Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

What we learned:

Floquet Theorem

$$X'' + K(s)X = 0 \qquad K(s) = K(s+L)$$

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \qquad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^{s} \frac{1}{\beta} ds$$

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2, s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu - \alpha_1 \sin\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

The values of the Courant–Snyder parameters α_2, β_2, γ_2 at s_2 are related to α_1, β_1, γ_1 at s_1 by

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 (s - \frac{\alpha_1}{\gamma_1})^2 = \beta^* + \frac{(s-s^*)^2}{\beta^*},$$

$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s-s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

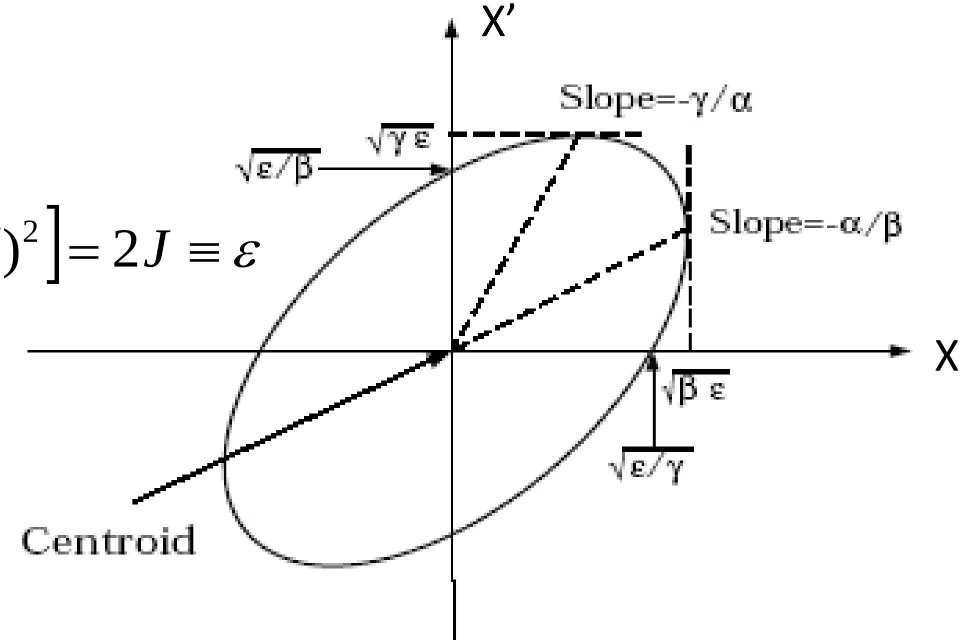
$$X = \sqrt{2\beta J}\cos\psi, \quad X' = -\sqrt{\frac{2J}{\beta}}(\sin\psi + \alpha\cos\psi)$$

$$P_X = \beta X' + \alpha X = -\sqrt{2\beta J}\sin\psi$$

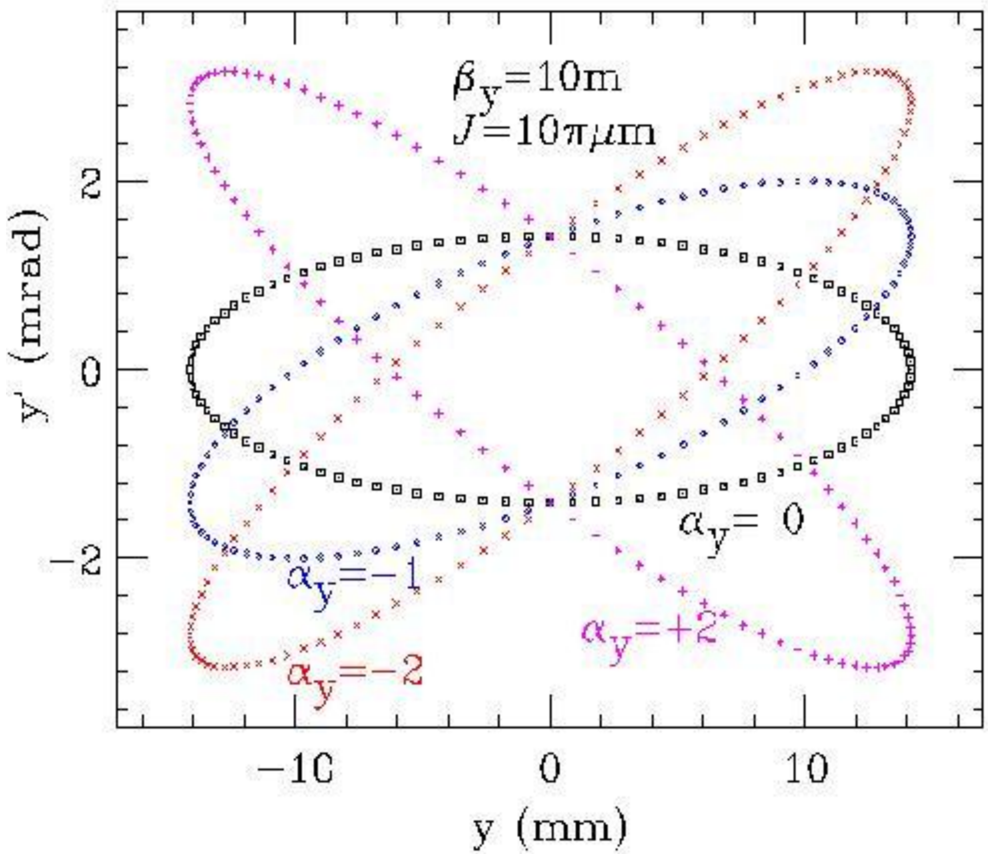
(X, P_X) form a normalized phase space coordinates with
$X^2+P_X^2=2\beta J$, here J is called **action**.

Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha X X' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



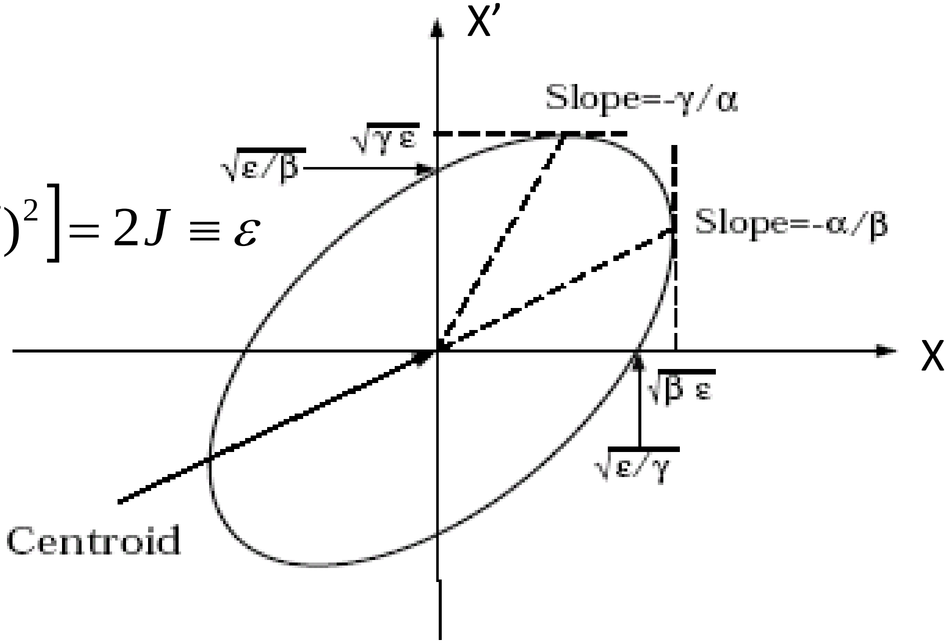
Example: Ellipses (vertical) with different optical parameters



The betatron phase space ellipses of a particle with actions J = 10π mm-mrad. The btatron parameters are β_y = 10m, and α_y shown by each curve. The scale for the ordinate y is mm, and y' in mrad. The betatron parameters for each ellipse are marked on the graph. All ellipses has the maximum y coordinate at $(2\beta_y J)^{1/2}$. The maximum anglular coordiante y' is $(2(1 + \alpha_y^2)J/\beta_y)^{1/2}$. All ellipses have the same phase space area of 2J.

Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha XX' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



Emittance of a beam

Given a normalized distribution function ρ(X, X’) with ∫ρ(X, X’)dXdX’ = 1, the moments of the beam distribution are

$$\langle X\rangle = \int X\rho(X,X')dXdX', \quad \langle X'\rangle = \int X'\rho(X,X')dXdX',$$

$$\sigma_X^2 = \int (X-\langle X\rangle)^2\rho(X,X')dXdX', \quad \sigma_{X'}^2 = \int (X'-\langle X'\rangle)^2\rho(X,X')dXdX',$$

$$\sigma_{XX'} = \int (X-\langle X\rangle)(X'-\langle X'\rangle)\rho(X,X')dXdX' = r\sigma_X\sigma_{X'}$$

Where σ_X and $\sigma_{X'}$ are the rms beam widths, $\sigma_{XX'}$ is the correlation, and r is the correlation coefficient. The rms beam emittance is then defined as

$$\varepsilon_{rms} = \sqrt{\sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2} = \sigma_X\sigma_{X'}\sqrt{1-r^2}$$

The rms emittance is invariant in linear transport:

$$\varepsilon^2 = \sigma_X^2 \sigma_{X'}^2 - \sigma_{XX'}^2$$

$$\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2, \quad \sigma_{X'}^2 = \langle X'^2 \rangle - \langle X' \rangle^2, \quad \sigma_{XX'} = \langle XX' \rangle - \langle X \rangle \langle X' \rangle$$

we find

$$\frac{d\sigma_X^2}{ds} = 2\langle XX' \rangle - 2\langle X \rangle \langle X' \rangle$$

$$\frac{d\sigma_{X'}^2}{ds} = 2\langle X'X'' \rangle - 2\langle X' \rangle \langle X'' \rangle$$

$$\frac{d\sigma_{XX'}}{ds} = \langle X'^2 \rangle - \langle X' \rangle^2 - \langle X \rangle \langle X'' \rangle + \langle XX'' \rangle$$

$$X'' + KX = 0$$

$$\frac{d\varepsilon^2}{ds} = \sigma_X^2 \frac{d\sigma_{X'}^2}{ds} + \sigma_{X'}^2 \frac{d\sigma_X^2}{ds} - 2\sigma_{XX'} \frac{d\sigma_{XX'}}{ds} = 0$$

The Gaussian distribution function

The equilibrium beam distribution in the linearized betatron phase space may be any function of the invariant action. However, the Gaussian distribution function is commonly used to evaluate the beam properties. Expressing the normalized Gaussian distribution in the normalized phase space, we obtain

$$\rho(X,P_X)=\frac{1}{2\pi\sigma_X^2}e^{-(X^2+P_X^2)/2\sigma_X^2}$$

where $\langle X^2\rangle = \langle P_X^2\rangle = \sigma_X^2 = \beta_X\varepsilon_{rms}$ with an rms emittance ε_{rms}. Transforming (X,P_X) into the action-angle variables (J, ψ) with

$$X=\sqrt{2\beta J}\cos\psi, \quad P_X=-\sqrt{2\beta J}\sin\psi$$

The Jacobian of the transformation is β_X, and the distribution function becomes

$$\rho(J)=\frac{1}{\varepsilon_{rms}}e^{-J/\varepsilon_{rms}}, \quad \rho(\varepsilon)=\frac{1}{2\varepsilon_{rms}}e^{-\varepsilon/2\varepsilon_{rms}}$$

The percentage of particles contained within $\varepsilon=n\varepsilon_{rms}$ is $1-e^{-n/2}$

$\epsilon/\epsilon_{\mathrm{rms}}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96

The maximum phase-space area that particles can survive in an accelerator is called the *admittance*, or the ***dynamic aperture***. The admittance is determined by the vacuum chamber size, the kicker aperture, and nonlinear magnetic fields.

Adiabatic damping and the normalized emittance: $\varepsilon_n=\varepsilon\beta\gamma$

The Courant–Snyder invariant, derived from the phase-space coordinate X, X′, is not invariant when the energy is changed. To obtain the Liouville invariant phase-space area, we should use the conjugate phase-space coordinates (X, P_X) in Hamiltonian. Since $p_X = p_X' = mc\beta\gamma X'$, where m is the particle's mass, p is its momentum, and βγ is the Lorentz relativistic factor, the *normalized emittance* defined by $\boldsymbol{\varepsilon_n=\varepsilon\beta\gamma}$ is invariant. The beam emittance decreases with increasing beam momentum, i.e. $\varepsilon=\varepsilon_n/\beta\gamma$. This is called ***adiabatic damping***. Since the transverse velocity of a particle does not change during acceleration, the transverse angle $X' = p_X/p$ becomes smaller at a higher particle momentum. Thus the beam emittance $\varepsilon=\varepsilon_n/\beta\gamma$ decreases with energy. The adiabatic damping also applies to beam emittance in proton or electron **linacs**.

Because of the quantum fluctuation, The beam emittance in electron storage rings **increases** with energy ($\sim\gamma^2$). The corresponding normalized emittance is proportional to γ^3.

Betatron motion: Effects of Linear Magnetic field Error

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho}$$

$$\Delta B_y + j\Delta B_x = B_0 \sum_n (b_n + ja_n)(x + jy)^n,$$

$$B_y = B_0 b_0, \quad B_x = B_0 a_0,$$

Dipole field error

$$B_y = B_0 b_1 x, \quad B_x = B_0 b_1 y,$$

Quadrupole field error

$$B_y = -B_0 a_1 y, \quad B_x = B_0 a_1 x,$$

Skew Quadrupole field error

$$B_y = B_0 b_2 (x^2 - y^2), \quad B_x = 2B_0 b_2 xy,$$

Sextupole field error

$$B_y = -2B_0 a_2 xy, \quad B_x = B_0 a_2 (x^2 - y^2),$$

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \qquad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

Effect of dipole field error:

We consider a single localized dipole error with the kick angle given by $\theta=\Delta B\ell/B\rho$. Because of the dipole field error, the reference orbit is perturbed! The idea is to find a new closed orbit that include the dipole field error.

$$X'' + K_X(s)X = \theta\delta(s-s_0)$$

The closed orbit is given by the following condition:

$$\begin{pmatrix} X_0 \\ X_0' - \theta \end{pmatrix} = M\begin{pmatrix} X_0 \\ X_0' \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha_0\sin\Phi & \beta_0\sin\Phi \\ -\gamma_0\sin\Phi & \cos\Phi - \alpha_0\sin\Phi \end{pmatrix}\begin{pmatrix} X_0 \\ X_0' \end{pmatrix}$$

Where $\Phi=2\pi\nu$, ν is the betatron tune, the parameters α_0, β_0, and γ_0 are values of the Courant-Snyder parameters at the kicker location. The solution is

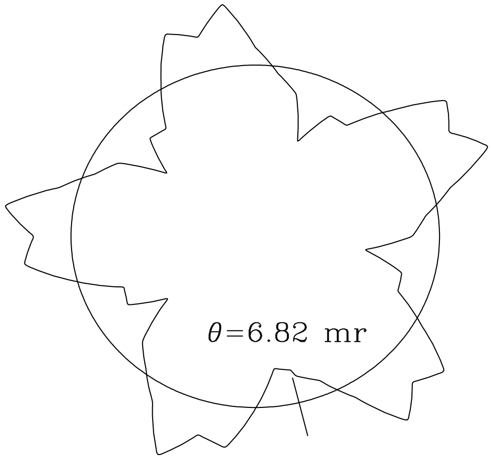
$$X_0 = \frac{\beta_0\theta}{2\sin\pi\nu}\cos\pi\nu,$$

$$X_0' = \frac{\theta}{2\sin\pi\nu}(\sin\pi\nu - \alpha_0\cos\pi\nu)$$

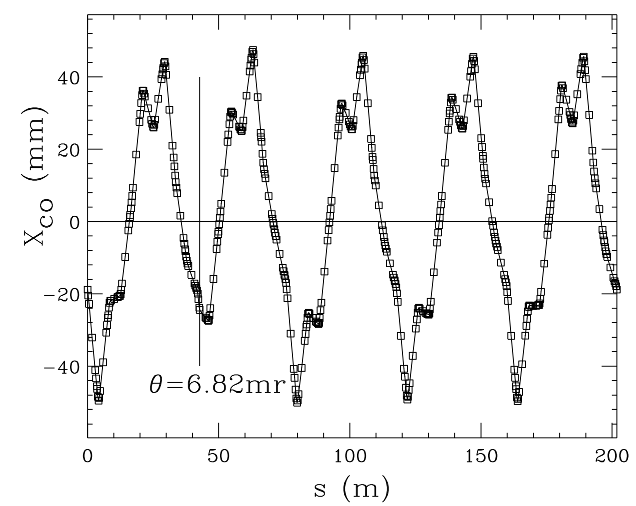
We have solved the closed orbit at one point s_0. The closed orbit of the accelerator can be obtained by making mapping matrix:

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix}_{co} = M(s, s_0) \begin{pmatrix} X_0 \\ X'_0 \end{pmatrix} \qquad X_{co}(s) = G(s, s_0)\theta$$

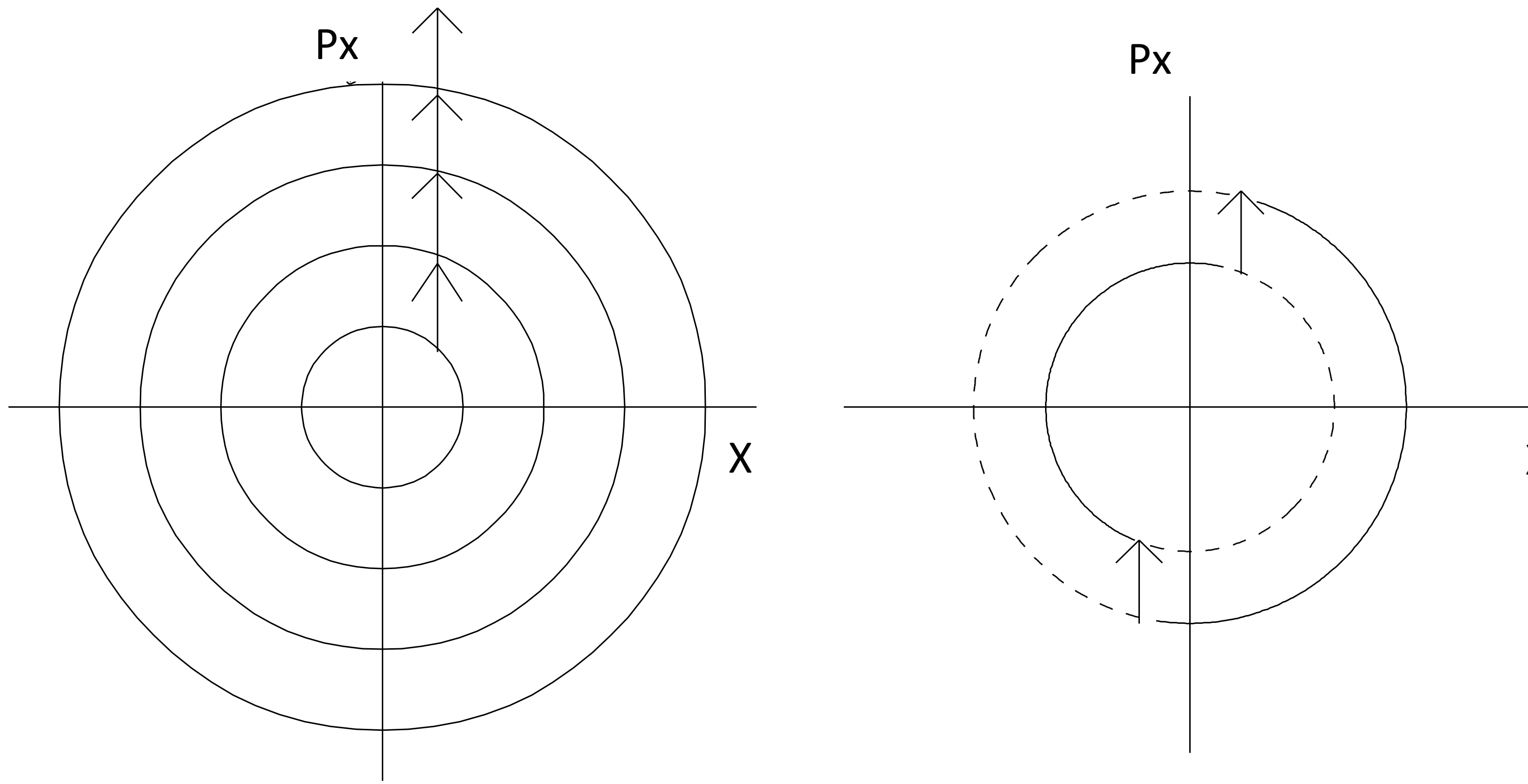
$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin \pi\nu} \cos[\pi\nu - |\psi(s) - \psi(s_0)|]$$



Note that the closed orbit is described by Green's function. When the betatron tune is an integer, the closed orbit diverges. Each time, when the particle arrives the same location will receive a coherent kick and the particle becomes unstable.

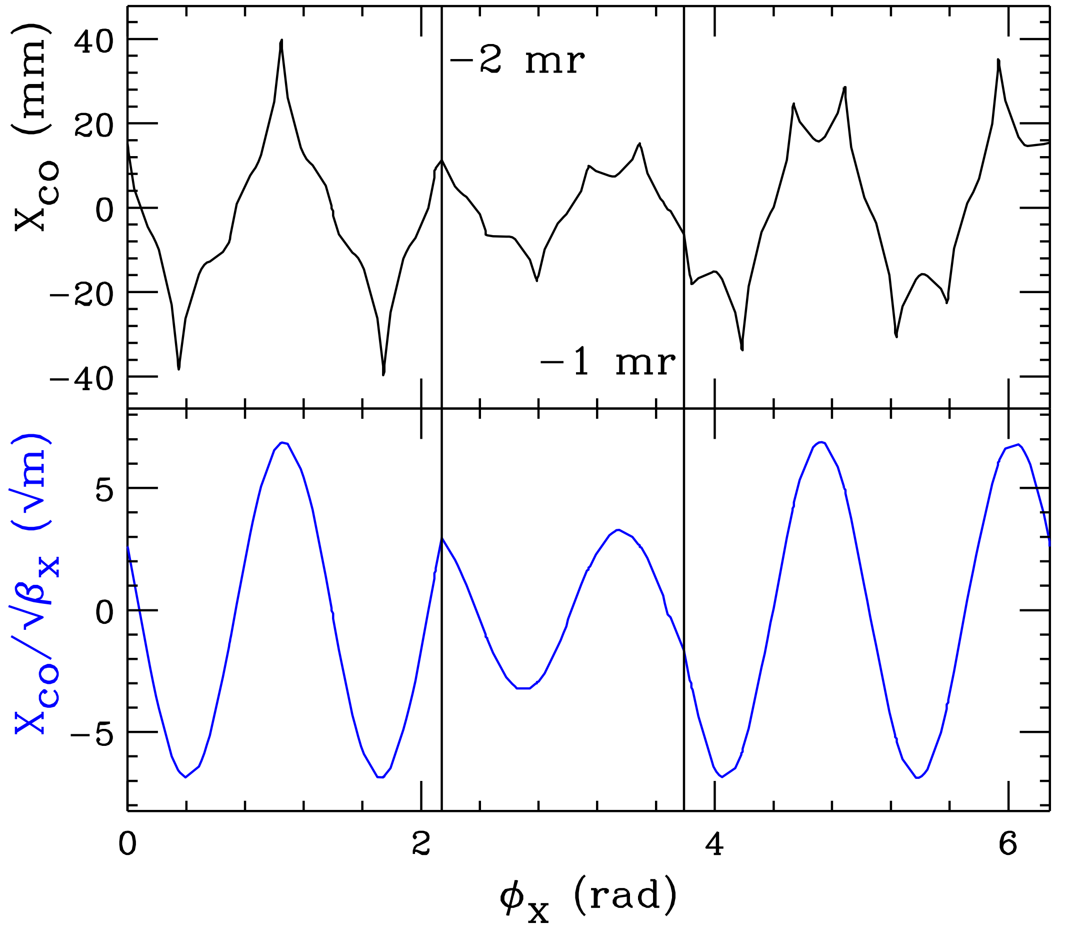


How? And Why

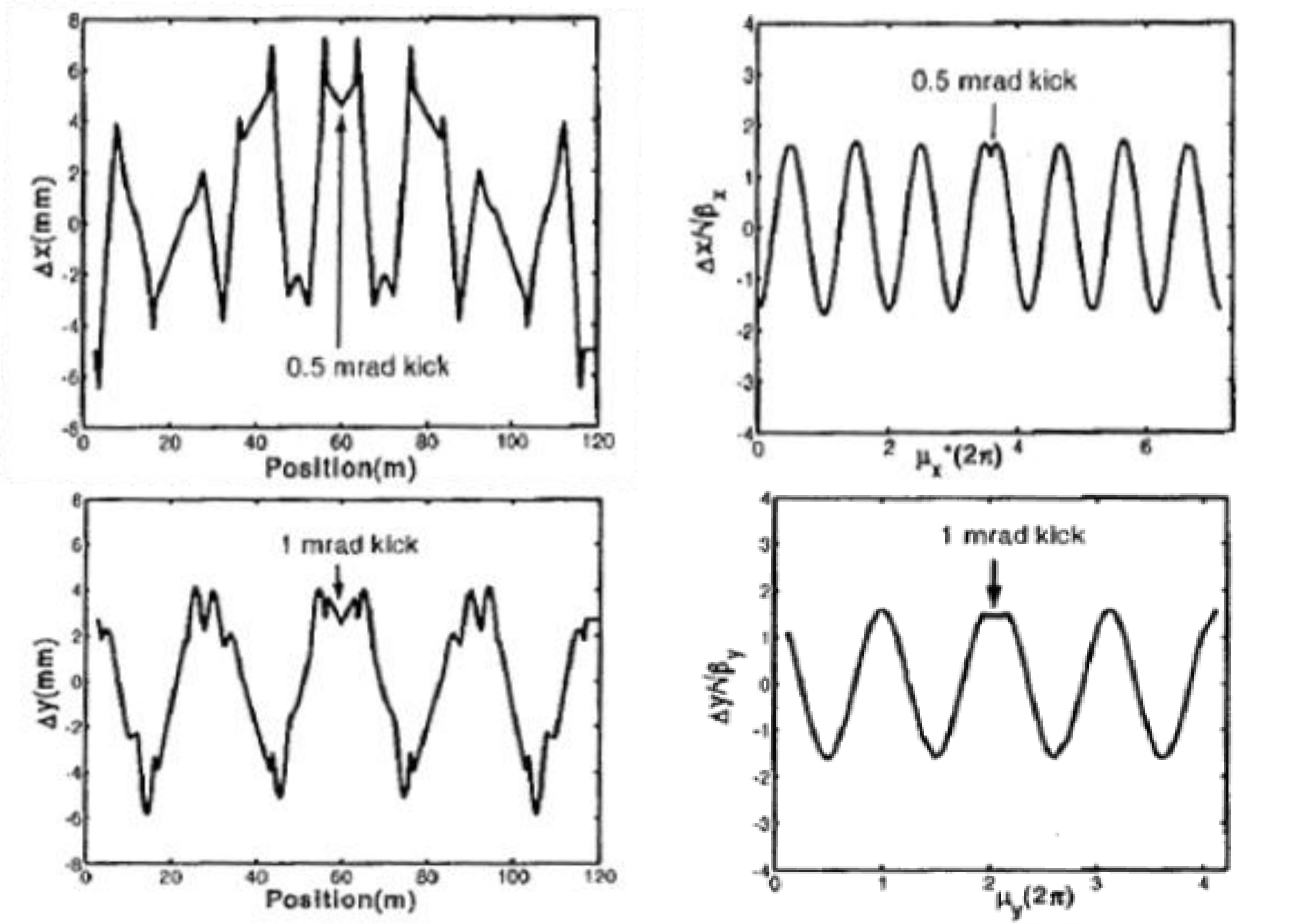


Left, a schematic plot of the closed-orbit perturbation due to an error dipole kick when the betatron tune is an integer. Here $p_X = \beta_X \Delta X' = \beta_X \theta$, where θ is the dipole kick angle and β_X is the betatron amplitude function value at the dipole. Right, a schematic plot of the particle trajectory resulting from a dipole kick when the betatron tune is a half-integer; here the angular kicks from two consecutive orbital revolutions cancel each other.

An accelerator with circumference 360 m is made of 18 FODO cells. The horizontal betatron tune of the synchrotron is ν_x=4.8. If one of the 36 dipoles has an error of -2 mrad and another has error of -1 mrad.



TLS orbit vs dipole field error: Lecture note by C.C. Kuo (2002 OCPA Singapore)



Thin lens – use with care

$$\cos\Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

In this FOFDO cell with Lq=1.0 m, L_dipole=2.0 m, drift length of 0.25 m, and thus L1=3.5 m. Thin lens approximation is good except when the focusing strength is high. The percentage error at high focusing gradient can be larger than 11%.

L_q=1m, K_q=1m^{-2}, L_1=3.5m