes $U = 0; F_x = F_y = 0;$ and it allows us to use $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$. It simplifies the system significantly. Since nothing depends on time,

$$\frac{d\delta}{ds} = -\frac{\partial h}{\partial \tau} = 0 \Rightarrow . \qquad (34)$$

We also should request that reference particle does not accelerate:

$$\frac{dE_o(s)}{ds} = eE_2(s, t_o(s)) = 0 \rightarrow p_o(s) = const$$

The value of this approach to matrix calculation is that we do not need to memorize all the different ways of deriving the matrices of various elements in accelerators, and ways of solving a myriad of systems of 2, 4, 6… linear differential equations. Just a smart "coach potato principle" allover again….

The elements of 6x6, 4x4, or 2x2 accelerator matrixes (often called R or T) are numerated as R_{ij}, where ***i*** is the line number and ***j*** is the column number. For example, R_{56} will signify an increment in τ (-arrival time by c) caused by the particle's energy change, δ. Let's look at most trivial case of decoupled transverse motion.

Since we fixed the particle's energy (momentum), using normalized/geometrical variable is convenient.

Matrix of a general DC accelerator element (including twisted quads or helical wiggler) can be found using our recipe. With all diversity of possible elements on accelerators, DC (or almost DC) magnets play the most prominent role. In this case energy of the particle stays constant and we can use reduced variables. Furthermore, large number of terms is the Hamiltonian simply disappear and from the previous lecture we have:

$$\tilde{h}_n = \frac{\pi_1^{\;2} + \pi_3^{\;2}}{2} + f\frac{x^2}{2} + n\cdot xy + g\frac{y^2}{2} + L(x\pi_3 - y\pi_1) + \frac{\pi_o^{\;2}}{2}\cdot\frac{m^2c^2}{p_o^{\;2}} + g_x x\pi_o + g_y y\pi_o; \qquad (36)$$

Even though it is tempting to remove electric field, it does not either helps or hurts in general case of an element. Hence, we will keep DC transverse electric fields. We also assume that these fields are in vacuum and $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$, $\frac{\partial E_x}{\partial x} + KE_x + \frac{\partial E_y}{\partial y} = 0$:

$$f = K^2 - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2 + \left(\frac{meE_x}{p_o^{\;2}}\right)^2;$$

$$g = \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2 + \left(\frac{meE_z}{p_o^{\;2}}\right)^2; \qquad ; \; (37)$$

$$2n = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_o c}B_x - \frac{e}{p_o \mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o \mathrm{v}_o} + \left(\frac{meE_z}{p_o^{\;2}}\right)\left(\frac{meE_x}{p_o^{\;2}}\right)$$

$$L = \kappa + \frac{e}{2p_o c}B_s; \qquad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3};$$

In the absence of longitudinal electric field, the momentum P_2 is constant as well $\pi_o = const$, δ=const. The fact that particle's energy does not changes in such element is rather obvious (It is completely correct for magnetic elements. Presence of electric field makes it less obvious, but it comes from the fact that Hamiltonian does not depend on time!): $\pi_o{}' = -\frac{\partial h}{\partial \tau} = 0$.

Equations of motion become specific:

$$\mathbf{X}^T = [x, \pi_1, y, \pi_3, \tau, \pi_o] = [X^T, \tau, \pi_o]; \; X^T = [x, \pi_1, y, \pi_3], \tag{38}$$

$$\frac{d\mathbf{X}}{ds} = \mathbf{D}(s)\cdot\mathbf{X}; \mathbf{D} = \mathbf{S}\cdot\mathbf{H}(s) = \begin{bmatrix} 0 & 1 & -L & 0 & 0 & 0 \\ -f & 0 & -n & -L & 0 & -g_x \\ L & 0 & 0 & 1 & 0 & 0 \\ -n & L & -g & 0 & 0 & -g_y \\ g_x & 0 & g_y & 0 & 0 & m^2c^2/p_o^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \tag{39}$$

and can be rewritten in a slightly different (just deceivingly looking better) way:

$$\frac{dX}{ds} = D \cdot X + \pi_o \cdot C;$$

$$\frac{d\tau}{ds} = g_x x + g_y y + \pi_o \cdot m^2 c^2 / p_o^2; D = \begin{bmatrix} 0 & 1 & -L & 0 \\ -f & 0 & -n & -L \\ L & 0 & 0 & 1 \\ -n & L & -g & 0 \end{bmatrix}; C = \begin{bmatrix} 0 \\ -g_x \\ 0 \\ -g_y \end{bmatrix}. \tag{40}$$

Hence, solution for transverse motion (4-vector) in such an element can be written as combination general solution of homogeneous equation plus specific solution of inhomogeneous one:

$$X = M(s) \cdot X_o + \pi_o \cdot R(s);\ \ M = e^{D(s-s_o)};\ \ R' = D \cdot R + C;\ \ R(s_o) = 0. \tag{41}$$

It worth noting that C=0 as soon as there is no field on the orbit – E=0, B=0. In this case R=0.

Before finding 4x4 matrixes M and vector R, let's see what we will know about the 6x6 matrix after that. First, the obvious:

$$\mathbf{M}_{6x6} = \begin{bmatrix} & \mathbf{M}_{4x4} & & & 0 & R \\ R_{51} & R_{52} & R_{53} & R_{54} & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (42)$$

with a natural question of what are non-trivial R_{5k} elements? Usually these elements, with exception of R_{56} are not even mentioned in most of textbooks. Fortunately for us, Mr. Hamiltonian gives us a hand in the form of symplecticity of transport matrixes. Using (18) and (18-1) we can find that:

$$\mathbf{M}^T_{6x6}\mathbf{S}\mathbf{M}_{6x6} == \begin{bmatrix} \mathbf{M}^T_{4x4} & Q^T & 0 \\ 0 & 1 & 0 \\ R^T & R_{56} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{S}_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{M}_{4x4} & 0 & R \\ Q & 1 & R_{56} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}^T_{4x4}\mathbf{S}_{4x4} & 0 & Q^T \\ 0 & 0 & 1 \\ R^T\mathbf{S}_{4x4} & -1 & R_{56} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{4x4} & 0 & R \\ Q & 1 & R_{56} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M}^T_{4x4}\mathbf{S}_{4x4}\mathbf{M}_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ R^T\mathbf{S}_{4x4}\mathbf{M}_{4x4} - Q & -1 & R^T\mathbf{S}_{4x4}R \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{4x4} \\ 0 \\ 0 \end{bmatrix}$$

where we used $Q = \left[R_{51}, R_{52}, R_{53}, R_{54}\right]$. We should note what $\mathbf{X^T S X = 0}$ for any vector, $\mathbf{M}^T{}_{4x4}\mathbf{S}_{4x4}\mathbf{M}_{4x4} = \mathbf{S}_{4x4}$ and only non-trivial condition from the equation above is:

$$R^T\mathbf{S}_{4x4}\mathbf{M}_{4x4} - Q = 0$$

which gives us very valuable dependence of the arrival time on the transverse motions:

$$Q = R^T\mathbf{S}_{4x4}\mathbf{M}_{4x4}; \ or \ Q^T = -\mathbf{M}^T_{4x4}\mathbf{S}_{4x4}R \ . \qquad (43)$$

Element R_{56} is decoupled form the symplectic condition in this case and should be determined by direct integration - no magic here:

$$\tau(s) = \tau(s_o) + \pi_o \cdot \left\{ m^2c^2 / p_o^2 (s - s_o) + \int_{s_o}^{s} \left(g_x R(s)_{16} + g_y R_{36}(s) \right) ds \right\}$$
$$R_{56} = m^2c^2 / p_o^2 (s - s_o) + \int_{s_o}^{s} \left(g_x R(s)_{16} + g_y R_{36}(s) \right) ds \qquad (44)$$

$$\det[D-\lambda I]=\det\begin{bmatrix} -\lambda & 1 & -L & 0 \\ -f & -\lambda & -n & -L \\ L & 0 & -\lambda & 1 \\ -n & L & -g & -\lambda \end{bmatrix}$$

Let's find the solutions for 4x4 matrixes of arbitrary element. First, let solve characteristic equation for D:

$$\det[D-\lambda I]=\lambda^4+\lambda^2\left(f+g+2L^2\right)+fg+L^4-L^2(f+g)-n^2=0 \tag{45}$$

with easy roots:

$$\lambda^2=a\pm b;\;\; a=-\frac{f+g+2L^2}{2};\;\; b^2=\frac{(f-g)^2}{4}+2L^2(f+g)+n^2 \tag{46}$$

Before starting classification of the cases, let's note that

$$f+g=K^2+2\left(\frac{eB_s}{2p_oc}\right)^2+\left(\frac{meE_x}{p_o{}^2}\right)^2+\left(\frac{meE_z}{p_o{}^2}\right)^2\geq 0$$

i.e. $a\leq 0;\;\; b^2\geq 0;\;\; \mathrm{Im}(b)=0$.

$$\lambda^2 = a \pm b; a = -\frac{f+g+2L^2}{2}; b^2 = \frac{(f-g)^2}{4} + 2L^2(f+g) + n^2$$

Before starting classification of the cases, let's note that

$$f+g = K^2 + 2\left(\frac{eB_s}{2p_oc}\right)^2 + \left(\frac{meE_x}{p_o{}^2}\right)^2 + \left(\frac{meE_z}{p_o{}^2}\right)^2 \geq 0$$

i.e. $a \leq 0;\ b^2 \geq 0;\ \mathrm{Im}(b) = 0$. The solutions can be classified as following: remember that the full set of eigen values is $\lambda_1, -\lambda_1, \lambda_2, -\lambda_2$:

I. $\lambda_1 = \lambda_2 = 0;\quad a = 0;\ b = 0;$

II. $\lambda_1 = \lambda_2 = i\omega;\quad a = -\omega^2;\ b = 0;$

III. $\lambda_1 = 0;\quad \lambda_2 = i\omega;\quad a + b = 0;\quad 2b = \omega^2$

IV. $\lambda_1 = i\omega_1;\quad \lambda_2 = i\omega_2;\quad \omega_1{}^2 = -a - b;\ \omega_2{}^2 = -a + b;\ |a| > b$

V. $\lambda_1 = i\omega_1;\quad \lambda_2 = \omega_2;\ \omega_1{}^2 = -a - b;\ \omega_2{}^2 = b - a;\ b > |a|$

Before going to case-by-case calculations, lets use Sylvester's formulae and try to find solution of inhomogeneous equation:

$$\frac{d\mathbf{R}}{ds} = \mathbf{D}\cdot \mathbf{R} + \mathbf{C};\ \mathbf{R}(0) = 0\,. \tag{47}$$

When matrix detD≠0, (47) can be inversed using a $\mathbf{R} = A + e^{Ds}\cdot B$ as a guess and the boundary condition $\mathbf{R}(0) = 0$:

$$\mathbf{R} = \left(\mathbf{M_{4x4}(s)} - \mathbf{I}\right)\cdot \mathbf{D}^{-1}\cdot \mathbf{C} \tag{48}$$

is the easiest solution. Prove is just straight forward:

$$\mathbf{R}' = \mathbf{D}\cdot \mathbf{M_{4x4}}^{-1}\cdot \mathbf{C};$$

$$\mathbf{D}\cdot \left(\mathbf{M} - \mathbf{I}\right)\cdot \mathbf{D}^{-1}\cdot \mathbf{C} + \mathbf{C} = \mathbf{D}\cdot \mathbf{M_{4x4}}^{-1}\cdot \mathbf{C} \quad \#$$

In all cases we can use method of variable constants to find it:

$$\frac{dR}{ds} = R' = \mathbf{D}\cdot R + \mathbf{C};\quad \mathbf{M}' = \mathbf{DM};$$
$$R = \mathbf{M}(s)A(s) \Rightarrow \mathbf{M}'A + \mathbf{M}A' = \mathbf{DM}A + C;\quad R(0) = 0 \Rightarrow A_o = 0 \tag{49}$$
$$A' = \mathbf{M}^{-1}(s)C \Rightarrow A = \int_0^s \mathbf{M}^{-1}(z)C\,dz = \left(\int_0^s e^{-\mathbf{D}z}dz\right)\cdot C;\quad R = e^{\mathbf{D}s}\left(\int_0^s e^{-\mathbf{D}z}dz\right)\cdot C$$

It is important to remember that $\mathbf{M}^{-1}(\mathbf{s})$ is just the $\mathbf{M}(-\mathbf{s}) = \mathbf{e}^{-\mathbf{Ds}}$. Hence in all our formulae for matrixes from previous lectures we need to replace s by –s to get $\mathbf{M}^{-1}(\mathbf{s})$**.** Other vice, we have to use general formula (33) for the homogeneous solution and use method of variable constants (see Appendix F) to find it:

$$R(s) = \sum_{k=1}^{m} \left\{ \prod_{i \neq k} \left[\frac{\mathbf{D} - \lambda_i \mathbf{I}}{\lambda_k - \lambda_i} \right] \sum_{j=0}^{n_k - 1} \left(\frac{\mathbf{D} - \lambda_k \mathbf{I}}{\lambda_i - \lambda_k} \right)^j \right\} \sum_{n=0}^{n_k - 1} (\mathbf{D} - \lambda_k \mathbf{I})^n \frac{s^n}{n!} \cdot \sum_{p=0}^{n_k - 1} (-1)^{p+1} (\mathbf{D} - \lambda_k \mathbf{I})^p \cdot \mathbf{C} \cdot \left[\sum_{q=0}^{p1} \frac{s^{p-q}}{(p-q)!\,\lambda_k^{\,q+1}} - \frac{e^{\lambda_k}}{\lambda_k^{\,p+1}} \right] \tag{50}$$

In all specific cases I, II, III, IV and V, integrating (L-53) directly is usually easier that using general form of (50).

$$\frac{f+g+2L^2}{2}=0;\ \frac{(f-g)^2}{4}+2L^2(f+g)+n^2=0;$$

Case I.

$$f+g=pos^2\geq 0\Rightarrow (f-g)^2=0;\ L^2(f+g)=0;\ n^2=0$$

$$f+g+2L^2=pos^2+2L^2=0\Rightarrow L=0;\ f+g=0\Rightarrow$$

$$f-g=0\Rightarrow f=g=L=n=0!!!$$

means that there is nothing in the Hamiltonian but p^2– is this the drift section matrix of which we already know. Hence, there is not curvature as well and R=0.

$$\mathbf{M_{4x4}}=\begin{bmatrix}1 & s & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & s\\ 0 & 0 & 0 & 0\end{bmatrix};\ \mathbf{R}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \tag{I-1}$$

The only not trivial (ha-ha – it is also as trivial as it can be) is R_{56}:

$$R_{56}=\frac{m^2c^2}{p_o^{\ 2}}s \tag{I-2}$$

we already had seen it when studied nilpotent case…

Case II: $b = \frac{(f-g)^2}{4} + 2L^2(f+g) + n^2 = 0;$

$$f = g;\; n = 0 \;\; and \;\; L^2(f+g) = L^2\left(K^2 + \Omega^2 + El^2\right) = 0; \Omega = eB_s / p_o c; E_\perp = 0.$$

i.e. there are two cases: L=0 or $f + g = 0$.

If both are equal zero, i.e. $f + g = 0;\;\; L = 0$, this is equivalent to the case I above.

Case II a: $f + g = 0$, K≠0, Bs=0 -> L=κ. Thus, this is just a drift (straight section) with rotation, whose matrix is trivial: Drift + rotation. There is not transverse force – hence R=0.

$$\mathbf{M_{4x4}} = \begin{bmatrix} M_d \cdot \cos\kappa s & -M_d \cdot \sin\kappa s \\ M_d \cdot \sin\kappa s & M_d \cdot \cos\kappa s \end{bmatrix}; M_d = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}; \;\; \mathbf{R} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(IIa-1)}$$

$R_{56}$is as for a drift:

$$R_{56} = \frac{m^2 c^2}{p_o{}^2} s \qquad \text{(IIa-2)}$$

Case II b: L=0; $f = g = \left(K^2+\Omega^2\right)/2; \kappa = -\Omega$; i.e. the motion is uncoupled:

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -f & 0 \end{bmatrix}; \; C = \begin{vmatrix} 0 \\ g_x \\ 0 \\ g_y \end{vmatrix}.$$

$$\mathbf{M}_{4x4} = \begin{vmatrix} M & 0 \\ 0 & M \end{vmatrix}; \; M = \begin{bmatrix} \cos\omega s & \sin\omega s/\omega \\ -\omega\sin\omega s & \cos\omega s \end{bmatrix} \qquad \text{(IIb-1)}$$

Here we may have non-zero R: yes, it may be! It is simple integrals to be taken care of:

$$C_{x,y} = -g_{x,y}\begin{bmatrix} 0 \\ 1 \end{bmatrix}; \; M^{-1}(z) = \begin{bmatrix} \cos\omega z & -\sin\omega z/\omega \\ \omega\sin\omega z & \cos\omega z \end{bmatrix} C_{x,y} = g_{x,y}\begin{bmatrix} \sin\omega z/\omega \\ -\cos\omega s \end{bmatrix};$$

$$\int_0^s \mathbf{M}^{-1}(z) C_{x,y} dz = g_{x,y}\begin{bmatrix} \int_0^s \sin(\omega z)dz/\omega \\ -\int_0^s \cos(\omega z)dz \end{bmatrix} = \frac{g_{x,y}}{\omega}\begin{bmatrix} (1-\cos\omega s)/\omega \\ -\sin(\omega s) \end{bmatrix}$$

$$C_{x,y} = -g_{x,y}\begin{bmatrix} 0 \\ 1 \end{bmatrix}; \int_0^s \mathbf{M}^{-1}(z)dz \cdot C_{x,y} = -g_{x,y}\begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$M(s)\int_0^s \mathbf{M}^{-1}(z) C_{x,y} dz = \frac{g_{x,y}}{\omega}\begin{bmatrix} \cos\omega s & \sin\omega s/\omega \\ -\omega\sin\omega s & \cos\omega s \end{bmatrix} \cdot \begin{bmatrix} (1-\cos\omega s)/\omega \\ -\sin(\omega z) \end{bmatrix} = \frac{g_{x,y}}{\omega^2}\begin{bmatrix} \cos\omega s - 1 \\ -\omega\sin\omega s \end{bmatrix}$$

$$R_{56} = s \cdot m^2c^2 / p_o + \int_0^s \left(g_x R(z)_{16} + g_y R_{36}(z) \right) dz =$$

$$\int_0^s \left(g_x R(z)_{16} + g_y R_{36}(z) \right) dz = \frac{g_x^{\ 2} + g_y^{\ 2}}{\omega^2} \int_0^s (\cos \omega z - 1) dz = \frac{g_x^{\ 2} + g_y^{\ 2}}{\omega^2} \left(\frac{\sin \omega s}{\omega} - s \right)$$

with the result:

$$R = \begin{bmatrix} \frac{g_x}{\omega^2}(\cos \omega s - 1) \\ -\frac{g_x}{\omega} \sin \omega s \\ \frac{g_y}{\omega^2}(\cos \omega s - 1) \\ -\frac{g_x}{\omega} \sin \omega s \end{bmatrix}; \quad R_{56} = \frac{m^2 c^2}{p_o^{\ 2}} s + \frac{g_x^{\ 2} + g_y^{\ 2}}{\omega^2} \left(\frac{\sin \omega s}{\omega} - s \right) \qquad \text{(IIb-2)}$$

Case III: $a+b=0;\ \det D=0;\ \ \omega^2=2b;\ \ \lambda_{1,2}=\pm i\omega;\lambda_3=0;\ \ m=3.$

We have to use degenerated case formula, but the maximum height of the eigen vector is 2 and only for 3-rd eigen value. Since it is not scary at all: $n_1=1;n_2=1;n_3=2$

Because of the Hamilton-Kelly theorem, $\mathbf{D}^2(\mathbf{D}^2+\omega^2 I)=0$. Let's do it

$$\exp[\mathbf{D}s]=\sum_{k=1}^{3}\left[e^{\lambda_k s}\prod_{i\neq k}\left\{\frac{\mathbf{D}-\lambda_i\mathbf{I}}{\lambda_k-\lambda_i}\sum_{j=0}^{n_k-1}\left(\frac{\mathbf{D}-\lambda_k\mathbf{I}}{\lambda_i-\lambda_k}\right)^j\right\}^{n_i}\sum_{p=0}^{n_k-1}\frac{s^p}{p!}(\mathbf{D}-\lambda_k\mathbf{I})^p\right]=$$

$$\lambda_1\lambda_2=\omega^2;\ i\omega$$

$$k=3;\ \left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)^2(\mathbf{I}+s\mathbf{D});\ \left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)^2=\left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)+\frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)\downarrow_0=\left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)$$

$$k=3;\ \left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)(\mathbf{I}+s\mathbf{D})$$

$$k=1+2;\ e^{i\omega s}\frac{\mathbf{D}+i\omega\mathbf{I}}{-2i\omega}\frac{\mathbf{D}^2}{\omega^2}+c.c.=-\frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s+\frac{\mathbf{D}}{\omega}\right)$$

$$M=\left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)(\mathbf{I}+s\mathbf{D})-\frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s+\frac{\mathbf{D}}{\omega}\sin\omega s\right)$$

$$\mathbf{M}_{\mathbf{4x4}}=\left(\mathbf{I}+\frac{\mathbf{D}^2}{\omega^2}\right)(\mathbf{I}+s\mathbf{D})-\frac{\mathbf{D}^2}{\omega^2}\left(\mathbf{I}\cos\omega s+\frac{\mathbf{D}}{\omega}\sin\omega s\right) \qquad \text{(III-1)}$$

Similarly

$$R = \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}s + \mathbf{D}\frac{s^2}{2} \right) + \frac{\mathbf{D}^2}{\omega^4} \left(\mathbf{D}(\cos\omega s - 1) - \mathbf{I}\omega \sin\omega s \right) \right\} C \qquad \text{(III-2)}$$

Next is just

$$\int_o^s C^T \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}z + \mathbf{D}\frac{z^2}{2} \right) + \frac{\mathbf{D}^2}{\omega^4} \left(\mathbf{D}(\cos\omega z - 1) - \mathbf{I}\omega \sin\omega z \right) \right\} C dz =$$

$$C^T \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}\frac{s^2}{2} + \mathbf{D}\frac{s^3}{6} \right) + \frac{\mathbf{D}^2}{\omega^4} \left(\mathbf{D}\left(\frac{\sin\omega s}{\omega} - s \right) \mathbf{I}(\cos\omega z - 1) \right) \right\} C$$

with result of:

$$R_{56} = m^2 c^2 / p_o s + C^T \left\{ \left(\mathbf{I} + \frac{\mathbf{D}^2}{\omega^2} \right) \left(\mathbf{I}\frac{s^2}{2} + \mathbf{D}\frac{s^3}{6} \right) + \frac{\mathbf{D}^2}{\omega^4} \left(\mathbf{D}\left(\frac{\sin\omega s}{\omega} - s \right) \mathbf{I}(\cos\omega z - 1) \right) \right\} C \qquad \text{(III-3)}$$

Case IV: all roots are different, no degeneration. Use formula (36)

$$\exp[\mathbf{D}s] = \sum_{k=1}^{2} \left(\frac{e^{\lambda_k s} + e^{-\lambda_k s}}{2}\mathbf{I} + \frac{e^{\lambda_k s} - e^{-\lambda_k s}}{2\lambda_k}\mathbf{D} \right) \prod \left(\frac{\mathbf{D}^2 - \lambda_j{}^2\mathbf{I}}{\lambda_k{}^2 - \lambda_j{}^2} \right)$$

with only one term in the product:

$$\mathbf{M_{4x4}} = \frac{1}{\omega_1{}^2 - \omega_2{}^2} \left\{ \left(\mathbf{I}\cos\omega_1 s + \mathbf{D}\frac{\sin\omega_1 s}{\omega_1} \right)\left(\mathbf{D}^2 + \omega_2{}^2\mathbf{I} \right) - \left(\mathbf{I}\cos\omega_2 s + \mathbf{D}\frac{\sin\omega_2 s}{\omega_2} \right)\left(\mathbf{D}^2 + \omega_1{}^2\mathbf{I} \right) \right\} \qquad \text{(IV-1)}$$

For R we invoke a simplest formula:

$$\mathbf{R} = \left(\mathbf{M_{4x4}(s)} - \mathbf{I} \right)\mathbf{D}^{-1} \cdot \mathbf{C} \qquad \text{(IV-2)}$$

For R56 it is tedious but easy:

$$R_{56} = m^2c^2 / p_o s + C^T \mathsf{M}\mathbf{D}^{-1}C;$$

$$\mathsf{M} = \frac{1}{\omega_1{}^2 - \omega_2{}^2} \left\{ \begin{array}{l} \left(\mathbf{I}\frac{\sin\omega_1 s}{\omega_1} + \mathbf{D}\frac{1-\cos\omega_1 s}{\omega_1{}^2} \right)\left(\mathbf{D}^2 + \omega_2{}^2\mathbf{I} \right) - \\ \left(\mathbf{I}\frac{\sin\omega_2 s}{\omega_2} + \mathbf{D}\frac{1-\cos\omega_2 s}{\omega_2{}^2} \right)\left(\mathbf{D}^2 + \omega_1{}^2\mathbf{I} \right) - \mathbf{I}\cdot s \end{array} \right\} \qquad \text{(IV-3)}$$

Case V: all roots are different, no degeneration. Use formula (36) again

$$\mathbf{M_{4x4}} = \frac{1}{\omega_1^2 + \omega_2^2}\left\{\left(\mathbf{I}\cos\omega_1 s + \mathbf{D}\frac{\sin\omega_1 s}{\omega_1}\right)\left(\mathbf{D}^2 - \omega_2^2\mathbf{I}\right) - \left(\mathbf{I}\cosh\omega_2 s + \mathbf{D}\frac{\sinh\omega_2 s}{\omega_2}\right)\left(\mathbf{D}^2 + \omega_1^2\mathbf{I}\right)\right\} \quad \text{(V-1)}$$

$$\mathbf{R} = \left(\mathbf{M_{4x4}(s)} - \mathbf{I}\right)\mathbf{D}^{-1}\cdot\mathbf{C} \quad \text{(V-2)}$$

$$R_{56} = m^2c^2/p_o s + C^T \mathsf{M}\mathbf{D}^{-1}C;$$

$$\mathsf{M} = \frac{1}{\omega_1^2 + \omega_2^2}\left\{\begin{aligned}&\left(\mathbf{I}\frac{\sin\omega_1 s}{\omega_1} + \mathbf{D}\frac{1-\cos\omega_1 s}{\omega_1^2}\right)\left(\mathbf{D}^2 - \omega_2^2\mathbf{I}\right) - \\ &\left(\mathbf{I}\frac{\sinh\omega_2 s}{\omega_2} + \mathbf{D}\frac{\cosh\omega_2 s - 1}{\omega_2^2}\right)\left(\mathbf{D}^2 + \omega_1^2\mathbf{I}\right) - \mathbf{I}\cdot s\end{aligned}\right\} \quad \text{(V-3)}$$

Before going into the discussion of the parameterization of the motion, we need to finish discussion of few remaining topics for 6x6 matrix of an accelerator. First is multiplication of the 6x6 matrixes for purely magnetic elements:

$$\mathbf{M}_{k\,(6x6)} = \begin{bmatrix} \mathbf{M}_{k\,(4x4)} & 0 & R_k \\ Q_k & 1 & R_{56_k} \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{M}_{2\,(6x6)}\mathbf{M}_{1\,(6x6)} = \begin{bmatrix} \mathbf{M}_{(4x4)} & 0 & R \\ Q & 1 & R_{56} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_2\mathbf{M}_1 & 0 & R_2 + \mathbf{M}_2 R_1 \\ Q_2 + Q_1\mathbf{M}_2 & 1 & R_{56_1} + R_{56_2} + Q_2 R_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{51}$$

i.e. having transformation rules for mixed size objects: a 4x4 matrix M, 4-elemetn column R, 4 element line L, and a number R_{56}. As you remember, L is dependent (L4-7) and expressed as $\mathbf{Q} = \mathbf{R}^{\mathbf{T}}\mathbf{SM}$. Thus:

$$\mathbf{M}_{(4x4)} = \mathbf{M}_2\mathbf{M}_1;\; R = \mathbf{M}_2 R_1 + R_2;\; Q = Q_2\mathbf{M}_1 + Q_1;\; R_{56} = R_{56_1} + R_{56_2} + Q_2 R_1 \tag{52}$$

One thing is left without discussion so far – the energy change. Thus, we should look into a particle passing through an RF cavity, which has alternating longitudinal field. Again, for simplicity we will assume that equilibrium particle does not gain energy, i.e. p_o stays constant and we can continue using reduced variables. We will also assume that the is no transverse field, neither AC or DC. In this case the Hamiltonian reduces to a simple, fully decoupled:

$$\tilde{h} = \frac{\pi_1^{\ 2} + \pi_3^{\ 2}}{2} + \frac{\pi_o^{\ 2}}{2} \cdot \frac{m^2c^2}{p_o^{\ 2}} + u\frac{\tau^2}{2};; \qquad (53)$$

$$\frac{dX}{ds} = \mathbf{D}\cdot X;\ \mathbf{D} = \begin{bmatrix} D_x & 0 & 0 \\ 0 & D_y & 0 \\ 0 & 0 & D_l \end{bmatrix};\ D_x = D_y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; D_l = \begin{bmatrix} 0 & \frac{m^2c^2}{p_o^2} \\ -u & 0 \end{bmatrix};$$

$$\mathbf{M} = \begin{bmatrix} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_l \end{bmatrix}; M_x = M_y = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}; \qquad \omega = \sqrt{\left|\det D_l\right|} = \frac{mc}{p_o}\sqrt{|u|} \qquad (53)$$

$$M_l = \begin{bmatrix} \cos\omega s & \frac{m^2c^2}{p_o^2}\sin\omega s/\omega \\ -u\sin\omega s/\omega & \cos\omega s \end{bmatrix}; u > 0;\ M_l = \begin{bmatrix} \cosh\omega s & \frac{m^2c^2}{p_o^2}\sinh\omega s/\omega \\ -u\sinh\omega s/\omega & \cosh\omega s \end{bmatrix}; u < 0;$$

In majority of the cases ωs<<1 ($mc/p_o \sim 1/\gamma$) and RF cavity can be represented as a thin lens located in its center:

$$\mathbf{M} = \begin{vmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_l \end{vmatrix}; \quad M_l = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix}; \quad q = u \cdot l_{RF} = -\frac{e}{p_o c}\frac{\partial V_{rf}}{\partial t} \tag{55}$$

Just for fun, let's look at 1D matrices of quadrupole:

$$\tilde{h} = \left(\frac{\pi_3^{\ 2}}{2} + K_1\frac{y^2}{2}\right) + \left(\frac{\pi_1^{\ 2}}{2} - K_1\frac{x^2}{2}\right) + \frac{\pi_\delta^{\ 2}}{2}\cdot\frac{m^2c^2}{p_o^{\ 2}};\ K_1 = \frac{e}{p_o c}\frac{\partial B_y}{\partial x}; \tag{56}$$

Thus, it is non-degenerated case only when $\det[D] \neq 0$ we have a simple two-piece expression :

$$\exp[\mathbf{D}s] = e^{\lambda s}\frac{\mathbf{D} - \lambda\mathbf{I}}{2\lambda} - e^{-\lambda s}\frac{\mathbf{D} + \lambda\mathrm{I}}{2\lambda} \tag{57}$$

while (37) bring it home right away:

$$\exp[\mathbf{D}s] = \mathbf{I}\cdot\frac{e^{\lambda s} + e^{-\lambda s}}{2} + \mathbf{D}\frac{e^{\lambda s} - e^{-\lambda s}}{2\lambda};$$

$$\exp[\mathbf{D}s] = \mathbf{I}\cdot\cosh|\lambda|s + \frac{\mathbf{D}\sinh|\lambda|s}{|\lambda|};\quad \det[\mathbf{D}] < 0;\quad |\lambda| = \sqrt{-\det[\mathbf{D}]} \tag{58}$$

$$\exp[\mathbf{D}s] = \mathbf{I}\cdot\cos|\lambda|s + \frac{\mathbf{D}\sin|\lambda|s}{|\lambda|};\qquad \det[\mathbf{D}] > 0;\quad |\lambda| = \sqrt{\det[\mathbf{D}]}$$

The case $\det[\mathbf{D}]=0$ means in this case that D is nilpotent: eqs (37) look like follows

$$\det\mathbf{D}=0 \Rightarrow \lambda_1=-\lambda_2=0;\ d(\lambda)=\det[\mathbf{D}-\lambda I]=(\lambda_1-\lambda)(-\lambda_1-\lambda)=\lambda^2 \quad \Rightarrow \mathbf{D}^2=0$$

hence

$$\exp[\mathbf{D}s]=\mathbf{I}+\mathbf{D}s; \quad \det[\mathbf{D}]=0; \tag{59}$$

For non-scaled case is just a change of variables:

$$\tilde{h}=\left(\frac{P_3^{\ 2}}{2p_o}+p_oK_1\frac{y^2}{2}\right)+\left(\frac{P_1^{\ 2}}{2p_o}-p_oK_1\frac{x^2}{2}\right)+\frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^{\ 2}};K_1=\frac{e}{p_oc}\frac{\partial B_y}{\partial x}; \tag{60}$$

$$D_x=\begin{bmatrix}0 & 1/p_o\\ p_oK_1 & 0\end{bmatrix};\ D_y=\begin{bmatrix}0 & 1/p_o\\ -p_oK_1 & 0\end{bmatrix};\ \phi=s\sqrt{K_1}$$

$$M_F=\begin{bmatrix}\cos\phi & \sin\phi/p_o\sqrt{K_1}\\ -p_o\sqrt{K_1}\sin\phi & \cos\phi\end{bmatrix};\ M_D=\begin{bmatrix}\cosh\phi & \sinh\phi/p_o\sqrt{K_1}\\ p_o\sqrt{K_1}\sinh\phi & \cosh\phi\end{bmatrix}$$

In a case when length of the quadrupole is very short, but the strength is finite. It is called thin-lens approximations:

$$\varphi = s\sqrt{K_1} \to 0; K_1 s = const = \frac{1}{F}$$

$$M_F \to \begin{bmatrix} 1 & 0 \\ -\frac{p_o}{F} & 1 \end{bmatrix}; M_D \to \begin{bmatrix} 1 & 0 \\ \frac{p_o}{F} & 1 \end{bmatrix}$$

$$\{x, x'\}, \{y, y'\}$$

$$M_F \to \begin{bmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{bmatrix}; M_D \to \begin{bmatrix} 1 & 0 \\ \frac{1}{F} & 1 \end{bmatrix} \tag{61}$$