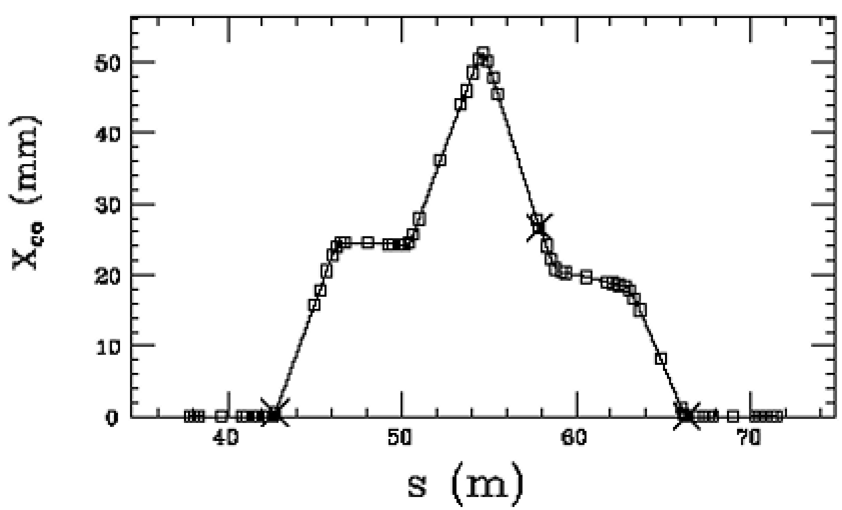
PHY 554

Fundamentals of Accelerator Physics

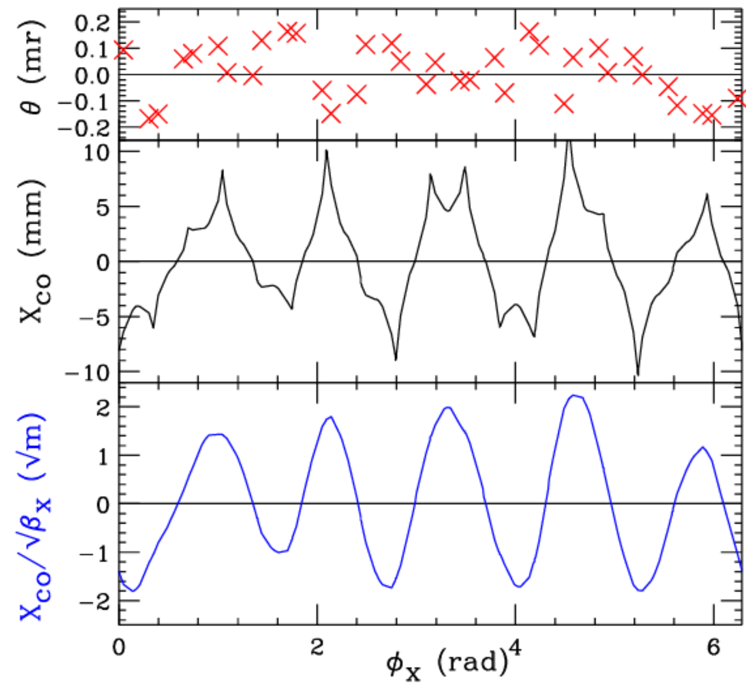
Lecture 7: Orbit correction and dispersion



From dipole field errors and orbit bumps to off-momentum closed orbits and dispersion in FODO lattices.

Outline: orbit response and dispersion

- Review emittance, normalized emittance, Gaussian beams and localized dipole errors.
- Move from one dipole error to distributed errors and orbit correction.
- Introduce closed orbit bumps for local orbit control, injection and extraction.
- Define off-momentum closed orbit and the dispersion function.
- Dispersion and lattice design.



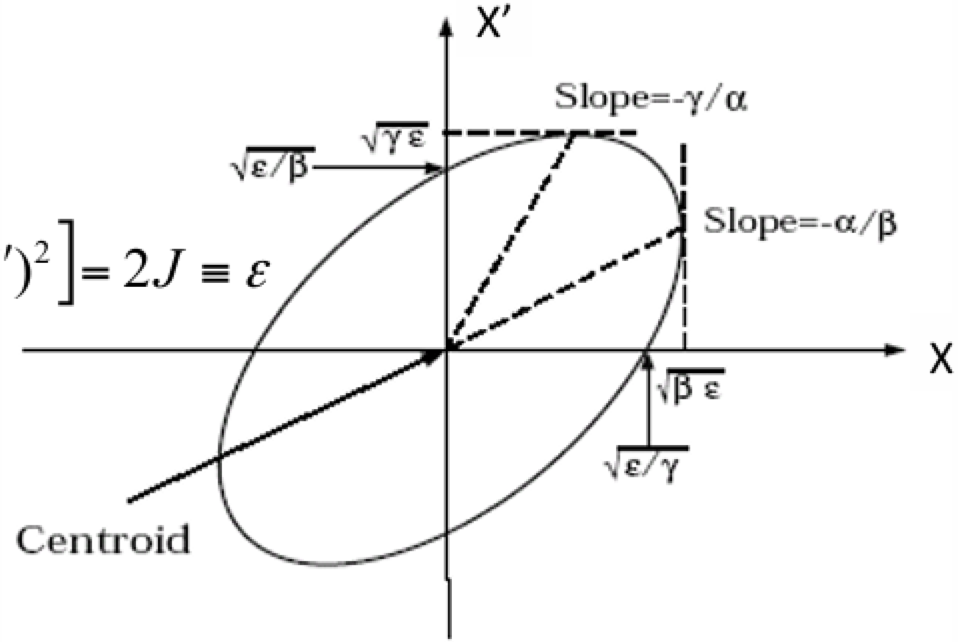
field errors → orbit response → correction

Core idea: linear optics lets us predict how local kicks and momentum errors become global orbit shifts.

What we learned:

Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha X X' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



Emittance of a beam

$$\langle X \rangle = \int X\rho(X,X')dXdX', \quad \langle X' \rangle = \int X'\rho(X,X')dXdX',$$

$$\sigma_X^2 = \int (X - \langle X \rangle)^2 \rho(X,X')dXdX', \quad \sigma_{X'}^2 = \int (X' - \langle X' \rangle)^2 \rho(X,X')dXdX',$$

$$\sigma_{XX'} = \int (X - \langle X \rangle)(X' - \langle X' \rangle)\rho(X,X')dXdX' = r\sigma_X\sigma_{X'}$$

$$\varepsilon_{rms} = \sqrt{\sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2} = \sigma_X\sigma_{X'}\sqrt{1-r^2}$$

The rms emittance is invariant in linear transport: $\dfrac{d\varepsilon^2}{ds} = 0$

Normalized emittance $\varepsilon_n = \varepsilon\beta\gamma$ is invariant when beam energy is changed.

Adiabatic damping – beam emittance decreases with increasing beam momentum, i.e. $\varepsilon = \varepsilon_n/\beta\gamma$, which applies to beam emittance in **linacs**.

In storage rings, the beam emittance **increases** with energy ($\sim\gamma^2$). The corresponding normalized emittance is proportional to γ^3.

The Gaussian distribution function

$$\rho(X, P_X) = \frac{1}{2\pi\sigma_X^2} e^{-(X^2+P_X^2)/2\sigma_X^2}$$

$$\rho(\varepsilon) = \frac{1}{2\varepsilon_{rms}} e^{-\varepsilon/2\varepsilon_{rms}}$$

$\epsilon/\epsilon_{\mathrm{rms}}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96

Reminder: emittance numbers are only meaningful with a definition (RMS?95%? Etc).

Effects of Linear Magnetic field Error

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \qquad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

For a localized dipole field error: θ=ΔBℓ/Bρ

$$X'' + K_X(s)X = \theta\delta(s - s_0)$$

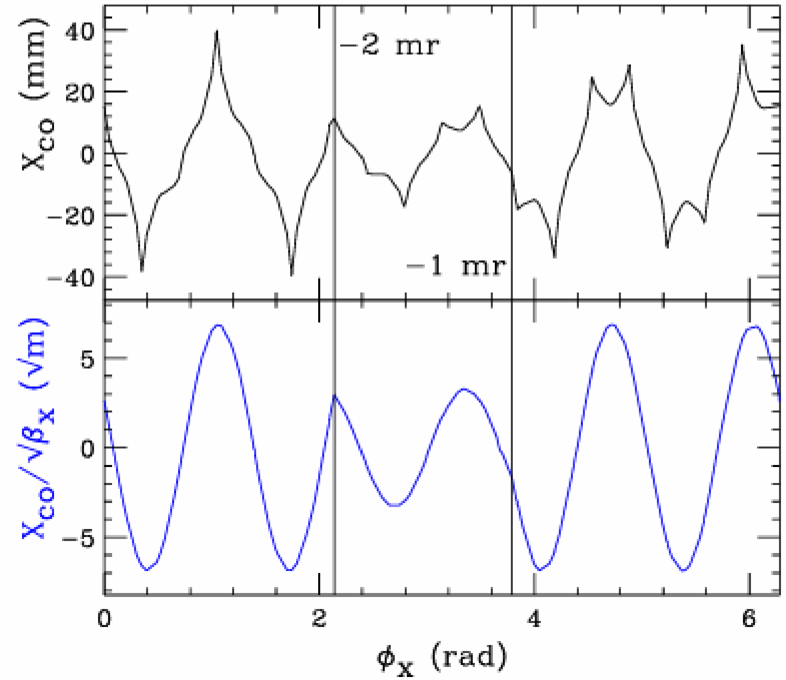
$$X_0 = \frac{\beta_0\theta}{2\sin\pi\nu}\cos\pi\nu,$$

$$X_0' = \frac{\theta}{2\sin\pi\nu}(\sin\pi\nu - \alpha_0\cos\pi\nu)$$

$$X_{co}(s) = G(s, s_0)\theta$$

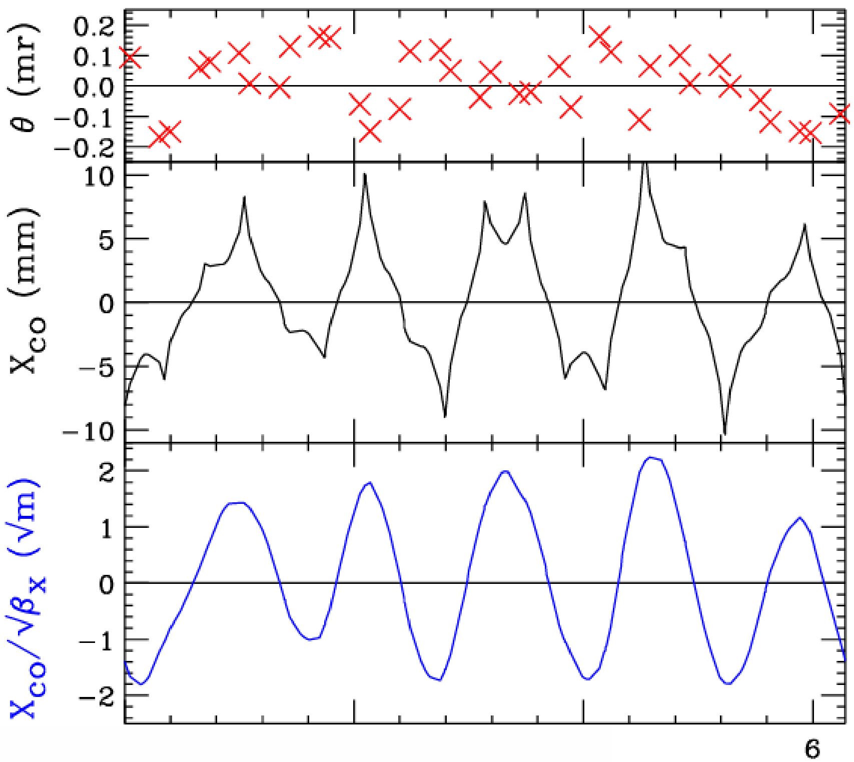
$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s) - \psi(s_0)|]$$

Normalized closed orbit can distinguish errors more easily.



How about many distributed errors?

- Real machines contain many dipole errors and alignment errors.
- In linear optics, the closed orbit is a superposition of all error responses.
- The pattern depends on the local beta function and the betatron phase from each error.
- Orbit correction uses BPMs and correctors to minimize the measured closed orbit.



$$X_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\int_s^{s+C} ds_0\sqrt{\beta(s_0)}\cos[\pi\nu - |\psi(s)-\psi(s_0)|]\theta\delta(s-s_0)$$

$$= \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\sum_i \sqrt{\beta(s_i)}\theta_i\cos[\pi\nu - |\psi(s)-\psi(s_i)|]$$

Orbit correction is global, even when each error is local.

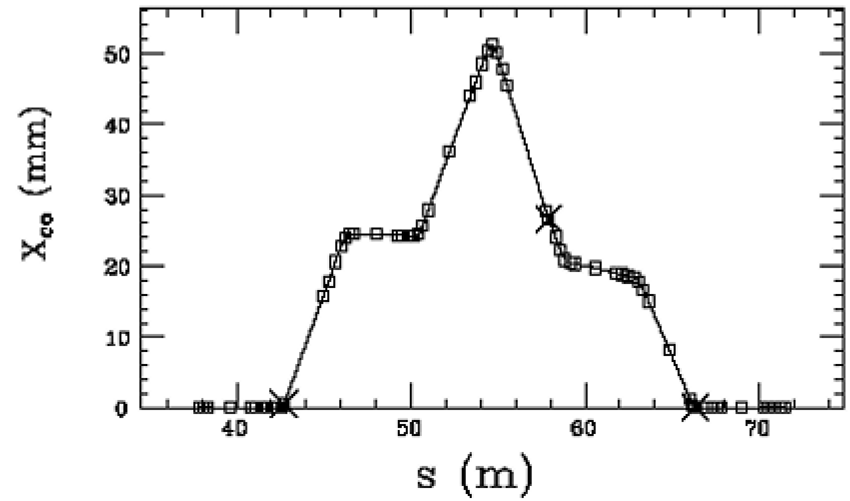
How to make use of dipole errors?

closed orbit bump:

$$X_{co}(s) = G(s,s_0)\theta \qquad G(s,s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s)-\psi(s_0)|]$$

$$X_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\sum_{i=1}^{4}\sqrt{\beta(s_i)}\theta_i\cos(\pi\nu - |\psi(s)-\psi(s_i)|)$$

where θi = $(\Delta Bs)_i/B\rho$ and $(\Delta Bs)_i$ are the kick-angle and the integrated dipole field strength of the i-th kicker. The conditions that the closed orbit is zero outside these four dipoles are $X_{co}(s_4) = 0$, $X'_{co}(s_4) = 0$.



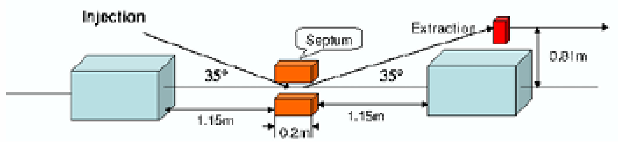
$$\sqrt{\beta_1}\theta_1\cos(\pi\nu-\psi_{41}) + \sqrt{\beta_2}\theta_2\cos(\pi\nu-\psi_{42}) + \sqrt{\beta_3}\theta_3\cos(\pi\nu-\psi_{43}) + \sqrt{\beta_4}\theta_4\cos\pi\nu = 0$$

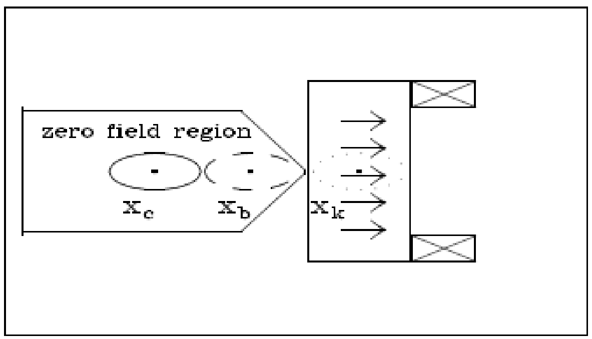
$$\sqrt{\beta_1}\theta_1\sin(\pi\nu-\psi_{41}) + \sqrt{\beta_2}\theta_2\sin(\pi\nu-\psi_{42}) + \sqrt{\beta_3}\theta_3\sin(\pi\nu-\psi_{43}) + \sqrt{\beta_4}\theta_4\sin\pi\nu = 0$$

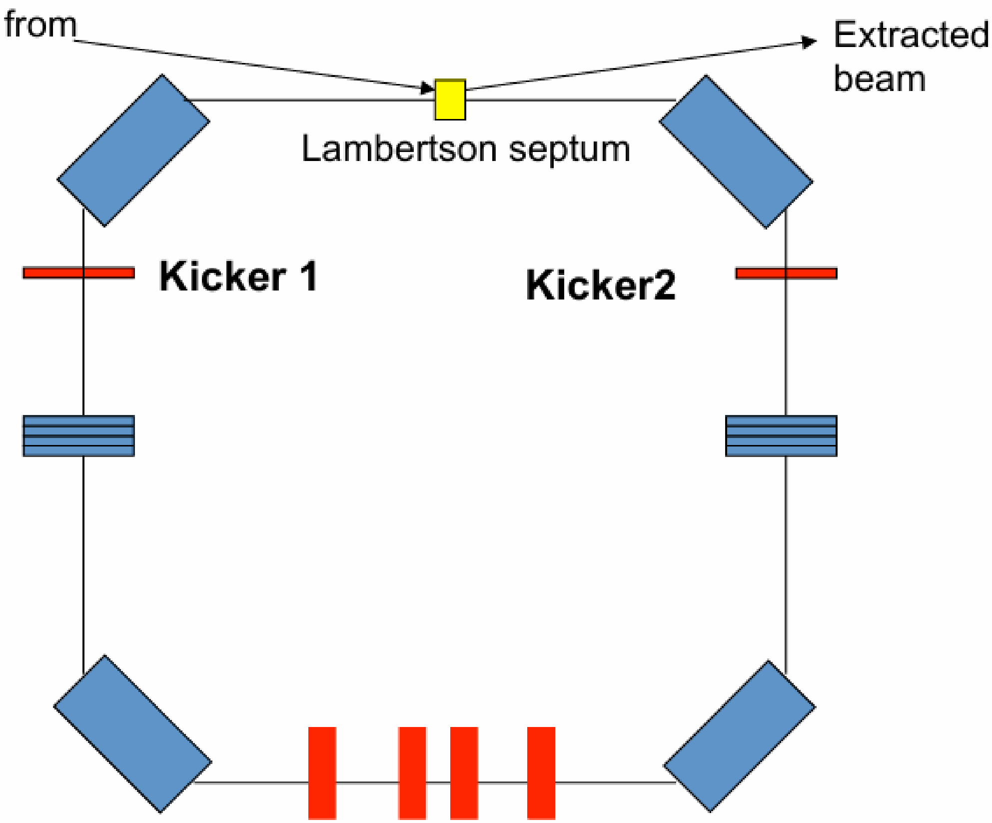
$$\Rightarrow \quad \sqrt{\beta_3}\theta_3 = -(\sqrt{\beta_1}\theta_1\sin\psi_{41} + \sqrt{\beta_2}\theta_2\sin\psi_{42})/\sin\psi_{43}$$

$$\sqrt{\beta_4}\theta_4 = (\sqrt{\beta_1}\theta_1\sin\psi_{31} + \sqrt{\beta_2}\theta_2\sin\psi_{32})/\sin\psi_{43}$$

Local bumps for injection/extraction







$$x_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \sum_{i=1}^{2} \sqrt{\beta_i}\theta_i \cos\left(\pi\nu - |\psi(s) - \psi(s_i)|\right)$$

Condition for localized closed orbit :

$$\sqrt{\beta_1}\theta_1 \cos(\pi\nu) + \sqrt{\beta_2}\theta_2 \cos(\pi\nu - \psi_{21}) = 0$$

$$\sqrt{\beta_1}\theta_1 \sin(\pi\nu) + \sqrt{\beta_2}\theta_2 \sin(\pi\nu - \psi_{21}) = 0$$

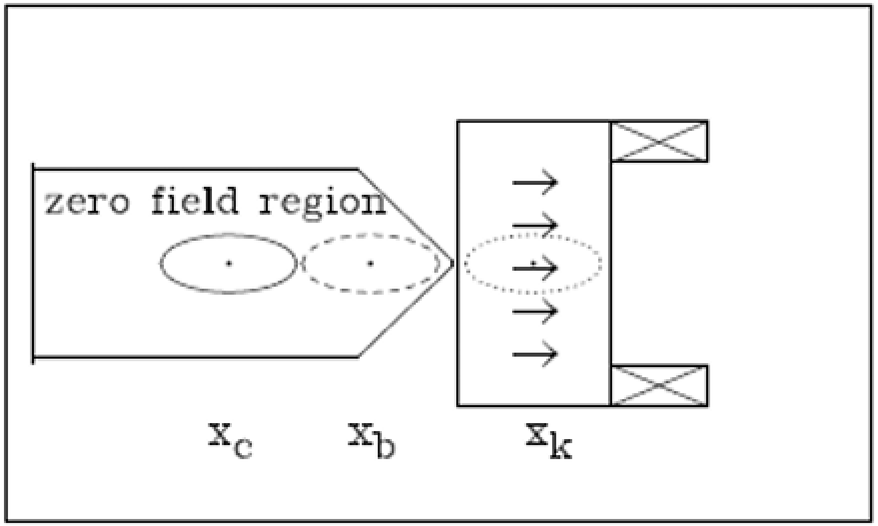
$$\boxed{\psi_{21} = \pi \quad \text{and} \quad \theta_2 = \theta_1\sqrt{\frac{\beta_1}{\beta_2}}}$$

Injection and extraction kicker

$$\Delta x_{co}(s) = \left\{ \sqrt{\beta_x(s_k)\beta_x(s)} \sin(\Delta\psi_x(s)) \right\} \theta_k$$

$\theta_k = \int B_k ds / B\rho$ is the kicker strength (angle), B_k is the kicker dipole field, $\beta_x(s_k)$ is the betatron amplitude function evaluated at the kicker location, $\beta_x(s)$ is the amplitude function at location s, and $\Delta\phi_x(s)$ is the phase advance from s_k of the kicker to location s. The quantity in curly brackets is called the **kicker lever arm**.

A schematic drawing of the central orbit x_c, bumped orbit x_b, and kicked orbit x_k in a Lambertson septum magnet. The blocks marked with X are conductor-coils, The ellipses marked beam ellipses with closed orbits x_c, x_b, and x_k. The arrows indicated a possible magnetic field direction for directing the kicked beams downward or upward in the extraction channel.



Kicker Strength

Electrostatic kicker:

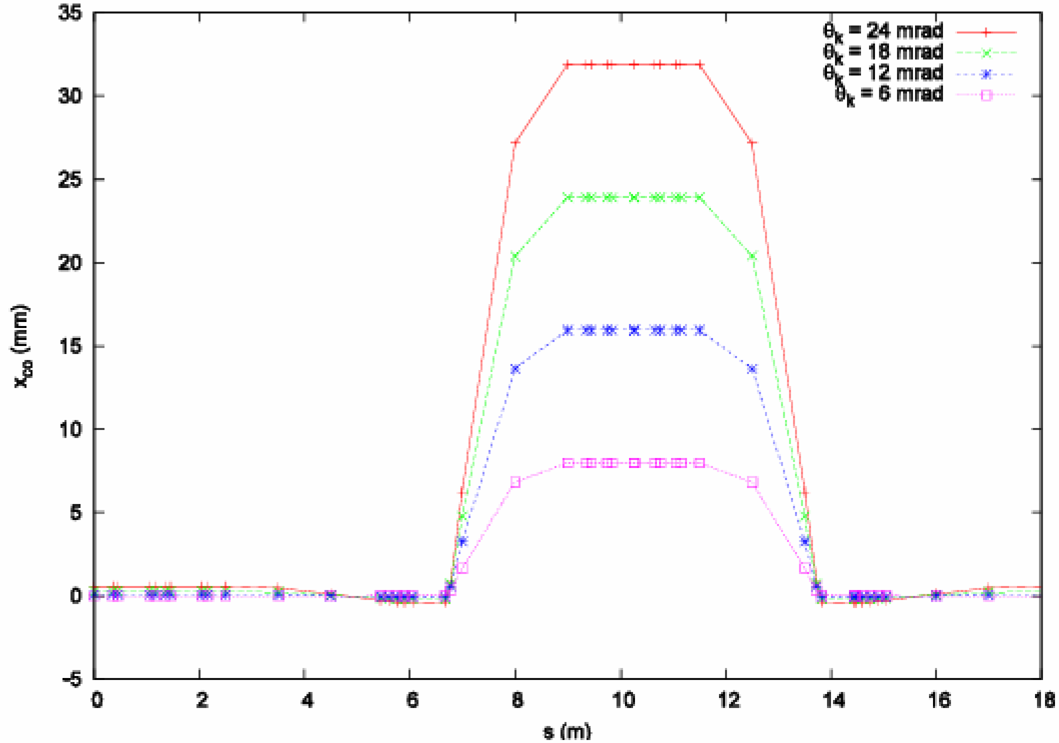
$$\theta_k = \frac{E \cdot L}{c \cdot B\rho} \text{ , where}$$

$$B\rho = 0.2[Tm] \text{ at 60 MeV}$$

$$\text{L} = \text{length of the kicker}$$

$$\text{c} = \text{speed of light}$$

$$\text{E} = \text{gap electric field}$$



For one turn injection and extraction, the integrated field strength is 0.60 MV at 25 MeV electron beam energy. Choosing a length of L=0.5 m, the applied voltage on two plate is 60 kV.

Off-momentum particles do not follow the same closed orbit

We have discussed the closed orbit for a reference particle with momentum p_0, including dipole field errors and quadrupole misalignment. By using closed-orbit correctors, we can achieve an optimized closed orbit that essentially passes through the center of all accelerator components. This closed orbit is called the "golden orbit," and a particle with momentum p_0 is called a **synchronous** particle. However, a beam is made of particles with momenta distributed around a synchronous momentum p_0. What happens to particles with momenta different from p_0? Here we study the effect of off-momentum on the closed orbit. For a particle with momentum p, the momentum deviation is $\Delta p = p - p_0$ and the fractional momentum deviation is $\delta = \Delta p / p_0$, which is typically small of the order of 10^{-6} to 10^{-3}. Since δ is small, we can study the motion of off-momentum particles perturbatively.

DISPERSION

$$p = p_0 + \Delta p,\ \ \delta = \frac{\Delta p}{p_0} \qquad x'' - \frac{\rho + x}{\rho^2} = \left(-\frac{1}{\rho} + Kx\right)\frac{1}{1+\delta}\left(1 + 2\frac{x}{\rho} + \frac{x^2}{\rho^2}\right)$$

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}$$

$$x'' + \left(\frac{1}{\rho^2} - K(s)\right)x = \frac{\delta}{\rho} \qquad K(s) = K_1(s) = \frac{B_1}{B\rho},\quad B_1 = \frac{\partial B_z}{\partial x}$$

The bending angle resulting from a dipole field is different for particles with different momenta. i.e. nonzero δ. The resulting betatron equation of motion is inhomogeneous. The solution of an in-homogeneous linear equation of motion is a linear superposition of the particular solution and the solution of the homogeneous equation, i.e.

$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s)x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s)D = \frac{1}{\rho}$$

The solution of the homogeneous equation is the betatron oscillation we have discussed earlier. The solution of the inhomogeneous equation is called the dispersion function, or the off-momentum closed orbit.

Superposition works again here

$$x = x_\beta + x_{co} = x_\beta + D\delta$$

$$D'' + \left(\frac{1}{\rho^2} - K(s)\right)D = \frac{1}{\rho},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

For a pure dipole (K=0):

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

Dispersion can be transported in a similar fashion as x and x prime.

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \mathrm{L} & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

When $\theta << 1 \quad i.e. \quad L << \rho$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For quadrupoles:

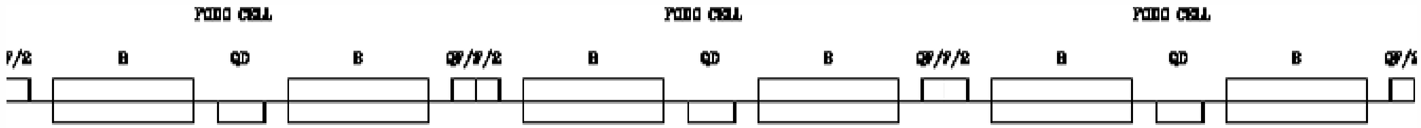
$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell & 0 \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For combined function magnets:

$$\bar{d} = \begin{pmatrix} \dfrac{1}{\rho K_x}(1-\cos\sqrt{K_x}\ell) \\ \dfrac{1}{\rho\sqrt{K_x}}\sin\sqrt{K_x}\ell \end{pmatrix}$$

Example: dispersion in a thin-lens FODO cell



$$\mathbf{M}=\begin{pmatrix}1&0&0\\-\frac{1}{2f}&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&L&\frac{1}{2}L\theta\\0&1&\theta\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\\frac{1}{f}&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&L&\frac{1}{2}L\theta\\0&1&\theta\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-\frac{1}{2f}&1&0\\0&0&1\end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix}D\\D'\\1\end{pmatrix}=\begin{pmatrix}1-\frac{L^2}{2f^2} & 2L(1+\frac{L}{2f}) & 2L\theta(1+\frac{L}{4f})\\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} & 2\theta(1-\frac{L}{4f}-\frac{L^2}{8f^2})\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}D\\D'\\1\end{pmatrix}$$

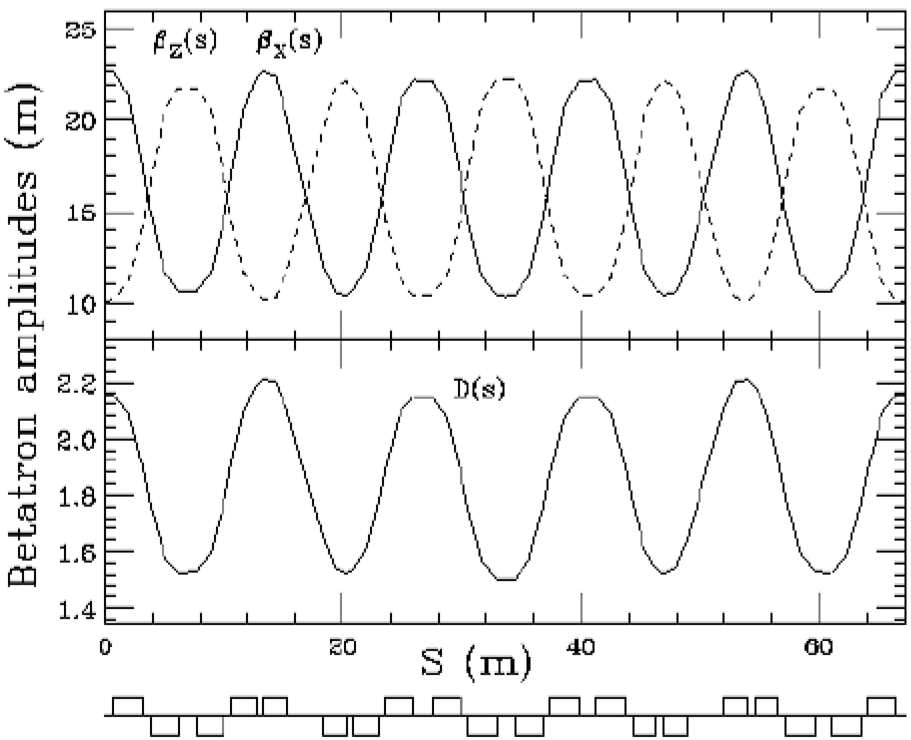
Using the Courant-Snyder parameterization for the transfer matrix, we obtain

$$\sin\frac{\Phi}{2}=\frac{L}{2f},\quad \beta_F=\frac{2L(1+\sin\frac{\Phi}{2})}{\sin\Phi},\quad \alpha_F=0\qquad D_F=\frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}},\quad D'_F=0$$

The dispersion is proportional to the cell length *L* times the bending angle θ, and inversely proportional to the square of the phase advance.

The dispersion at other locations can be obtained by using the 3×3 transfer matrix $M(s_2,s_1)$.

AGS: machine example in a FODO-like lattice

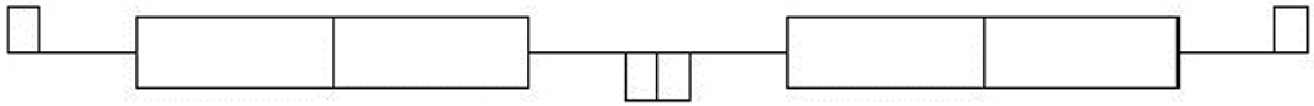


- The AGS is an example of a large proton synchrotron with a repeated FODO-like structure.
- The upper plot shows horizontal and vertical beta functions.
- The middle plot shows horizontal dispersion around one superperiod.
- Combined-function magnets make the relation between bending and focusing especially visible.

In real machine, both beta function and dispersion play a role in beam size

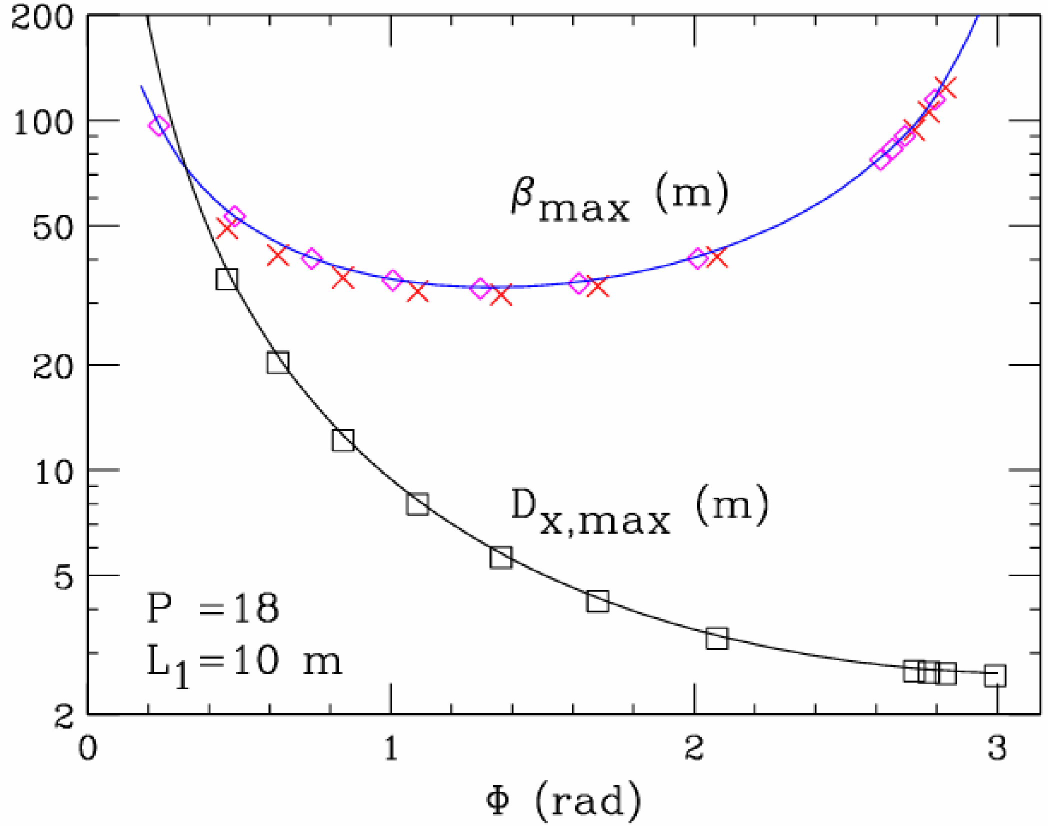
$$x = x_\beta + x_{co} = x_\beta + D\delta$$

$$\sigma_X^2 = \beta\varepsilon_{rms} + (D\delta)^2$$



$$\beta_{max} = \frac{2L_1(1+\frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$

$$D_F = \frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}, \quad D_F' = 0$$



$$\sigma_X^2 = \beta\varepsilon_{rms} + (D\delta)^2$$

lattice design = beta control + dispersion control + aperture + hardware

What to remember from Lecture 7

- Closed-orbit distortion is the Green-function response of the ring to dipole kicks.
- Orbit bumps deliberately combine kicks to create a local displacement while closing the orbit elsewhere.
- Kicker strength depends on integrated field and beam rigidity; useful separation also depends on the lattice lever arm.
- Dispersion is the off-momentum closed orbit: $x = x_\beta + D\delta$.
- Dipoles generate dispersion, quadrupoles transport it and FODO design sets its scale.

Questions to think about: where should dispersion be large, where must it be small, and what errors make the orbit hard to correct?