PHY 564
Advanced Accelerator Physics
Lecture 3
Particles In Electromagnetic Fields

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In likely event of us running out of time in class – the home reading is paragraphs 1-7 of Classical theory of fields of L.D. Landau and E.M. Lifshitz

1.1 Relativistic Mechanics

From here further: i=0,1,2,3 in Minkowki space with (1,-1,-1,-1) metric.

Let's use Principle of Least Action for a relativistic particle. To determine *the action integral for a free particle* (which does not interact with the rest of the world), we must ensure that the action integral does not depend on our choice of the inertial system. Otherwise, the laws of the particle motion also will depend on the choice of the reference system, which contradicts the first principle of relativity. Therefore, the action must be invariant of Lorentz transformations and rotation in 3D space; i.e., it must depend on a 4D scalar. So far, *from Appendix A*, we know of one 4D scalar for a free particle: the interval. We can employ it as trial function for the action integral, and, by comparing the result with classical mechanics find a constant α connecting the action with the integral of the interval:

$$ds^2 = dx^i dx_i \equiv \sum_{i=1}^{4} dx^i dx_i = (cdt)^2 - (d\vec{r})^2$$

$$S = -\alpha \int_A^B ds = -\alpha \int_A^B \sqrt{(cdt)^2 - d\vec{r}^2}. \qquad (16)$$

The minus sign before the integral reflects a natural phenomenon: the law of inertia requires a resting free particle to stay at rest in inertial system. The interval $ds = cdt$ has a maximum possible value ($cdt \geq \sqrt{(cdt)^2 - d\vec{r}^2}$) and requires for the action to be minimal, that the sign is set to be "-".

The integral (16) is taken along the world line of the particle. The initial point A (event) determines the particle's start time and position, while the final point B (event) determines its final time and position. The action integral (16) can be represented as integral with respect to the time:

$$S = -\alpha \int_A^B \sqrt{(cdt)^2 - d\vec{r}^2} = -\alpha c \int_A^B dt \sqrt{1 - \vec{v}^2 / c^2} = \int_A^B \mathbf{L} dt; \qquad \mathbf{L} = -\alpha c \sqrt{1 - \frac{\vec{v}^2}{c^2}}; \ \vec{v} = \frac{d\vec{r}}{dt};$$

where $\mathbf{L}$ signifies the Lagrangian function of the mechanical system. It is important to note that while the action is an invariant of the Lorentz transformation, the Lagrangian is not. It must depend on the reference system because time depends on it. To find coefficient α, we compare the relativistic form with the known classical form by expanding $\mathbf{L}$ by $\vec{v}^2 / c^2$:

$$\mathbf{L} = -\alpha c \sqrt{1 - \frac{\vec{v}^2}{c^2}} \approx -\alpha c + \alpha \frac{\vec{v}^2}{2c}; \qquad \mathbf{L}_{classical} = m \frac{\vec{v}^2}{2};$$

which confirms that α is positive and $\alpha = mc$, where m is the mass of the particle.

Thus, we found the action and the Lagrangian for a relativistic particle:

$$S = -mc\int_A^B ds\ ; \tag{17}$$

$$\mathbf{L} = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}}; \tag{18}$$

The energy and momentum of the particles are defined by the standard relations eqs. (4) and (5):

$$\vec{p} = \frac{\partial \mathbf{L}}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} = \gamma m\vec{v}; \tag{19}$$

$$E = \vec{p}\vec{v} - L = \gamma mc^2;\ \ \gamma = 1/\sqrt{1-\vec{v}^2/c^2} \tag{20}$$

with ratio between them of

$$E^2 = \vec{p}^{\,2}c^2 + \left(mc^2\right)^2. \tag{21}$$

The energy of the resting particle does not go to zero as in classical mechanics but is equal to the famous Einstein value, $E = mc^2$; with the standard classical additions at low velocities ($v << c; p << mc$): $E \cong mc^2 + m\dfrac{\vec{v}^2}{2} \cong mc^2 + \dfrac{\vec{p}^2}{2m}$.

Four-momentum, conservation laws. The least-action principle gives us the equations of motion and an expression for the momentum of a system. Let us consider the total variation of an action for a single particle:

$$\delta S = -mc\delta\int_A^B ds = -mc\delta\int_A^B \sqrt{dx^i dx_i} = -mc\{\int_A^B \sqrt{d\delta x^i dx_i} + \sqrt{dx^i d\delta x_i}\} =$$

$$-mc\{\int_A^B \frac{d\delta x^i dx_i}{2\sqrt{dx^i dx_i}} + \frac{dx^i d\delta x_i}{2\sqrt{dx^i dx_i}}\} = -mc\int_A^B \frac{dx^i d\delta x_i}{ds} = -mc\int_A^B u^i d\delta x_i;$$

where $u^i \equiv dx^i / ds$ is 4-velocity. Integrating by parts,

$$\delta S = -mcu^i \delta x_i \Big|_A^B + mc\int_A^B \delta x_i \frac{du^i}{ds} ds; \qquad (22)$$

we obtain the expression that can be used for all purposes. First, using the least-action principle with fixed A and B $\delta x_i(A) = \delta x_i(B) = 0$, to derive the conservation of 4-velocity for a free particle: $\frac{du^i}{ds} = 0;\ u^i = const$ or **the inertia law.**

Along a real trajectory $mc\int_A^B \delta x_i \frac{du^i}{ds} ds = 0$ the action is a function of the limits A and B (see eq. (12):

$\delta S_{real\ traj} = (-E\delta t + \vec{P}\delta\vec{r})\Big|_A^B$, i.e., $dS_{real\,traj} = -Edt + \vec{P}d\vec{r}$ is the full differential of t and $\vec{r}$ with energy and momentum as the parameters. We note that this form of the action already is a Lorentz invariant:

$$\delta S_{real\ traj} = (-E\delta t + \vec{P}\delta\vec{r})\Big|_A^B = (-P^i \delta x_i)\Big|_A^B;$$

i.e. classical Hamiltonian mechanics always encompassed a relativistic form and a metric: a scalar δS is a 4-product of P^i and δx_i with the metric (1,-1,-1,-1). Probably one of most remarkable things in physics is that its classic approach detected the metric of 4-D space and time at least a century before Einstein and Poincaré.

To get 4-momentum, we consider a real trajectory $mc\int_A^B \delta x_i \frac{du^i}{ds} ds = 0$ and set $\delta x_i(B) = \delta x_i$:

$$p^i = -\frac{\partial S}{\partial x_i} = -\partial^i S = mcu^i = \left(\gamma mc, \gamma m\vec{v}\right) = \left(E/c, \vec{p}\right) \qquad (23)$$

with an obvious scalar product ($u^i u_i = 1$, see Appendix A. eq. (A.42))

$$p^i p_i = E^2/c^2 - \vec{p}^{\,2} = m^2c^2u^iu_i = m^2c^2\,. \qquad (24)$$

Equivalent forms of presentation are

$$p^i = (E/c, \vec{p}) \equiv m\gamma_v(c, \vec{v}) \equiv \frac{(mc, m\vec{v})}{\sqrt{1 - v^2/c^2}} \qquad (25)$$

and, Lorentz transformation (P^i is a 4-vector, K' moves with $\vec{V} = \hat{e}_x V$):

$$E = \gamma_V(E' + c\beta_V p'_x); p_x = \gamma_V(p'_x + \beta_V E'/c); p_{y,z} = p'_{y,z}; \gamma_V = 1/\sqrt{1 - \beta_V^{\,2}}; \beta_V = V/c; \qquad (26)$$

where subscripts are used for γ, β to define the velocity to which they are related. .

Equation (24) expresses energy, velocity, and the like in terms of momenta and allows us to calculate all differentials:

$$E = c\sqrt{\vec{p}^2 + m^2c^2}; dE = cd\sqrt{\vec{p}^2 + m^2c^2} = \frac{d\vec{p}\cdot c\vec{p}}{\sqrt{\vec{p}^2 + m^2c^2}} = \frac{c^2\vec{p}\cdot d\vec{p}}{E} = \vec{v}\cdot d\vec{p}; \tag{27}$$

$$\vec{v} = \frac{c\vec{p}}{\sqrt{\vec{p}^2 + m^2c^2}}; \vec{a}dt = d\vec{v} = d\frac{c\vec{p}}{\sqrt{\vec{p}^2 + m^2c^2}} =$$
$$\frac{c\left(d\vec{p}(\vec{p}^2 + m^2c^2) - \vec{p}(\vec{p}d\vec{p})\right)}{\left(\sqrt{\vec{p}^2 + m^2c^2}\right)^3} = c\frac{d\vec{p}\cdot m^2c^2 + \left[\vec{p}\times\left[d\vec{p}\times\vec{p}\right]\right]}{\left(\sqrt{\vec{p}^2 + m^2c^2}\right)^3}; \tag{28}$$

Coefficients $\gamma = E/mc^2$; $\vec{\beta} = \vec{v}/c$ differ from the above by constants, and satisfy similar relations.

The conservation laws reflect the homogeneity of space and time (see Mechanics): these natural laws do not change even if the origin of the coordinate system is shifted by δx. Then, $\delta x_i(A) = \delta x_i(B) = \delta x_i$. We can consider a closed system of particles (without continuous interaction, i.e., for most of the time they are free). Their action is sum of the individual actions, and

$$\sum_a \delta S_a = -(\sum_a m_a c u^i{}_a)\delta x_i\Big|_A^B = -(\sum_a m_a c u^i{}_a)\delta x_i\Big|_A^B = \{\sum_a p^i{}_a(A) - \sum_a p^i{}_a(B)\}\delta x_i = 0 \tag{29}$$

$$\sum_a p^i{}_a(A) = \sum_a p^i{}_a(B) = \left(\sum_a E_a / c, \sum_a \vec{p}\right) = const. \tag{30}$$

1.2 Particles in the 4-potential of the EM field.

The EM field propagates with the speed of light, i.e., it is a natural product of relativistic 4-D space-time; hence, the 4-potential is not an odd notion!

In contrast with the natural use of the interval for deriving the motion of the free relativistic particle, there is no clear guideline on what type of term should be added into action integral to describe a field. It is possible to consider some type of scalar function $\int A(x^i)ds$ * to describe electromagnetic fields, but this would result in wrong equations of motion. Nevertheless, the next guess is to use a product of 4-vectors $A^i dx_i$, and surprisingly it does work, even though we do not know why? **Hence, the fact that electromagnetic fields are fully described by the 4-vector of potential $\boxed{A^i = (A^0, \vec{A})}$ must be considered as an experimental fact!**

Nevertheless, it looks natural that the interaction of a charge with electromagnetic field is represented by the scalar product of two 4-vectors with the $-e/c$ coefficient chosen by convention:

$$S_{\text{int}} = -\frac{e}{c}\int_A^B A^i dx_i \,;\; A^i \equiv (A^o, \vec{A}) \equiv (\varphi, \vec{A}) \tag{31}$$

where the integral is taken along the particle's world line. A charge e and speed of the light c are moved outside the integral because they are constant; hence, **we use the conservation of the charge e and constancy of the speed of the light !**

It is essential that field is GIVEN, SINCE we are CONSIDERING a particle interacting with a given field.

*You can check that this function will give the equations of motion $(mc - A)\dfrac{du^i}{ds} + \partial^i A = 0$

Turning our attention back to the Least-Action Principle and Hamiltonian Mechanics

The standard presentation of 4-potential is

$$A^i \equiv (A^0, \vec{A}) \equiv (\varphi, \vec{A})\ ; \tag{32}$$

where φ is called the scalar potential and $\vec{A}$ is termed the vector potential of electromagnetic field.

Gauge Invariance. As we discussed earlier the action integral is not uniquely defined; we can add to it an arbitrary function of coordinates and time without changing the motion: $S' = S + f(x_i)$. This corresponds to adding the full differential of f in the integral (31)

$$S' = \int_A^B \left(-mcds - \frac{e}{c} A^i dx_i + dx_i \partial^i f \right).$$

This signifies that the 4-potential is defined with sufficient flexibility to allow the addition of any 4-gradient to it (let us choose $f(x_i) = \frac{e}{c} g(x_i)$)

$$A'^i = A^i - \partial^i g(x_i) = A^i - \frac{\partial g}{\partial x_i}; \tag{33}$$

without affecting the motion of the charge, a fact called **THE GAUGE INVARIANCE .**

We should be aware that the evolution of the system does not change but appearance of the equation of the motion for the system could change. For example, as follows from (33), the canonical momenta will change:

$$P'^i = P^i - \partial^i f\ .$$

Nevertheless, only the appearance of the system is altered, not its evolution. Measurable values (such as fields, mechanical momentum) do not depend upon it. One might consider Gauge invariance as an inconvenience, but, in practice, it provides a great opportunity to find a gauge in which the problem becomes more comprehensible and solvable.

The action is an additive function: therefore, the action of a charge in electromagnetic field is simply the direct sum of a free particle's action and action of interaction: (remember $ds = ds^2 / ds = dx^i dx_i / ds = u^i dx_i$)

$$S = \int_A^B \left(-mcds - \frac{e}{c} A^i dx_i \right) = \int_A^B \left(-mcu^i - \frac{e}{c} A^i \right) dx_i \tag{34}$$

Then the total variation of the action is

$$\delta S = \delta \int_A^B \left(-mcds - \frac{e}{c} A^i dx_i \right) = \int_A^B \left(-mc \frac{dx^i d\delta x_i}{ds} - \frac{e}{c} A^i d\delta x_i - \frac{e}{c} \delta A^i dx_i \right) =$$
$$-\left[\left(mcu^i + \frac{e}{c} A^i \right) \delta x_i \right]_A^B + \int_A^B \left(mc \frac{du^i}{ds} \delta x_i ds + \frac{e}{c} \delta x_i dA^i - \frac{e}{c} \delta A^i dx_i \right) = 0. \tag{35}$$

That gives us a 4-momentum

$$P^i = -\frac{\delta S}{\delta x_i} = \left(mcu^i + \frac{e}{c} A^i \right) = \left(H / c, \vec{P} \right) = p^i + \frac{e}{c} A^i; \tag{36}$$

with

$$H = E = c(mcu^0 + \frac{e}{c} A^0) = \gamma mc^2 + e\varphi = c\sqrt{m^2 c^2 + \vec{p}^2} + e\varphi;$$

$$\vec{P} = \gamma m\vec{v} + \frac{e}{c} \vec{A} = \vec{p} + \frac{e}{c} \vec{A}; \Rightarrow \vec{p} = \vec{P} - \frac{e}{c} \vec{A}. \tag{37}$$

The Hamiltonian must be expressed in terms of generalized 3-D momentum, $\vec{P} = \vec{p} + \frac{e}{c}\vec{A}$ and it is

$$H(\vec{r},\vec{P},t) = c\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\varphi; \tag{38}$$

with Hamiltonian equation following from it:

$$\vec{\mathrm{v}} = \frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{P}} = \frac{\vec{P}c - e\vec{A}}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}}$$

$$\frac{d\vec{P}}{dt} = \frac{d\vec{p}}{dt} + \frac{e}{c}\frac{d\vec{A}}{dt} = -\frac{\partial H}{\partial \vec{r}} = -e\vec{\nabla}\varphi - e\frac{\{(\vec{P} - \frac{e}{c}\vec{A})\cdot\vec{\nabla}\}\vec{A}}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} = -e\vec{\nabla}\varphi - \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A};$$

From this equation we can derive (without any elegance!) the equation for mechanical momentum $\vec{p} = \gamma m\vec{v}$. We will not do it here, but rather we will use easier way to obtain the 4D equation of motion via the least-action principle. We fix A and B to get from equation (35)

$$\delta S = \int_A^B \left(mc\frac{du^i}{ds}\delta x_i ds + \frac{e}{c}\delta x_i dA^i - \frac{e}{c}\delta A^k dx_k \right) = \int_A^B \left(mc\frac{du^i}{ds}\delta x_i ds + \frac{e}{c}\frac{\partial A^i}{\partial x_k}\delta x_i dx_k - \frac{e}{c}\frac{\partial A^k}{\partial x_i}\delta x_i dx_k \right) =$$
$$\int_A^B \left(\frac{dp^i}{ds} + \frac{e}{c}\left\{\frac{\partial A^i}{\partial x_k} - \frac{\partial A^k}{\partial x_i}\right\}u_k \right)\delta x_i ds = 0. \tag{39}$$

As usual, the expression inside the round brackets must be set at zero to satisfy (39); i.e., we have the equations of charge motion in an electromagnetic field:

$$mc\frac{du^i}{ds} \equiv \frac{dp^i}{ds} = \frac{e}{c}F^{ik}u_k; \tag{40}$$

wherein we introduce an anti-symmetric **electromagnetic field tensor**

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}. \tag{41}$$

Electromagnetic field tensor: The Gauge Invariance can be verified very easily:

$$F'^{ik} = \frac{\partial A'^k}{\partial x_i} - \frac{\partial A'^i}{\partial x_k} = F^{ik} - \frac{\partial^2 g}{\partial x_i \partial x_k} + \frac{\partial^2 g}{\partial x_k \partial x_i} = F^{ik};$$

which means that the equation of motion (40) is not affected by the choice of the gauge, and the **electromagnetic field tensor is defined uniquely!** Using the Landau convention, we can represent the asymmetric tensor by two 3-vectors (see Appendix A):

$$F^{ik} = (-\vec{E},\vec{B}); F_{ik} = (\vec{E},\vec{B});$$

$$F^{ik} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{42}$$

$\vec{E}$ is the so-called vector of the electric field and $\vec{B}$ is the vector of the magnetic field. Note the occurrence of the Lorentz group generator (see special material for Lecture 2) in (42).

The 3D expressions of the field vectors can be obtained readily:

$$E^{\alpha} = F^{\alpha 0} = \frac{\partial A^0}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_0} - = -\frac{\partial \varphi}{\partial r_\alpha} - \frac{1}{c}\frac{\partial A^\alpha}{\partial t};\ \alpha = 1,2,3; \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - grad\varphi; \quad (43)$$

$$B^{\alpha} = -\frac{1}{2} e^{\alpha\kappa\lambda} F^{\kappa\lambda} = e^{\alpha\kappa\lambda}\left(\frac{\partial A^\lambda}{\partial x_\kappa} - \frac{\partial A^\kappa}{\partial x_\lambda}\right);\ \vec{B} = curl\vec{A};\ F^{\kappa\lambda} = e^{\lambda\kappa\alpha} H_\alpha. \quad (44)$$

A 3D asymmetric tensor $e^{\alpha\kappa\lambda}$ and the *curl* definition are used to derive last equation and use Greek symbols for the spatial 3D components. The electric and magnetic fields are also Gauge invariant being components of Gauge invariant tensor.

We have the first pair of Maxwell's equations without further calculation using the fact that differentiation is symmetric operator ($\partial^i \partial^k \equiv \partial^k \partial^i$):

$$e_{iklm}\partial^k F^{lm} = e_{iklm}\partial^k\left(\partial^l A^m - \partial^m A^l\right) = 2e_{iklm}(\partial^k \partial^l) A^m = 0; \quad (45)$$

or explicitly:

$$\partial^k F^{lm} + \partial^l F^{mk} + \partial^m F^{kl} = 0. \quad (46)$$

A simple exercise gives the 3D form of the first pair of Maxwell equations. They also can be attained using (43) and (44) and known 3D equivalencies: $div(curl\vec{A}) \equiv 0$; $curl(grad\varphi) \equiv 0$:

$$\begin{aligned} &\vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}; && curl\vec{E} = -curl(grad\varphi) - \frac{1}{c} curl\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}; \\ &\vec{B} = curl\vec{A}; && div\vec{B} = div(curl\vec{A}) \equiv 0; \end{aligned} \quad (47)$$

I note that (47) is the exact 3D equivalent of invariant 4D Maxwell equations (45) that you may wish to verify yourself. There are 4 equations in (45): i=0,1,2,3. The *div* is one equation and *curl* gives three (vector components) equations. Even the 3D form looks very familiar; the beauty and relativistic invariance of the 4D form makes it easy to remember and to use.

EM Fields transformation, Invariants of the EM field. The 4-potential was defined as 4-vector and it transforms as 4-vector. The electric and magnetic fields, as components of the asymmetric tensor, follow its transformation rules (See Appendix A).

$$\begin{aligned} &\varphi = \gamma(\varphi' + \beta A'_x); \;\; A_x = \gamma(A'_x + \beta\varphi'); \\ &E_y = \gamma(E'_y + \beta B'_z); \; E_z = \gamma(E'_z - \beta B'_y); \\ &B_y = \gamma(B'_y - \beta E'_z); \; B_z = \gamma(B'_z + \beta E'_y). \end{aligned} \tag{48}$$

and the rest is unchanged. An important repercussion from these transformations is that the separation of the electromagnetic field in two components is an artificial one. They translate into each other when the system of observation changes and **MUST** be measured in the same units (Gaussian). The rationalized international system of units (SI) system measures them in V/m, Oe, A/m and T. Why not use also a horse power per square mile an hour, the old British thermal units as well? This makes about the same sense as using Tesla or A/m.

While the values and directions of 3D field components are frame-dependent, two 4-scalars can be build from the EM 4-tensor $F^{ik} = (-\vec{E}, \vec{B})$

$$F^{ik}F_{ik} = inv; \quad e^{iklm}F_{ik}F_{lm} = inv; \tag{49}$$

which in the 3D-form appear as

$$\vec{B}^2 - \vec{E}^2 = inv; \; (\vec{E}\cdot\vec{B}) = inv. \tag{50}$$

This conveys a good sense what can and cannot be done with the 3D components of electromagnetic fields. Any reference frame can be chosen and both fields transferred in a minimal number of components limited by (50). For example; 1) if $|\vec{E}| > |\vec{B}|$ in one system it is true in all systems and vice versa; and (2) if fields are perpendicular in one frame, $(\vec{E}\cdot\vec{B}) = 0$, this is true in all frames. When $(\vec{E}\cdot\vec{B}) = 0$ a frame can always be found where E or B are equal to zero (locally!).

Lorentz form of equation of a charged particle's motion.

The equations of motion (40) can rewritten in the form:

$$\frac{d\mathrm{E}}{dt}=c\frac{dp^0}{dt}=eF^{0k}\mathrm{v}_k=e\vec{E}\cdot\vec{\mathrm{v}};\qquad \mathrm{v}_k=\frac{dx_k}{dt}=(c,-\vec{\mathrm{v}})$$
$$\frac{d\vec{p}}{dt}=e\left(\hat{e}_\alpha F^{\alpha k}\frac{\mathrm{v}_k}{c}\right)=\frac{e}{c}\left(\hat{e}_\alpha\cdot cF^{\alpha 0}-\hat{e}_\alpha\cdot F^{\alpha\kappa}\mathrm{v}_k\right)=e\vec{E}+\hat{e}_\alpha e^{\alpha\kappa\lambda}B_\lambda\frac{\mathrm{v}_k}{c}=e\vec{E}+\frac{e}{c}\left[\vec{\mathrm{v}}\times\vec{B}\right]. \tag{51}$$

So, we have expressions for the generalized momentum and energy of the particle in an electromagnetic field. Generalized momentum is equal to the particle's mechanical momentum plus the vector potential scaled by *e/c*. The total energy of the charged particle is its mechanical energy, γmc^2, plus its potential energy ,$e\varphi$, in an electromagnetic field. The Standard Lorentz (not Hamiltonian!) equations of motion for $\vec{p}=\gamma m\vec{v}$ are

$$\frac{d\vec{p}}{dt}=e\vec{E}+\frac{e}{c}\left[\vec{\mathrm{v}}\times\vec{B}\right]. \tag{52}$$

with the force caused by the electromagnetic field (Lorentz force) comprised of two terms: the electric force, which does not depend on particle's motion, and, the magnetic force that is proportional to the vector product of particle velocity and the magnetic field, i.e., it is perpendicular to the velocity. Accordingly, the magnetic field does not change the particle's energy. We derived it in Eq. (51):

$$mc^2\frac{d\gamma}{dt}=e\vec{E}\cdot\vec{\mathrm{v}}; \tag{53}$$

Eqs. (52) and (53) are generalized equations. Using directly standard Lorentz equations of motion in a 3D form is a poor option. The 4D form is much better (see below) and, from all points of view, the Hamiltonian method is much more powerful! For fun, see the last slide of this lecture

First pair of Maxwell's equations (a little more of juice)

We will derive full set of Maxwell equations using the least action principle. Nevertheless, you can consider the Maxwell equation as given - in any case they were derived originally from numerous experimental laws!

First pair of Maxwell's equations is the consequence of definitions of electric and magnetic field through the 4-potential:

$$\vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \qquad \vec{H} = curl\vec{A};$$

it is equivalent to

$$curl\vec{E} = -curl(grad\varphi) - \frac{1}{c}curl\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}; \qquad div\vec{H} = div(curl\vec{A}) \equiv 0; \tag{59}$$

Nevertheless, it is very important to remember that they are actually originated from experiment. First Maxwell equation is the Faraday law and the second is nothing else that absence of magnetic charge! You should remember all time that inclusion of the term $S_{\text{int}} = -\frac{e}{c}\int_A^B A^i dx_i$ into action integral is consequence of experiment! Thus, the first pair of Maxwell equations governing the electromagnetic fields is:

$$curl\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}; \tag{60}$$

$$div\vec{H} = 0; \tag{61}$$

with well known integral ratios following it:

Gauss' theorem:
$$\oint \vec{H}d\vec{a} = \int div\vec{H}dV = 0; \tag{62}$$

Stokes' theorem:
$$\oint \vec{E}d\vec{l} = \int curl\vec{E}d\vec{a} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{H}d\vec{a}; \tag{63}$$

where $d\vec{a}$ is vector of the element of the surface and $d\vec{l}$ is a vector of a contour length. Integral equations read: the

1) Flux of the of the magnetic field though the surface covering any volume V is equal zero;
2) The circulation of electric field around the contour (electromotive force) is equal to the derivative of the magnetic flux though the contour scaled down by "-c" - the Faraday law.

7.2 Action of the electromagnetic field.

As we discussed earlier, in the relativistic picture of the world, the field acquires its it's own physical reality. Therefore, the action of whole system including a particle and a field must consist of three parts: the action of free particle, the action of free field and the action their interaction:

$$S = S_p + S_f + S_{pf}. \tag{64}$$

We already got first and last term. For a several free particles, the action is the direct sum of individual actions:

$$S_p = -\sum_p mc \int ds; \tag{65}$$

and interaction with the field is the sum of their individual interactions:

$$S_{pf} = -\sum_p \frac{e}{c} \int A^i dx_i. \tag{66}$$

The sum of (65) and (66) gives us equation of particle's motion in "external", i.e. pre-defined electromagnetic fields. Now we want to know how charged particles influence the EM field and how EM field evolves on its own? We do not know, also, what defines properties of a free field? First pair of Maxwell equations gives us only two connections: the time derivative of the magnetic field and its divergence (zero). We still don't know what is time derivative of electric field and what is its divergence?

Please remember that all following discussion must be considered as a logical excise. Final form of the field action has to have the most important property: it must satisfy the experimental observations! Where to start to get them?

One of the most important properties of the field confirmed by experiments is **the Principle of Superposition:**

the resulting field produced by various sources is a simple composition (the direct sum) of the fields produced by individual sources! It means that resulting electric and magnetic fields are vector sum of individual fields. Thus, we have a clue that we should look for type of equations, which allows superposition of solutions, i.e. linear differential field equations*. In order to generate linear differential equations, the action should contain quadratic expression of the field components**, which described by field 4-tensor F^{ik}.

**In field theory the 4-vector of the field A^i is coordinate of the field. Therefore, field's 4-tensor is first order derivative of the coordinates. According to Hamiltonian principle, the action could have under integral only coordinates and their first derivatives. This requirement excludes derivatives of F^{ik} from the action's integral.*

*** 4-vector of the field A^i is not unique (Gauge transformation) and trial function comprising 4-vector of the field will give non-unique equation of the field. The difference with interaction term is that last includes first order of 4-potential and non-uniqueness does not affect equation of motions. Situation is not the same for quadratic term! A variation acts in similar manner as a differentiation - "to get linear ($2x^1$) we need to differentiate (x^2)".*

In addition, the action must be 4-invariant (4-scalar, not pseudo-scalar!), which leaves us with $F^{ik}F_{ik} = 2(\vec{H}^2 - \vec{E}^2)$. Finally, the field is "an entity leaving" in space and time coordinates. In order to describe total field we should integrate over all space between two "time" events $d\Omega = dx^0 dx^1 dx^2 dx^3 = c dt dV$ which is 4-invariant: $d\Omega = e_{iklm} dx_a^i dx_b^k dx_c^l dx_d^m$ where a,b,c,d four 4-vectors defining element of 4-volume. Therefore, a probable form of the action of the EM field is:

$$S_f = -a\int F^{ik} F_{ik} d\Omega. \tag{67}$$

The choice of the coefficient before integral is equivalent to the choice of the units to measure the field. In the Gaussian system of units, which we are using, fields are measured in Gs and coefficient is

$$a = \frac{1}{16\pi \cdot c}. \tag{68}$$

The total action is:

$$S = -\sum_p mc\int ds - \sum_p \frac{e}{c}\int A^i dx_i - \frac{1}{16\pi c}\int F^{ik} F_{ik} d\Omega. \tag{69}$$

4-current and equation of continuity. The conservation of the charge should affect our equations. Let's make a glance on this issue and write a charge conservation law in the form useful for future derivation of the field equations. It is very useful to describe charges by a distribution function. The charge density ρ is defined as the charge contained in unit volume:

$$de = \rho dV; \tag{70}$$

and microscopic (exact in classical EM) definition of ρ is sum of Dirac's delta-functions:

$$\rho = \sum_a e_a \delta(\vec{r} - \vec{r}_a); \tag{71}$$

where index *a* is index to count particles. 4-vector of current is defined as:

$$j^i = \rho \frac{dx^i}{dt}. \tag{72}$$

The fact that j^i is a 4-vector comes from equivalence:

$$dedx^i = \rho \frac{dx^i}{dt} \cdot dtdV = \rho \frac{dx^i}{dt} \cdot d\Omega; \tag{73}$$

and the fact that charge is 4-scalar or invariant (experimental fact) and $d\Omega \equiv dVdt$ is the 4-scalar. Thus:

$$j^i = (\rho c, \vec{j}); \vec{j} = \rho \vec{v}. \tag{74}$$

To be exact, for point charges, the 4--current is:

$$j^i = \sum_a e_a \delta(\vec{r} - \vec{r}_a) \frac{dx_a^i}{dt}. \tag{75}$$

It is the microscopic 4-current for ensemble of particles. When it is necessary, it can be averaged over a "small volume" for macroscopic description. We do not need averaging now and can comfortably use Eq. (75). Our goal is to get the equation of continuity:

$$\partial_i j^i = \frac{\partial j^i}{\partial x^i} = (\frac{\partial \rho}{\partial t} + div\vec{j}) = 0; \qquad (76)$$

which is resulting from charge conservation. It is easy to do for microscopic distribution (75):

$$\partial_i j^i = \sum_a e_a \left\{ \frac{\partial}{c\partial t}\big(\delta(\vec{r}-\vec{r}_a(t))c\big) + div\big(\big(\delta(\vec{r}-\vec{r}_a(t))\vec{v}_a\big)\big) \right\} = \sum_a e_a \vec{\nabla}\delta(\vec{r}-\vec{r}_a)\cdot\left\{ -\frac{\partial \vec{r}_a(t)}{\partial t} + \vec{v}_a \right\} \equiv 0; \qquad (77)$$

with $\partial^i = (\partial / \partial ct, \partial / \partial\vec{r}); \partial / \partial\vec{r}(r_a(t)) \equiv 0$ and we use derivative of Dirac's delta-function. Now we are ready for next trick, i.e. to present action of the interaction as integral of 4-current:

$$e_a = \int e_a \delta(\vec{r}-\vec{r}_a) dV; \quad \frac{1}{c}\int \sum_a e_a A_k dx^k = \frac{1}{c}\int A_k \sum_a e_a dx^k \delta(\vec{r}-\vec{r}_a) dV$$
$$= \frac{1}{c}\int A_k \sum_a e_a \frac{dx^k}{dt}\delta(\vec{r}-\vec{r}_a) dV\, dt = \frac{1}{c}\int A_k j^k \, dt\, dV = \frac{1}{c^2}\int A_k j^k \, d\Omega \qquad (78)$$

Side note: Today we are using the method, which is standard for all modern filed theories: QED, QCD, SUSY, etc. In self-consistent theories, particles become fields as well. In QED, an electron is not a point particle but a "wave" described by 4-spinor ψ. We can include this into our action very easy by writing correct QED current in the interaction term (78)

$$j^i = \overline{\psi}\gamma^i\psi\,.$$

In this case, the current is a continuous function of the space and time. It is a better way that having Dirac's delta-function. The nature of the current, as we would see, does not change equation of the field motion. It means that Maxwell equations do not change when we introduce quantum description of charges! In this case, the equation of the charges motion should be also proper, i.e. those derived by Dirac:

$$\left[\gamma^i\left(p_i - \frac{eA_i}{c}\right) - m\right]\gamma^0\psi = 0\,.$$

I would not go into details of Dirac's description of electron and his 4x4 γ-matrices. If you are interested, look through one of many QED books. Thus, equivalent form of (7-12) is:

$$S = -\sum_p mc\int ds - \frac{1}{c^2}\int A_k j^k d\Omega - \frac{1}{16\pi c}\int F^{ik}F_{ik}d\Omega. \qquad (79)$$

Second pair of Maxwell's equations: more of the least action…

We already found equation of charges motion in the field. Let's consider all charges following their equation of motion

$$\delta_{for\ particles}\left(\sum_p mc\int ds+\int A_k j^k d\Omega\right)=0. \qquad (80)$$

Let's changes move along their real trajectories. Now we will vary only the field to find its equations of motion:

$$\delta S=-\frac{1}{16\pi c^2}\int\left(16\pi\delta A_i j^i+c\delta(F^{ik}F_{ik})\right)d\Omega=-\frac{1}{8\pi c^2}\int\left(8\pi\delta A_i j^i+cF^{ik}\delta F_{ik}\right)d\Omega=0; \qquad (81)$$

where we use

$$F^{ik}\delta F_{ik}=\delta F^{ik}F_{ik}. \qquad (82)$$

It is important to remember that we can vary both particle's trajectories and field if we wish. It will give us two terms in the variation of the action: one containing variation of the trajectories

$$\delta S_{part}=\sum_a\int_A^B\left(\frac{dp_a{}^i}{ds}+\frac{e}{c}\left\{\frac{\partial A^i}{\partial x_k}-\frac{\partial A^k}{\partial x_i}\right\}u_k\right)\delta x_{a\,i}ds \qquad (83)$$

and the other containing variation of the field. Variations for each particle and the field are independent.

Therefore, each independent component of action's variation must be equal zero. (83) will give us again equation of particle's motion, while field terms (81) will bring us to the field equations. Let's rewrite second term in (81):

$$F_{ik} = \partial_i A_k - \partial_k A_i$$

$$F^{ik}\delta F_{ik} = F^{ik}\partial_i \delta A_k - F^{ik}\partial_k \delta A_i = -F^{ki}\partial_i \delta A_k - F^{ik}\partial_k \delta A_i = -2F^{ik}\partial_k \delta A_i \ . \tag{84}$$

Now we can integrate by parts:

$$\begin{aligned} \delta S = -\frac{1}{4\pi c^2}\int\left(4\pi \delta A_i j^i - cF^{ik}\,\partial_k \delta A_i\right)d\Omega = \\ -\frac{1}{4\pi c^2}\int\left(4\pi\, j^i + c\,\partial_k F^{ik}\right)\delta A_i\, d\Omega - \frac{1}{4\pi c}\int F^{ik}\,\delta A_i\, dS_k\Big|_{suraface\, of\, \Omega} = 0; \end{aligned} \tag{85}$$

with second integral obtained by 4D Gauss theorem:

$$\int div_4 A^i d\Omega = \oint A^i dS_i, \tag{86}$$

where dS_i is element of hyper-surface surrounding 4-volume Ω. It is not so essential, how it looks. One simple case: we integrate over all space and fixed time interval *(t_1,t_2)*. Surface of the *W* is full 3D space at moments of t_1,t_2. The least action method calls for zero variations on the boundaries $\delta A_i\big|_{surafaceof\,\Omega} = 0$ and second integral in (85) disappears leaving us with:

$$\delta S = -\frac{1}{4\pi c^2}\int\left(4\pi j^i + c\partial_k F^{ik}\right)\delta A_i d\Omega = 0\,. \tag{87}$$

Please notice that we are left only with variations of 4-potential. It is very natural because variations of 4-potential fully define field's variations. Equation (87) gives us "second pair" of Maxwell equations in 4D form:

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i \tag{88}$$

3D form follows directly from (88) and form of the field tensor:

$$F^{lm} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{bmatrix}; \tag{89}$$

and yields:

$$div\,\vec{E} = 4\pi\rho; \tag{90}$$

$$curl\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}. \tag{91}$$

Integral equations are obvious applications of Stokes and Gauss theorems to Eqs (90-91)

$$\oint \vec{E} d\vec{a} = 4\pi \int \rho dV; \tag{92}$$

$$\oint \vec{H} d\vec{l} = \frac{1}{c} \int (4\pi \vec{j} + \frac{\partial \vec{E}}{\partial t}) d\vec{a}. \tag{93}$$

Equivalent forms of Maxwell equations:

$$\vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \quad \vec{H} = curl\vec{A}; \quad \Leftrightarrow \quad F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}. \tag{94}$$

Compact 4-D:

$$e^{iklm} \frac{\partial F_{lm}}{\partial x^k} = 0; \quad \oplus \quad \frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i; \tag{95}$$

What we rehashed

- We continued using the Least Action Principle and obtained all necessary equations
  - For motion of relativistic charged particles in E&M fields
  - And their influence on E&M field, e.g. E&M field evolution
- We continued path of using Hamiltonian formalism and defined Canonical momentum of charged particle in E&M field

- This ends out refresher classes - now we are ready to take on real accelerator physics

- Again, during this class we covered few months worth of material
- You should refresh your memory by flipping through your favorite E&M and relativistic mechanics books

Next slide is for most curious of you

It is worth noting that the 4D form of the charge motion (40) and its matrix form is the most compact one,

$$u^i = \frac{dx^i}{ds}; \;\; mc\frac{du^i}{ds} = \frac{e}{c}F^i{}_k u^k; \Rightarrow \frac{d}{ds}[x] = [I]\cdot[u]; \;\; \frac{d}{ds}[u] = \frac{e}{mc^2}[F]\cdot[u] \tag{54}$$

and, in many cases, it is very useful. We treat the ***x, u*** as a vectors, and ***[F]*** as the 4x4 matrix. ***[I]*** is just the unit 4x4 matrix It has interesting formal solution in the matrix form:

$$[u] = e^{\int \frac{e}{mc^2}[F]ds}[u_0]; \; [x] = [x_o] + \left[\int ds e^{\int \frac{e}{mc^2}[F]ds}\right][u_0] \tag{55}$$

Its resolution is well defined when applied to the motion of a charged particle in uniform, constant EM field:

$$[u] = e^{\frac{e}{mc^2}[F](s-s_0)}[u_0]; \; [x] = [x_o] + \left[\int e^{\frac{e}{mc^2}[F](s-s_0)}ds\right][u_0] \tag{56}$$

The Lorentz group of theoretical physics (see Appendix B) is fascinating, and the fact that EM field tensor has the same structure as the generator of Lorentz group is no coincidence – rather, it is indication that physicists have probably come very close to the roots of nature in this specific direction. This statement is far from truth for other fundamental forces and interactions.

To conclude this subsection, we will take one step further from (54) and write a totally linear evolution equation for a combination of 4D vectors

$$\frac{d}{ds}\begin{bmatrix} x \\ u \end{bmatrix} = [\Lambda]\cdot\begin{bmatrix} x \\ u \end{bmatrix}; \;\; [\Lambda] = \begin{bmatrix} 0 & I \\ 0 & \frac{e}{mc^2}F \end{bmatrix} \tag{57}$$

where [Λ] is an 8x8 degenerated matrix. Similarly to (55) and (56)

$$\begin{bmatrix} x \\ u \end{bmatrix} = e^{\int [\Lambda]ds}\cdot\begin{bmatrix} x \\ u \end{bmatrix}_o; \; \begin{bmatrix} x \\ u \end{bmatrix} = e^{[\Lambda](s-s_o I}\cdot\begin{bmatrix} x \\ u \end{bmatrix}_o \; for\,[\Lambda] = const; \tag{58}$$