Homework 7. Due October 9

Problem 1. 10 points. FODO cell.

Consider a general FODO cell comprised of two quadrupoles F and D separated by two drift sections, e.g. the structure below:

$$F:\ K_F = \frac{e}{pc}\frac{\partial B_y}{\partial x}, l_F;$$
$$O1: l_1$$
$$D:\ K_D = \frac{e}{pc}\frac{\partial B_y}{\partial x}, l_D;$$
$$O2: l_2$$

(a) write matrix (both x and y or 4x4) of general FODO cell (not assuming any limitations on K F,D).

(b) write stability criteria (for x and y) for periodic lattice built of this FOD cell. Hint – do not try to solve it!

(c,d) make transition to short lens approximation and assume equal strength of

$$l_F K_F = -K_D l_D = \frac{1}{f} = const, l_{F,D} \to 0$$
$$l = l_1 = l_2$$

and

(c) show that both x and y motion can be stable (e.g. prove so called strong focusing: combination of focusing and defocusing length can provide focusing in both directions);

(d) define (e.g solve) the stability criteria for such cell.

Problem 2. 10 points. Matrix Gymnastics.

For a one-dimensional motion consider parameterization via the eigen vectors and their propagation along *s*:

$$Y(s) = \begin{bmatrix} \mathrm{w(s)} \\ \mathrm{w}'(s) + \dfrac{i}{\mathrm{w(s)}} \end{bmatrix}; Y^*(s) = \begin{bmatrix} \mathrm{w(s)} \\ \mathrm{w}'(s) - \dfrac{i}{\mathrm{w(s)}} \end{bmatrix};$$

$$\beta(s) \equiv \mathrm{w}^2(s);\ \ \alpha(s) = -\frac{\beta'(s)}{2} \equiv -\mathrm{w(s)w'(s)};\ \ \frac{d\psi}{ds} = \frac{1}{\mathrm{w}^2(s)} \equiv \frac{1}{\beta(s)};$$

$$Y(s_2)e^{i\Delta\psi} = \mathrm{M}(s_1|s_2)Y(s_1) \Leftrightarrow Y^*(s_2)e^{-i\Delta\psi} = \mathrm{M}(s_1|s_2)Y^*(s_1); \Delta\psi = \psi(s_2) - \psi(s_1).$$

or in other form as:

$$\left[Y(s_2)e^{i\Delta\psi}, Y^*(s_2)e^{-i\Delta\psi}\right] = \mathrm{M}(s_1|s_2)\left[Y(s_1), Y^*(s_1)\right];$$
$$\tilde{W}(s) = \left[Y(s)e^{i\psi(s)}, Y^*(s)e^{-i\psi(s)}\right]; \tilde{W}(s_2) = \mathrm{M}(s_1|s_2)\tilde{W}(s_1);$$

Show that then matrix can be expressed through the values of envelopes **w**, and its derivatives **w'** and the "betatron" phase advance $\Delta\psi = \psi(s_2)\text{-}\psi(s_1)$ as:

$$M\left(s_1|s_2\right) = \begin{bmatrix} \frac{w_2}{w_1}\cos\Delta\psi - w_1' w_2 \sin\Delta\psi & w_1 w_2 \sin\Delta\psi \\ -\sin\Delta\psi\left(\frac{1}{w_1 w_2} + w_1' w_2'\right) - \cos\Delta\psi\left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right) & \frac{w_1}{w_2}\cos\Delta\psi + w_2' w_1 \sin\Delta\psi \end{bmatrix}$$

or in traditional terms:

$$M\left(s_1|s_2\right) = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi + \alpha_1\sin\Delta\psi\right) & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\frac{\sin\Delta\psi\left(1+\alpha_1\alpha_2\right)+\cos\Delta\psi\left(\alpha_2-\alpha_1\right)}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi - \alpha_2\sin\Delta\psi\right) \end{bmatrix}$$

Hint: if you are using $\tilde{W}(s)$*, than use that* $\tilde{W}^T S W = -2iS$*. If you hate complex variables, than construct* $\tilde{U}(s)=\left[\tilde{R}(s),\tilde{Q}(s)\right]; \tilde{R}(s) = \operatorname{Re}\tilde{Y}(s); \tilde{Q}(s) = \operatorname{Im}\tilde{Y}(s);$ *show that it is symplectic, and define how it transformed with s.*