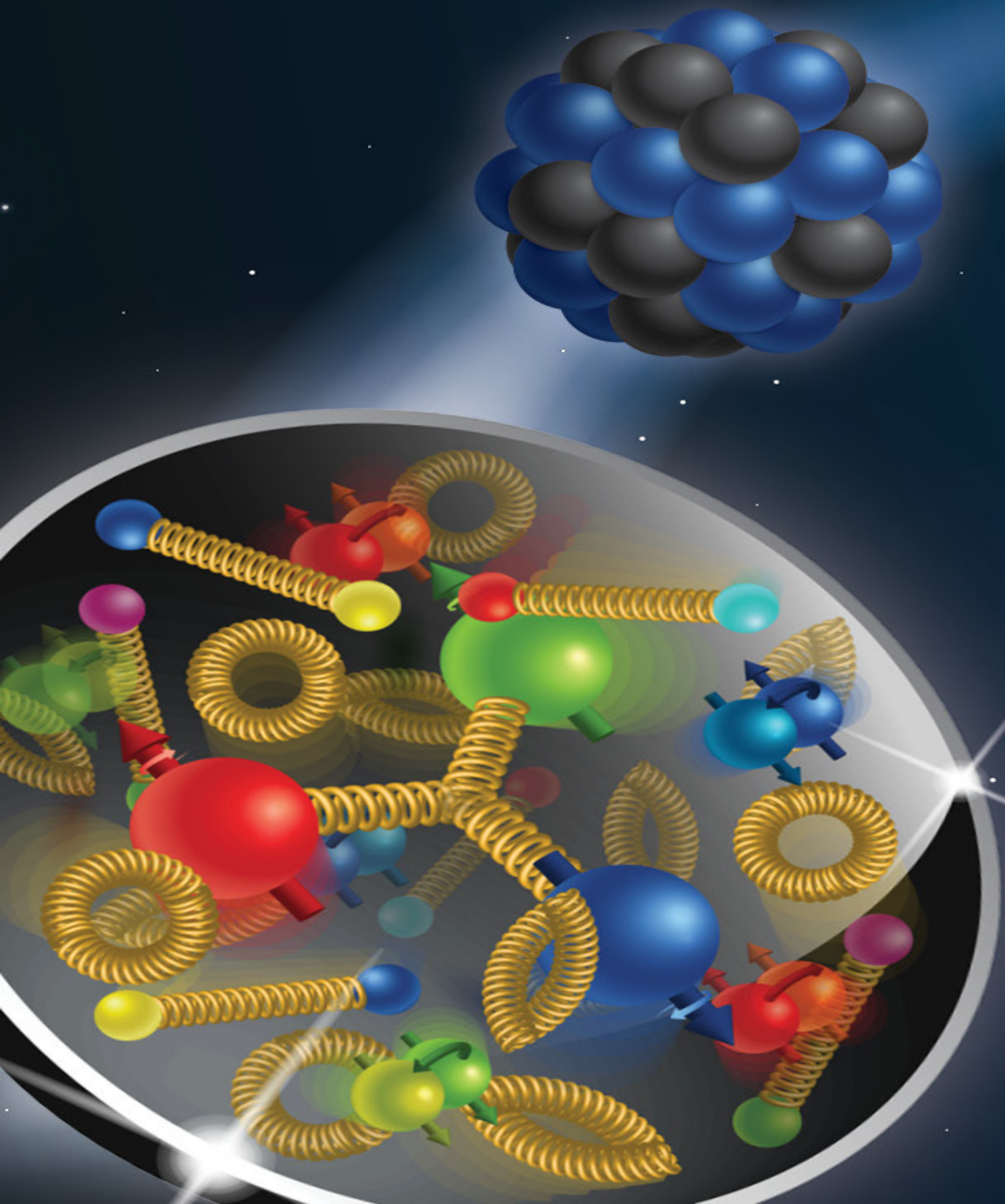

High Power RF Engineering -Cavity (2)

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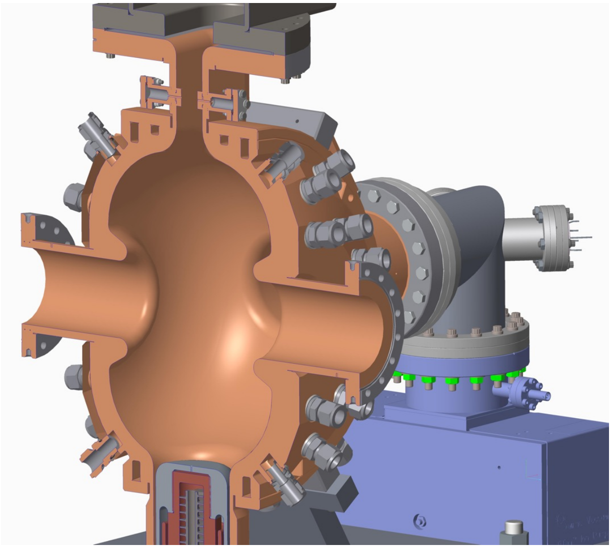
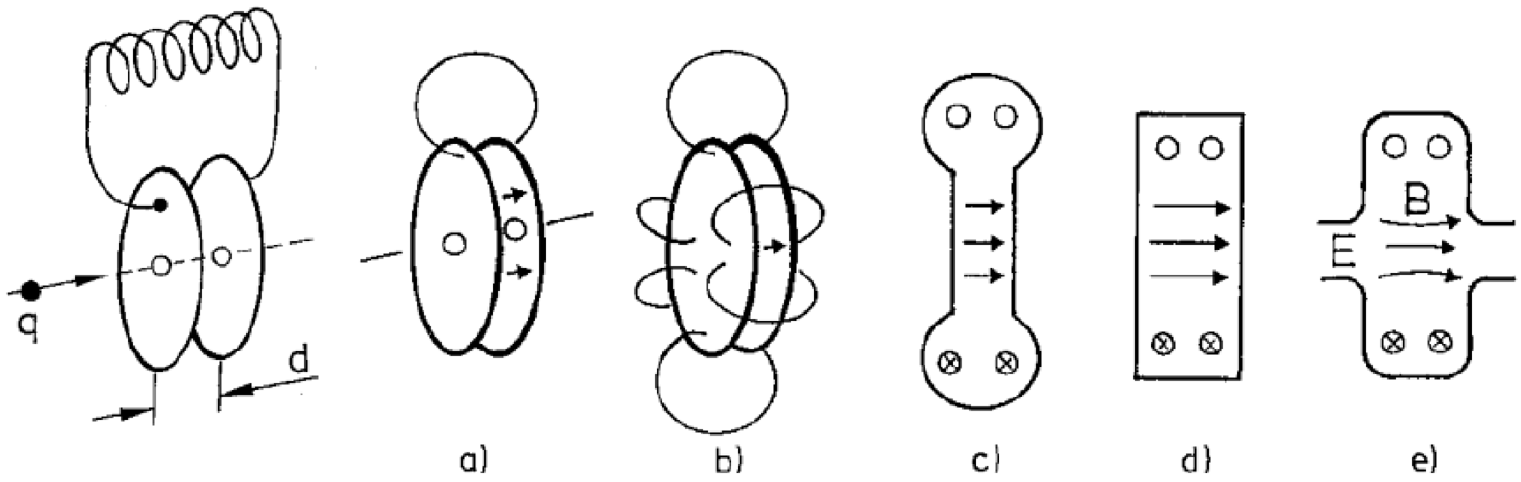
Pillbox cavity

Pillbox cavity



RF Resonator

- Capacitor for DC-> LC circuit for RF -> Pillbox RF cavity



- Before we go deep into it, let us start with EM field

Recall – TM in Circular Waveguide

$$E_\rho = -j\beta\frac{P_{nm}}{a} A\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_\varphi = j\beta\frac{n}{\rho} A\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_z = \left(\frac{P_{nm}}{a}\right)^2 A\cos n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

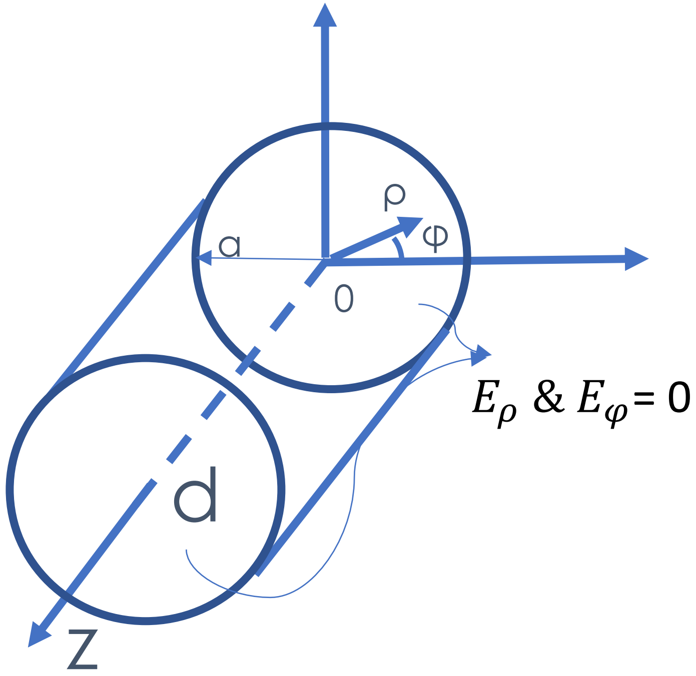
$$H_\rho = -j\omega\varepsilon\frac{n}{\rho} A\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\varphi = -j\omega\varepsilon\frac{P_{nm}}{a} A\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_z = 0$$

$$k_c^2 = k^2 - \beta^2, k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} \;\&\; k_c = \frac{P_{nm}}{a}$$

TM_{nml} in Pillbox Cavity



Circular waveguide with two endplates spaced d. Wave can travel in both directions, $e^{-j\beta z}$ in circular waveguide becomes $C'e^{-j\beta z} + D'e^{j\beta z}$ or $C\cos\beta z + jD\sin\beta z$.

Notice that ∂/∂z now is not –jβ, but
∂sinβz /∂z = -βcosβz or ∂cosβz /∂z = -βsinβz.
Additional boundary condition: $E_\rho \,\&\, E_\varphi|_{at\ z=0,d} = 0$
The term $C\cos\beta z + jD\sin\beta z$ for E_ρ & E_φ should have $C = 0$
& $\sin\beta d = 0$, so $\beta d = l\pi$, l=0,1,2…, it is in the form of $\sin\frac{l\pi}{d}z$
So E_ρ is in the form $B\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$

TM_{nml}

$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ & $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$

$$\nabla \times \mathbf{E} = \boldsymbol{\rho}\left(\frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z}\right) + \boldsymbol{\varphi}\left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right) + \mathbf{z}\frac{1}{\rho}\left(\frac{\partial(\rho E_\varphi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \varphi}\right)$$

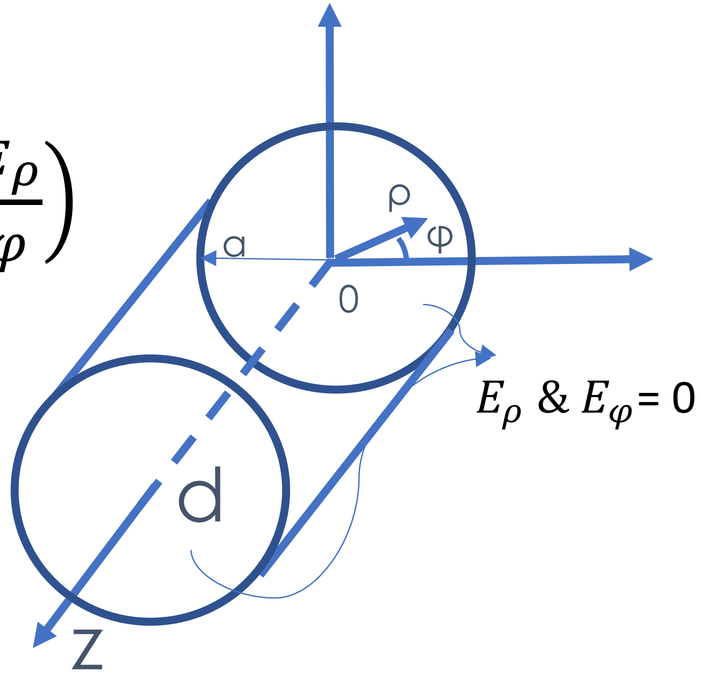
$$= \boldsymbol{\rho}(-j\omega\mu H_\rho) + \boldsymbol{\varphi}(-j\omega\mu H_\varphi) + \mathbf{z}(-j\omega\mu H_z)$$

Similarly

$$\boldsymbol{\rho}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z}\right) + \boldsymbol{\varphi}\left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right) + \mathbf{z}\frac{1}{\rho}\left(\frac{\partial(\rho H_\varphi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \varphi}\right)$$

$$= \boldsymbol{\rho}(j\omega\varepsilon E_\rho) + \boldsymbol{\varphi}(j\omega\varepsilon E_\varphi) + \mathbf{z}(j\omega\varepsilon E_z)$$

and $H_z = 0$ for TM_{nml}



TM_{nml}

$$E_\rho = B\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$$

$$\frac{\partial(\rho E_\varphi)}{\partial\rho} - \frac{\partial E_\rho}{\partial\varphi} = 0 \rightarrow E_\varphi = \frac{-n}{\frac{P_{nm}}{a}\rho} B\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$$

$$-\frac{\partial H_\varphi}{\partial z} = j\omega\varepsilon E_\rho \rightarrow H_\varphi = \frac{j\omega\varepsilon}{\frac{l\pi}{d}} B\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial\rho} = -j\omega\mu H_\varphi \rightarrow E_z = -\frac{\frac{P_{nm}}{a}}{\frac{l\pi}{d}} B\cos n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

$$-j\omega\mu H_\rho = \frac{1}{\rho}\frac{\partial E_z}{\partial\varphi} - \frac{\partial E_\varphi}{\partial z} \rightarrow H_\rho = \frac{j\omega\varepsilon}{\frac{l\pi}{d}\frac{P_{nm}}{a}}\frac{n}{\rho} B\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

TM_{nml}

$$E_\rho = -\frac{l\pi}{d}\frac{P_{nm}}{a} A\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$$

$$E_\varphi = \frac{l\pi}{d}\frac{n}{\rho} A\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$$

$$E_z = \left(\frac{P_{nm}}{a}\right)^2 A\cos n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

$$H_\rho = -j\omega\varepsilon\frac{n}{\rho} A\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

$$H_\varphi = -j\omega\varepsilon\frac{P_{nm}}{a} A\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

$$H_z = 0$$

Circular waveguide:

$$E_\rho = -j\beta\frac{P_{nm}}{a} A\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_\varphi = j\beta\frac{n}{\rho} A\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_z = \left(\frac{P_{nm}}{a}\right)^2 A\cos n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\rho = -j\omega\varepsilon\frac{n}{\rho} A\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\varphi = -j\omega\varepsilon\frac{P_{nm}}{a} A\cos n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_z = 0$$

$$k_c^2 = k^2 - \beta^2,\ k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\ \&\ k_c = \frac{P_{nm}}{a}$$

$$-j\beta e^{-j\beta z} \rightarrow -\frac{l\pi}{d}\sin\frac{l\pi}{d}z\ \&\ e^{-j\beta z} \rightarrow \cos\frac{l\pi}{d}z$$

Pillbox Cavity – TE_{nml}

There is no TE_{nm0} mode

$$E_\rho = j\omega\mu\frac{n}{\rho}A\sin n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$$

$$E_\varphi = j\omega\mu\frac{P'_{nm}}{a}A\cos n\varphi J'_n\left(\frac{P'_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$$

$$E_z = 0$$

$$H_\rho = \frac{l\pi}{d}\frac{P'_{nm}}{a}A\cos n\varphi J'_n\left(\frac{P'_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

$$H_\varphi = -\frac{l\pi}{d}\frac{n}{\rho}A\sin n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)\cos\frac{l\pi}{d}z$$

$$H_z = \left(\frac{P'_{nm}}{a}\right)^2 A\cos n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)\sin\frac{l\pi}{d}z$$

Circular waveguide:

$$E_\rho = \frac{j\omega\mu n}{\rho}A\sin n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_\varphi = j\omega\mu\frac{P'_{nm}}{a}A\cos n\varphi J'_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_z = 0$$

$$H_\rho = -j\beta\frac{P'_{nm}}{a}A\cos n\varphi J'_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\varphi = \frac{j\beta n}{\rho}\sin n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_z = \left(\frac{P'_{nm}}{a}\right)^2 A\cos n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$k_c^2 = k^2 - \beta^2,\ k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\ \&\ k_c = \frac{P'_{nm}}{a}$$

$$-j\beta e^{-j\beta z} \rightarrow \frac{l\pi}{d}\cos\frac{l\pi}{d}z\ \&\ e^{-j\beta z} \rightarrow \sin\frac{l\pi}{d}z$$

TM_{010}

Field pattern

$$E_\rho = 0$$

$$E_\varphi = 0$$

$$E_z = \left(\frac{P_{01}}{a}\right)^2 A J_0\left(\frac{P_{01}}{a}\rho\right)$$

$$H_\rho = 0$$

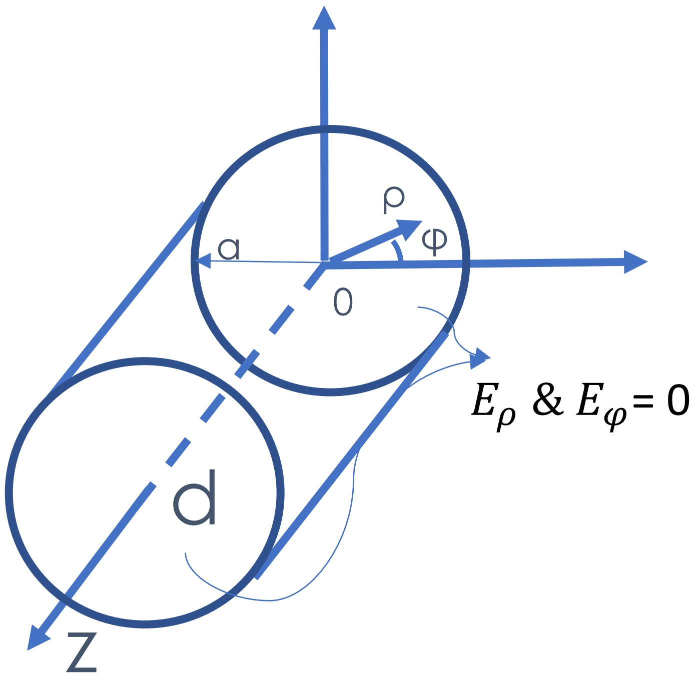
$$H_\varphi = -j\omega\varepsilon\frac{P_{01}}{a} A J_0'\left(\frac{P_{01}}{a}\rho\right)$$

$$H_z = 0$$

$E_0 = \left(\frac{P_{01}}{a}\right)^2 A$ (E field along beamline)

$$V_0 = E_0 \mathrm{d} = \left(\frac{P_{01}}{a}\right)^2 dA$$

$$P_{01} = 2.4048$$

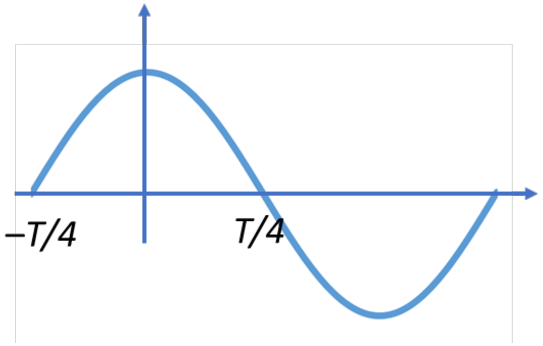


Resonant Frequency

$$k_c^2 = k^2 - \beta^2, k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}, k_c = \frac{P_{01}}{a} \ \& \ \beta = \frac{l\pi}{d} = 0$$

- Resonant frequency $f = \frac{\omega}{2\pi} = \frac{c}{2\pi}k = \frac{c}{2\pi}\frac{P_{01}}{a}$
- Wavelength $\lambda = \frac{2\pi}{P_{01}}a = 2.613a$
- For TM_{010}, resonant frequency is determined by the cavity diameter, it is not related to the cavity length.
- For a 1GHz cavity, the cavity diameter is 0.23m (radius a=0.115m). For a 100MHz cavity, it is 2.3m, it is huge.

Cavity Length



- Beam passes through the cavity center.
- Recall that $E_z = (\frac{P_{01}}{a})^2 A J_0\left(\frac{P_{01}}{a}\rho\right)$, it is a constant $(\frac{P_{01}}{a})^2 A J_0(0)$ in the cavity center, with a factor containing time $e^{j\omega t}$. This is also the peak E field.
- For $V(t) = V_0 e^{j\omega t} = \frac{P_{01}{}^2}{a^2} d A J_0(0) e^{j\omega t}$, to get the maximum accelerating voltage, the (positively) charged particles/beam (with velocity v) enter the capacitor at time –*T*/4 and exist at time *T*/4. Accelerating force is at maximum while particles are in the center, so called on-crest. Cavity may also work at off-crest though.
- The length of the capacitor is thus *d*=λ/2 for v close to c, with λ=c/f_0 the wavelength.
- For low β (=v/c) cavity, the length is normally less than λ/2.
- Transit time factor $T = sin\left(\frac{\omega d}{2v}\right) / \left(\frac{\omega d}{2v}\right)$, for $d = \lambda/2$ & β=1 it is $\frac{2}{\pi}$.

Stored Energy

$$\int_0^a J_0^2\left(\frac{P_{01}}{a}\rho\right)\rho d\rho = \frac{a^2}{2}[J_0'^2(P_{01}) + J_0^2(P_{01})] = \frac{a^2}{2}J_0'^2(P_{01})$$
$$\int_0^a J_0'^2\left(\frac{P_{01}}{a}\rho\right)\rho d\rho = \frac{a^2}{2}J_0'^2(P_{01})$$
$$J_0'(P_{01}) = -J_1(P_{01})$$

- $U = \frac{\mu}{2}\int_v |H_\varphi|^2\, dv = \frac{\varepsilon}{2}\int_v |E_z|^2\, dv$
- $H_\varphi = -j\omega\varepsilon\frac{P_{01}}{a}AJ_0'\left(\frac{P_{01}}{a}\rho\right)$ & $E_z = (\frac{P_{01}}{a})^2 AJ_0\left(\frac{P_{01}}{a}\rho\right)$
- $\int_v\, dv = \int_{z=0}^{d}\int_{\varphi=0}^{2\pi}\int_{\rho=0}^{a}\rho d\rho\, d\varphi\, dz = 2\pi d\int_{\rho=0}^{a}\rho d\rho$ for TM_{010}
- $U = \frac{\mu}{2}\int_v |H_\varphi|^2\, dv = \frac{\mu}{2}\left[\omega\varepsilon\frac{P_{01}}{a}A\right]^2 2\pi d\frac{a^2}{2}J_0'^2(P_{01})$
- $U = \frac{\varepsilon}{2}\int_v |E_z|^2\, dv = \frac{\varepsilon}{2}\left[(\frac{P_{01}}{a})^2 A\right]^2 2\pi d\frac{a^2}{2}J_0'^2(P_{01}) = \frac{\varepsilon}{2}E_0{}^2 J_1^2(P_{01})\times Volume$
- Note that $\frac{\omega}{c} = \frac{P_{01}}{a}$ & $\varepsilon_0\mu_0 = 1/c^2$, the above two equations are equal.

L, C & Shunt impedance over Q

- $U = \frac{1}{2} C V_0^2$ with $V_0 = \frac{{P_{01}}^2}{a^2} d A J_0(0) = \frac{{P_{01}}^2}{a^2} dA$ thus $C = \varepsilon\pi \frac{a^2}{d} J_1^2(P_{01})$
- $\mathrm{L} = \frac{1}{\omega^2 C} = \frac{d}{\omega^2 \mathcal{E}\pi a^2 J_1^2(P_{01})}$
- Shunt impedance over Q: $\frac{R_{sh}}{Q} = \omega L T^2 = \frac{d}{\omega \mathcal{E}\pi a^2 J_1^2(P_{01})} T^2$, with transit time factor $T = sin\left(\frac{\omega d}{2v}\right) / \left(\frac{\omega d}{2v}\right)$

Power dissipation

$$\int_0^a J_0'^2\left(\frac{P_{01}}{a}\rho\right)\rho d\rho = \frac{a^2}{2}J_0'^2(P_{01})$$

- $H_\varphi = -j\omega\varepsilon\frac{P_{01}}{a}AJ_0'\left(\frac{P_{01}}{a}\rho\right)$
- $P_{rf} = \frac{R_s}{2}\iint|H|^2 dS = \frac{R_s}{2}\left(2\int_{\varphi=0}^{2\pi}\int_{\rho=0}^{a}\left(\omega\varepsilon\frac{P_{01}}{a}AJ_0'\left(\frac{P_{01}}{a}\rho\right)\right)^2\rho d\rho\, d\varphi + a\int_{z=0}^{d}\int_{\varphi=0}^{2\pi}\left(\omega\varepsilon\frac{P_{01}}{a}AJ_0'(P_{01})\right)^2 d\varphi\, dz\right) =$

$\frac{R_s}{2}\left(\omega\varepsilon\frac{P_{01}}{a}A\right)^2\left(4\pi\int_{\rho=0}^{a}J_0'^2\left(\frac{P_{01}}{a}\rho\right)\rho d\rho + 2\pi a d J_0'^2(P_{01})\right) =$

$\frac{R_s}{2}\left(\omega\varepsilon\frac{P_{01}}{a}A\right)^2\left(4\pi\frac{a^2}{2}J_0'^2(P_{01}) + 2\pi a d J_0'^2(P_{01})\right) =$

$\left(\omega\varepsilon\frac{P_{01}}{a}A\right)^2\pi R_s J_0'^2(P_{01})(a^2+ad) = \left(\omega\varepsilon\frac{P_{01}}{a}A\right)^2\pi R_s J_1^2(P_{01})(a^2+ad)$

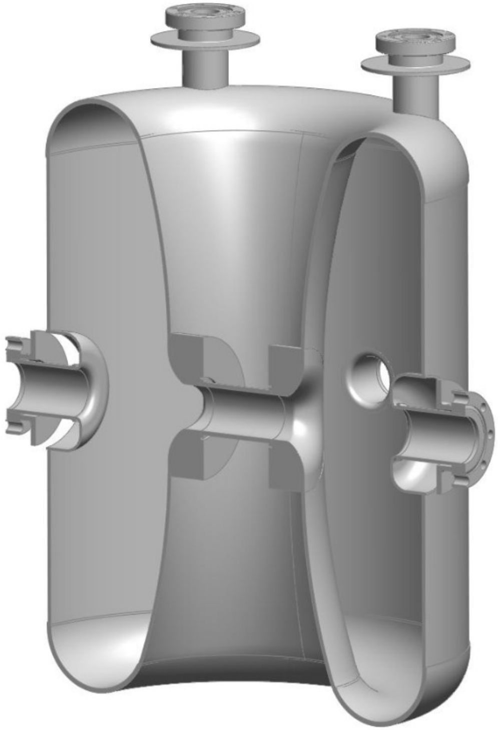
Quality factor

- $Q = \frac{\omega U}{P_{rf}} = \frac{\omega \frac{\mu}{2}\left(\omega\varepsilon \frac{P_{01}}{a} A\right)^2 2\pi d \frac{a^2}{2} J_1^2(P_{01})}{\left(\omega\varepsilon \frac{P_{01}}{a} A\right)^2 \pi R_s J_1^2(P_{01})(a^2+ad)} = \frac{\frac{\omega\mu}{2}}{R_s} \frac{a}{\frac{a}{d}+1}$
- For a normal conductor with skin depth $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$ and surface resistance $R_s = \sqrt{\frac{\mu\omega}{2\sigma}}$, $Q = \frac{1}{\delta} \frac{a}{\frac{a}{d}+1}$
- Geometry factor $G = \frac{\omega\mu}{2} \frac{a}{\frac{a}{d}+1}$

HWR

https://uspas.fnal.gov/materials/12MSU/JPH_HWR_Design.pdf

Half wave resonator



Recall - TEM Field Pattern in coax line

- $k_c = 0,\ k = \beta = \omega\sqrt{\mu\varepsilon}$, no cutoff frequency.

$$E_\rho = \frac{A}{\sqrt{\varepsilon}\rho} e^{-jkz}$$

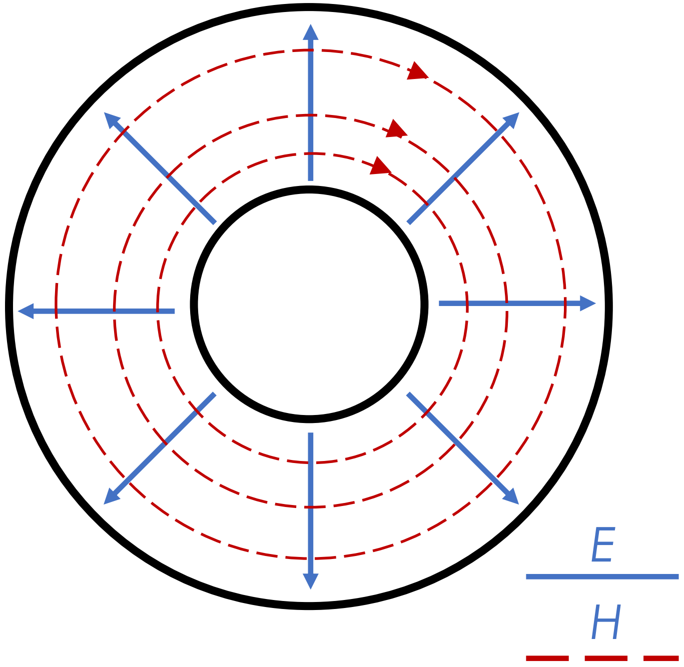
$$E_\varphi = 0$$

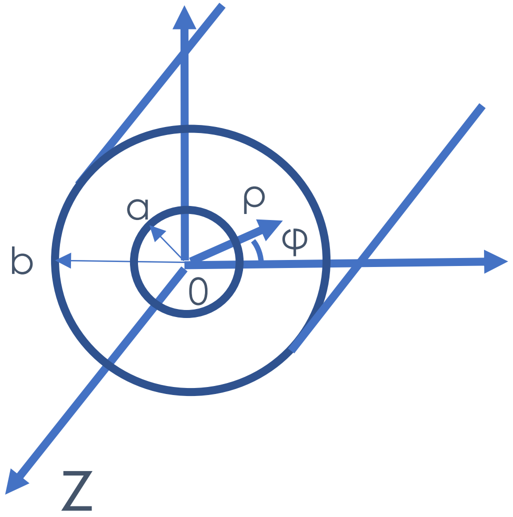
$$E_z = 0$$

$$H_\rho = 0$$

$$H_\varphi = \frac{A}{\sqrt{\mu}\rho} e^{-jkz}$$

$$H_z = 0$$





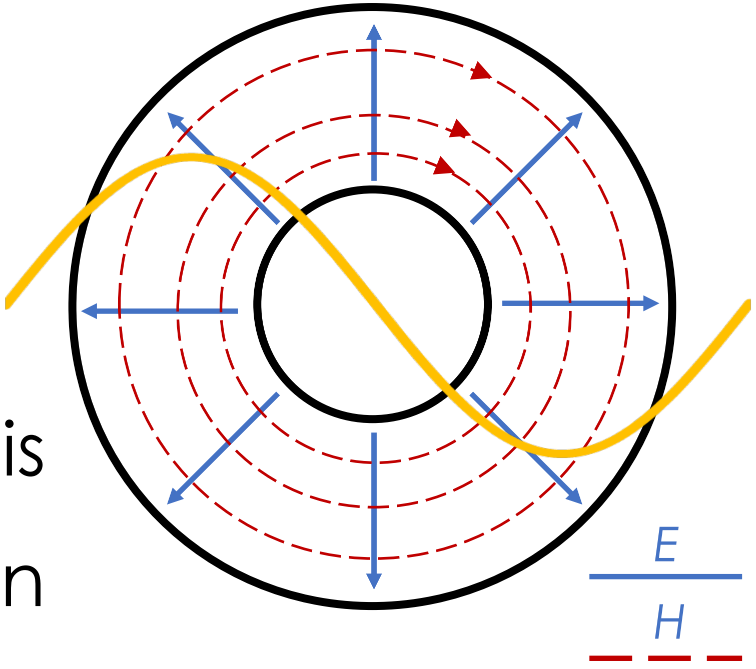
Additional boundary conditions

- e^{-jkz} in E_ρ now becomes cos(kz) and it should be zero at z = -d/2 and z = d/2.
- The mode with lowest resonant frequency should satisfy $kd/2 = \pi/2$, and $d = \lambda/2$, therefore it is called HWR.
- Resonant frequency $f = \frac{c}{2d}$ is solely determined by the cavity height *d*, and is not related to *a* & *b*.

$$E_\rho = \frac{A}{\sqrt{\varepsilon}\rho}\cos(\frac{\pi z}{d})$$
$$E_\varphi = 0$$
$$E_z = 0$$
$$H_\rho = 0$$
$$H_\varphi = \frac{A}{\sqrt{\mu}\rho} sin(\frac{\pi z}{d})$$
$$H_z = 0$$

Accelerating voltage

- Beam passes through the center z = 0, with $E_{\rho,z=0} = \frac{A}{\sqrt{\mathcal{E}}\rho} e^{j\omega t}$
- Ideally, one would like the beam to "see" the maximum *E* field while beam is at [-b,-a], and when beam passes through the center hole, *E* field flips the sign and when beam is at [a,b], beam "sees" the maximum *E* field again.



- Accelerating voltage in this case is $2\left|\int_a^b \frac{A}{\sqrt{\mathcal{E}}\rho}\sin\left(\frac{\omega\rho}{v}\right)d\rho\right| = \frac{2A}{\sqrt{\mathcal{E}}}\left|\int_{\frac{\omega a}{v}}^{\frac{\omega b}{v}} \frac{\sin\alpha}{\alpha}d\alpha\right|$, it can be integrated numerically.
- HWRs are normally thin and tall, with b a small fraction of λ/2, and for low β applications. Ideally $a + b = \beta\lambda/2 = \beta d$

HWR – RLC properties (see coax line)

- Inductance per unit length $L = \frac{\mu}{2\pi} ln \frac{b}{a}$ H/m.
- Capacitance per unit length $C = \frac{2\pi\mathcal{E}}{ln\frac{b}{a}}$ F/m.
- Center conductor voltage $V = \frac{A}{\sqrt{\mathcal{E}}} ln \frac{b}{a} \cos\left(\frac{2\pi z}{\lambda}\right) = V_0 \cos\left(\frac{2\pi z}{\lambda}\right)$
- Center conductor current $I = \frac{2\pi A}{\sqrt{\mu}} \sin(\frac{2\pi z}{\lambda}) = I_0 \sin\left(\frac{2\pi z}{\lambda}\right)$
- $Z_0 = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}} = \frac{\eta}{2\pi} ln \frac{b}{a}$

HWR – Peak fields

Normalize to $V_0 = \frac{A}{\sqrt{\varepsilon}} ln\frac{b}{a}$

- Peak electric field $E_{pk} = \frac{A}{\sqrt{\varepsilon}a} = \frac{V_0}{aln\frac{b}{a}}$
- Peak magnetic field $H_{pk} = \frac{A}{\sqrt{\mu}a} = \frac{V_0}{\eta aln\frac{b}{a}}$

HWR – Stored energy

Normalize to $V_0 = \frac{A}{\sqrt{\varepsilon}} ln\frac{b}{a}$

- $H_\varphi = \frac{A}{\sqrt{\mu}\rho} sin(\frac{\pi z}{d})$
- Stored energy $U = \frac{\mu}{2}\int_v \left|H_\varphi\right|^2 dv = \frac{\mu}{2}\int_v \left|\frac{A}{\sqrt{\mu}\rho} sin(\frac{\pi z}{d})\right|^2 dv =$
$\frac{\mu}{2}\int_{z=-d/2}^{d/2}\int_{\varphi=0}^{2\pi}\int_{\rho=a}^{b}\left|\frac{A}{\sqrt{\mu}\rho} sin(\frac{\pi z}{d})\right|^2 \rho d\rho\, d\varphi\, dz = \frac{1}{2}\pi d A^2 ln\frac{b}{a} = \frac{\pi\varepsilon\lambda}{4}\frac{V_0^2}{ln\frac{b}{a}}$

HWR – Power dissipation

Normalize to $V_0 = \frac{A}{\sqrt{\varepsilon}} ln\frac{b}{a}$

- Power dissipation $P_{rf} = \frac{R_s}{2}\iint |H|^2 dS = \frac{R_s}{2}\left(2\int_{\varphi=0}^{2\pi}\int_{\rho=a}^{b}\left(\frac{A}{\sqrt{\mu}\rho}sin(\frac{\pi d/2}{d})\right)^2 \rho d\rho\, d\varphi + a\int_{z=-d/2}^{d}\int_{\varphi=0}^{2\pi}\left(\frac{A}{\sqrt{\mu}a}sin(\frac{\pi z}{d})\right)^2 d\varphi\, dz + b\int_{z=-d/2}^{d}\int_{\varphi=0}^{2\pi}\left(\frac{A}{\sqrt{\mu}b}sin(\frac{\pi z}{d})\right)^2 d\varphi\, dz\right) = \frac{R_s}{2}\left(2\times 2\pi\frac{A^2}{\mu}ln\frac{b}{a} + \pi\frac{A^2 d}{\mu}(\frac{1}{a}+\frac{1}{b})\right) = \frac{R_s}{4}\frac{\pi V_0^2}{\eta^2}\frac{1}{(ln\frac{b}{a})^2}\left[8ln\frac{b}{a} + \lambda(\frac{1}{a}+\frac{1}{b})\right]$
- HWR is normally a thin tall cylinder, the first term above (loss on the endplates) can be ignored. $P_{rf} = \frac{R_s}{4}\frac{\pi V_0^2}{\eta^2}\frac{\lambda}{(ln\frac{b}{a})^2}(\frac{1}{a}+\frac{1}{b})$

HWR – Quality factor

- $Q = \frac{\omega U}{P_{rf}} = \frac{\omega \frac{\pi \mathcal{E} \lambda V_0^2}{4 \; ln\frac{b}{a}}}{\frac{R_S \pi V_0^2}{4 \; \eta^2} \frac{\lambda}{(ln\frac{b}{a})^2}(\frac{1}{a}+\frac{1}{b})} = \frac{2\pi\eta ln\frac{b}{a}}{\lambda R_S(\frac{1}{a}+\frac{1}{b})}$ thus $G = \frac{2\pi\eta ln\frac{b}{a}}{\lambda\,(\frac{1}{a}+\frac{1}{b})}$

HWR – Shunt impedance

- There are 2 gaps thus the voltage should be $2V_0$
- $R_{sh} = \frac{(2V_0)^2}{P_{rf}} = \frac{4V_0^2}{\frac{R_S}{4}\frac{\pi V_0^2}{\eta^2}\frac{\lambda}{(ln\frac{b}{a})^2}(\frac{1}{a}+\frac{1}{b})} = \frac{16\eta^2(ln\frac{b}{a})^2}{\pi\lambda R_S(\frac{1}{a}+\frac{1}{b})}$ thus $R_{sh}R_s = \frac{16\eta^2(ln\frac{b}{a})^2}{\pi\lambda\,(\frac{1}{a}+\frac{1}{b})}$
- $\frac{R_{sh}}{Q} = \frac{\frac{16\eta^2(ln\frac{b}{a})^2}{\pi\lambda R_S(\frac{1}{a}+\frac{1}{b})}}{\frac{2\pi\eta ln\frac{b}{a}}{\lambda R_S(\frac{1}{a}+\frac{1}{b})}} = \frac{8}{\pi^2}\eta ln\frac{b}{a}$

HWR – estimation

- Practically $ln\frac{b}{a}$~1, with b/a~3.
- $V_0 = bln\frac{b}{a}E_0 \sim bE_0$ with E_0 amplitude of the E field on outer wall.
- $a + b \sim b \sim \beta\lambda/2$
- $U = \frac{\pi\mathcal{E}\lambda}{4}\frac{V_0^2}{ln\frac{b}{a}} \sim \frac{\pi\mathcal{E}}{16}E_0^2\beta^2\lambda^3$
- $P_{rf} = \frac{R_s}{4}\frac{\pi V_0^2}{\eta^2}\frac{\lambda}{(ln\frac{b}{a})^2}\left(\frac{1}{a}+\frac{1}{b}\right) \sim \frac{\pi R_s}{2\eta^2}E_0^2\beta\lambda^2$
- $G = \frac{2\pi\eta ln\frac{b}{a}}{\lambda\,(\frac{1}{a}+\frac{1}{b})} \sim \frac{\pi}{4}\eta\beta$
- $R_{sh}R_s = \frac{16\eta^2(ln\frac{b}{a})^2}{\pi\lambda\,(\frac{1}{a}+\frac{1}{b})} \sim \frac{2}{\pi}\eta\beta^2$
- $\frac{R_{sh}}{Q} = \frac{8}{\pi^2}\eta ln\frac{b}{a} \sim \frac{8}{\pi^2}\eta$

HOM

Unwanted modes

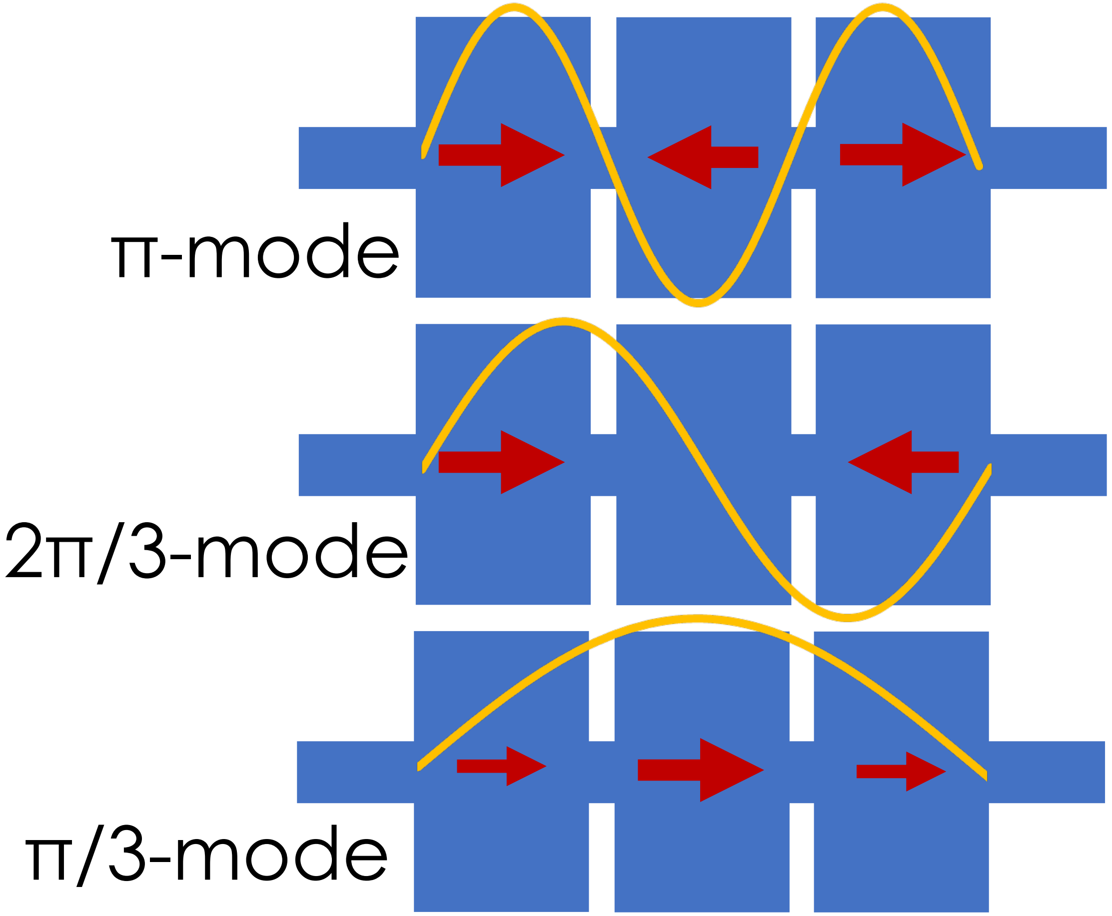
- The working mode (also called fundamental mode) in the cavity is the mode we want the beam to "see" and to interact with.
- Some cavities have multiple working modes*, this is not the major topic of this course though.
- The modes other than the working mode may disturb the beam (beam dynamics consideration) and cause energy degradation (power consideration), thus they are unwanted.

*https://doi.org/10.1103/PhysRevAccelBeams.19.122001

Multicell cavity

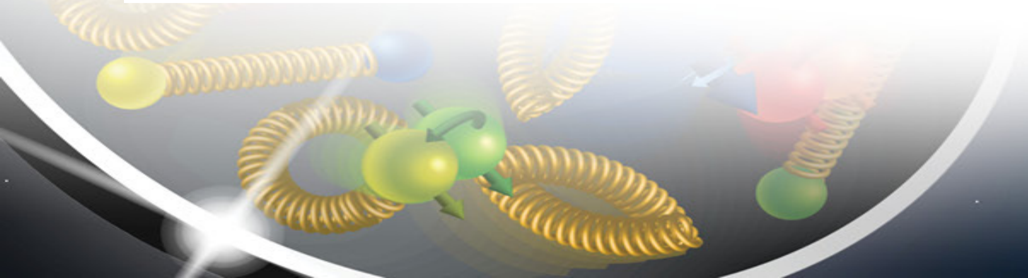
- Sometimes multicell cavity is used to save space, components (money), power needed etc.
- The working mode now split to *n* modes (called passband modes), with *n* the cell number.
- The passband modes are named by the phase advance between two adjacent cells (or by the phase advance from beginning to end over the number of cells): $k\pi/n$, with $k=1,2,\ldots,n$
- π-mode is usually the working mode.

Note: the yellow sin curve does not represent the wavelength of the resonance.



HOM, SOM, LOM

- The modes that are in the same passband as the working mode are called Same Order Modes (SOMs).
- Modes with frequencies lower than the working mode/passband are called Lower Order Modes (LOMs).
- Modes with frequencies higher than the working mode/passband are called Hihger Order Modes (HOMs).
- For single-cell λ/2 TM_{010} cavity, there are no SOMs or LOMs, only HOMs exist.



HOMs (single-cell λ/2 TM_{010} pillbox cavity)

- TM_{nml} $k^2 = k_c^2 + \beta^2 = (\frac{P_{nm}}{a})^2 + (\frac{l\pi}{d})^2 = (\frac{\omega}{c})^2$
- TE_{nml} $k^2 = k_c^2 + \beta^2 = (\frac{P'_{nm}}{a})^2 + (\frac{l\pi}{d})^2 = (\frac{\omega}{c})^2$
- λ/2 TM_{010} pillbox cavity length $d = \frac{\lambda}{2} = \frac{\pi}{P_{01}} a$
- TM_{nml} $f = \frac{c}{2\pi a}\sqrt{{P_{nm}}^2 + (lP_{01})^2}$
- TE_{nml} $f = \frac{c}{2\pi a}\sqrt{{P'_{nm}}^2 + (lP_{01})^2}$

HOMs (single-cell λ/2 TM_{010} pillbox cavity)

- TM_{nml} $f=\frac{c}{2\pi a}\sqrt{{P_{nm}}^2+(lP_{01})^2}$ & TE_{nml} $f=\frac{c}{2\pi a}\sqrt{{P'_{nm}}^2+(lP_{01})^2}$
- There is no TE_{nm0} mode
- HOMs with frequency from low to high (normalize to TM_{010} $f_0=\frac{c}{2\pi a}P_{01}$):

TE_{111} 1.259 TM_{011} 1.414 TM_{110} 1.593 TE_{211} 1.616 TM_{111}/TE_{011} 1.881 TE_{112} 2.141 TM_{012} 2.236 TE_{212} 2.369 TM_{112} 2.557

n	P'_{n1}	P'_{n2}	P'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

n	P_{n1}	P_{n2}	P_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

HOMs

- TM monopoles produce most of the HOM power.
- Monopoles and dipoles perturb the beam more than sextupoles, octupoles…, the so called “(shunt) impedance budget” for these two need to be considered during the cavity design.