PHY 564
Advanced Accelerator Physics
Lectures 21

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Lecture 21

Synchrotron oscillations.

Since we will consider momentum of particle constant here, we can switch to a geometrical variables using p_o for normalization $\pi_\tau = \frac{\delta}{p_o}; \pi_{x,y} = \frac{P_{1,3}}{p_o}$. We finished last class with establishing periodic transverse motion (orbits) for particles with constant energy $\pi_\tau = \frac{\delta}{p_o}$. In addition particles will execute transverse betatron oscillation with respect to this orbit:

$$Z = Z_\beta + \pi_\tau \cdot \eta(s); \eta(s+C) = \eta(s); \eta' = D\eta + C;$$

$$Z'_\beta = DZ'_\beta; Z_\beta = \mathrm{Re}\sum_{k=1}^{2} a_k Y_{k\beta}(s) e^{i(\psi_k(s)+\varphi_k)}; \tag{21-01}$$

where are periodic eigen vectors of the transverse oscillations:

$$T_{4x4} Y_{k\beta} = e^{i\mu_k} Y_{k\beta}.$$

In addition, we found that particles with energy deviation are slipping in time as follows:

$$\tau(s) = \pi_\tau\left(\eta_\tau \cdot s + \chi_\tau(s)\right) + \tau_\beta(s); \chi_\tau(s+C) = \chi_\tau(s)$$

$$\eta_\tau \cdot s + \chi_\tau(s) = \left(\frac{mc}{p_o}\right)^2 \cdot s + \int_0^s \left(g_x(\xi)\eta_x(\xi) + g_y(\xi)\eta_y(\xi)\right) d\xi; \tag{21-02}$$

$$\eta_\tau = \frac{1}{C}\int_0^C \left(g_x\eta_x + g_y\eta_y\right) ds + \left(\frac{mc}{p_o}\right)^2;$$

with τ_β is the contribution from the betatron motion.

To be exact, we just separated two parts of the linear motion using the fact that solution o linear differential equation are additive (linear combination of solution is a solution) and that there no time dependence.

Now, let's find the full set of eigen vectors for 3D motion using $\mathbf{T_{6x6}}$ one turn transport matrix.

Let's start from obvious eigen vector:

$$Y_\tau = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; T_{6x6} Y_\tau = Y_\tau; \lambda_\tau = 1. \qquad (21\text{-}03)$$

nothing depends on the time shift! A particle following the reference particle with some time delay follow the same trajectory but with the given time delay. Next eigen vector is not a simple vector but a root vector:

$$Y_\delta = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix}; \quad T_{6x6} Y_\delta = Y_\delta + \eta_\tau Y_\tau; \quad \lambda_\delta = 1. \qquad (21\text{-}04)$$

Note, this is clearly degenerated case when matrix $\mathbf{T_{6x6}}$ can not be diagonalized and we have to use root vectors, but their symplectic product

$$Y_\tau{}^T S Y_\delta = 1 \tag{21-05}$$

is well behaving. What it left is to define the structure of 6-compont betatron eigen vectors. Again, since energy is constant, it does not depend on the transverse motion, e.g. the corresponding element is simply zero:

$$Y_k = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + \dfrac{iq_k}{\mathrm{w}_{kx}}\right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + \dfrac{i(1-q_k)}{\mathrm{w}_{ky}}\right) e^{i\chi_{ky}} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} \tag{21-06}$$

which is generally not true for the time component. While it can be calculated directly and after long manipulations brought to the form we derive easily using symplectic orthogonality of eigen vector of symplectic matrix T:

$$\begin{gathered} Y_i^T \left(T^T S T\right) Y_k = Y_i^T S Y_k \to \lambda_i \lambda_k \cdot Y_i^T S Y_k = Y_i^T S Y_k; \\ \left(\lambda_i \lambda_k - 1\right) Y_i^T S Y_k = 0. \end{gathered} \tag{21-07}$$

With root vectors is just a bit different, but still trivial. Note that betatron eigen vectors is a

With root vectors is just a bit different, but still trivial. Note that betatron eigen vectors is a regular are symplectic-orthogonal to Y_τ is a regular eigen vector with eigen value of 1 and, naturally,

$$Y_\tau{}^T S Y_k = 0;\ \ k = 1,2. \tag{21-08}$$

You can check directly that this is true using explicit expressions (21-06) and (21-03). Note, that this is also requirement is equivalent to requirement that 6th component of betatron eigen vectors Y_k is equal zero. It takes one extra step to prove that for root eigen vector Y_δ:

$$\begin{gathered}
Y_k{}^T\left(T^T S T\right)Y_\delta = Y_k{}^T S Y_\delta;\ \ T Y_\delta = Y_\delta + \eta_\tau Y_\tau \\
\lambda_k \cdot \left(Y_k{}^T S Y_\delta + \eta_\tau Y_k{}^T S Y_\tau\right) = Y_k{}^T S Y_\delta;\ \ Y_k{}^T S Y_\tau = 0;. \\
\left(\lambda_k - 1\right) Y_k{}^T S Y_\delta = 0 \rightarrow Y_k{}^T S Y_\delta = 0.
\end{gathered} \tag{21-09}$$

This gives us automatically explicit expression for 5th component of the betatron eigen vectors:

$$\begin{gathered}
Y_k{}^T S Y_\delta = 0 \rightarrow \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix}^T \begin{bmatrix} S_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = Y_{k\beta}{}^T S\eta + \mathrm{y}_{k\tau} = 0; \\
\mathrm{y}_{k\tau} = -Y_{k\beta}{}^T S\eta = \eta^T S Y_{k\beta} = . \\
\left(\eta_x\left(\mathrm{v}_{kx} + \frac{iq_k}{\mathrm{w}_{kx}}\right) - \eta_{px}\mathrm{w}_{kx}\right)e^{i\chi_{kx}} + \left(\eta_y\left(\mathrm{v}_{ky} + \frac{i(1-q_k)}{\mathrm{w}_{ky}}\right) - \eta_{py}\mathrm{w}_{ky}\right)e^{i\chi_{ky}}.
\end{gathered} \tag{21-10}$$

This equation makes explicit the dependence of the arrival time on amplitudes and phases of betatron oscillation. In locations where dispersion is zero (called achromatic), this dependence vanishes

$$\eta = 0 \Leftrightarrow \mathrm{y}_{k\tau} = 0. \tag{21-11}$$

Now we can formally separate energy dependent motion from transverse betatron oscillations using a Canonical transformation:

$$\begin{gathered}\tilde{H}\left(X_\beta\right) = H\left(\tilde{X} + X_\delta\right) - \frac{\partial F}{\partial s} = H\left(\tilde{X} + X_\delta\right) + \\ +\eta_x'\tilde{\pi}_\tau\left(\tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau\right) - \eta_{px}'\tilde{\pi}_\tau\tilde{x} + \eta_y'\tilde{\pi}_\tau\left(\tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau\right) - \eta_{py}'\tilde{\pi}_\tau\tilde{y}\end{gathered} \tag{21-12}$$

while we can prove that matrix of such transformation is symplectic, it is also very easy to do using a generation function noticing that $\tilde{\pi}_\tau = \pi_\tau$ is not changing during the transformation

$$\begin{gathered}F\left(q,\tilde{P}\right) = \left(x - \eta_x\tilde{\pi}_\tau\right)\left(\tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau\right) + \left(y - \eta_y\tilde{\pi}_\tau\right)\left(\tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau\right) \\ +\tau\tilde{\pi}_\tau - \left(\eta_x\eta_{px} + \eta_y\eta_{py}\right)\frac{\tilde{\pi}_\tau^{\ 2}}{2} \\ \pi_\tau = \frac{\partial F}{\partial \tau} = \tilde{\pi}_\tau;\ \ \tilde{\tau} = \frac{\partial F}{\partial \tilde{\pi}_\tau} = \tau - \eta_x\tilde{\pi}_x + \eta_{px}\tilde{x} - \eta_y\tilde{\pi}_y + \eta_{py}\tilde{y}; \\ x_\beta = \tilde{x} = \frac{\partial F}{\partial \tilde{\pi}_x} = x - \eta_x\tilde{\pi}_\tau; y_\beta = \tilde{y} = \frac{\partial F}{\partial \tilde{\pi}_y} = y - \eta_y\tilde{\pi}_\tau; \\ \pi_x = \frac{\partial F}{\partial x} = \tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau;\ \ \pi_y = \frac{\partial F}{\partial y} = \tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau.\end{gathered} \tag{21-13}$$

We should note that transformation (21-13) is linear:

$$\tilde{X} = \mathbf{I}\cdot X - \pi_\tau \begin{bmatrix} \eta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} O \\ -\eta_x\left(\pi_x - \eta_{px}\pi_\tau\right) + \eta_{px}\left(x - \eta_x \pi_x\right) \\ -\eta_y\left(\pi_y - \eta_{py}\pi_\tau\right) + \eta_{py}\left(y - \eta_y \pi_\tau\right) \\ 0 \end{bmatrix} = \mathbf{L}\cdot X \qquad (21\text{-}14)$$

with matrix L naturally being symplectic – you can directly check that it is correct. We can follow a direct way of finding form of Hamiltonian in the new variables (e.g. transforming coefficients in Hamiltonian (20-21) and adding *s*-derivative of the transfer function), but since the transformation is simply linear we can use a short-cut rewriting (2-14) as

$$\tilde{X} = \begin{bmatrix} \tilde{Z} \\ \tilde{\tau} \\ \tilde{\pi}_\tau \end{bmatrix} = X - \begin{bmatrix} \pi_\tau \eta \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} O \\ \eta^T S \tilde{Z} \\ 0 \end{bmatrix}; X = \begin{bmatrix} Z \\ \tau \\ \pi_\tau \end{bmatrix}; \tilde{Z} = Z - \pi_\tau \eta;$$

$$X = \tilde{X} + \begin{bmatrix} \tilde{\pi}_\tau \eta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} O \\ \eta^T S \tilde{Z} \\ 0 \end{bmatrix}; \qquad (21\text{-}15)$$

where we used that $\eta^T S \eta = 0$.

Now it is relatively straight-forward to write equations of motion is new variables:

$$\frac{dX}{ds} = \mathbf{D}X; \frac{d\tilde{X}}{ds} = \tilde{\mathbf{D}}\tilde{X}; \mathbf{D} = \begin{bmatrix} D_{4\text{x}4} & 0 \quad C \\ .... & .... \end{bmatrix}; \frac{dZ}{ds} = D_{4\text{x}4}Z + C\pi_\tau; \eta' = D_{4\text{x}4}\eta + C;$$

$$\frac{d\tilde{X}}{ds} = \frac{dX}{ds} - \pi_\tau \begin{bmatrix} \eta' \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} O \\ \frac{d}{ds}\left(\eta^T S\tilde{Z}\right) \\ 0 \end{bmatrix} = \tag{21-16}$$

$$\begin{bmatrix} Z' - \pi_\tau \eta' \\ \frac{d}{ds}\left(\tau - \eta^T S\tilde{Z}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} D_{4x4}\left(Z - \pi_\tau \eta\right) \\ \frac{d}{ds}\tilde{\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} D_{4x4}\tilde{Z} \\ \tilde{\tau}' \\ 0 \end{bmatrix}.$$

It means that the transverse part of the Hamiltonian remain the same since $D_{4x4} = S_{4x4} \cdot H_{4x4}$

$$\frac{d\tilde{Z}}{ds} = D_{4\text{x}4}\tilde{Z}; \quad \tilde{D}_{4\text{x}4} \equiv D_{4\text{x}4}; \tag{21-17}$$

but the C components, as expected, vanish. It means that in new Hamiltonian mixed components between longitudinal $\left\{\tilde{\tau}, \tilde{\pi}_\tau\right\}$ and $\tilde{Z}^T = \left\{x_\beta, \pi_{x\beta}, y_\beta, \pi_{y\beta}\right\}$. A non-zero component of type $\left(a\tilde{\tau} + b\tilde{\pi}_\tau\right)\tilde{z}_i$ in the new Hamiltonian will generate non-zero additional component in (21-17):

$$\frac{d\tilde{z}_k}{ds} = S_{ki}\frac{\partial \tilde{H}}{\partial \tilde{z}_i} = S_{ki}\left(a\tilde{\tau} + b\tilde{\pi}_\tau\right)$$

which contradict the findings.

Hence, both the Hamiltonian and new D matrix have a block diagonal form:

$$\tilde{\mathbf{H}} = \begin{bmatrix} H_{4x4} & O_{4x2} \\ O_{4x4} & \tilde{H}_{l2x2} \end{bmatrix}; \tilde{\mathbf{D}} = \mathbf{S}\cdot\tilde{\mathbf{H}} = \begin{bmatrix} D_{4x4} & O_{4x2} \\ O_{4x4} & \tilde{D}_{l2x2} \end{bmatrix}; \ \tilde{H}_{4x4} = H_{4x4} \qquad (21\text{-}18)$$

It is also possible to prove it explicitly by considering in detail the only remaining equation in (21-16) for $\tilde{\tau}'$. This also allows us to find explicitly expression for the longitudinal Hamiltonian $\tilde{H}_l$.

$$\frac{d\tilde{\tau}}{ds} = \frac{d}{ds}\left(\tau - \eta^T SZ\right) = \frac{d}{ds}\left(\tau - \eta^T S\tilde{Z}\right)$$

$$\frac{d\tau}{ds} == \left(g_x x + g_y y + \left(\frac{mc}{p_o}\right)^2 \right)\pi_\tau = \pi_\tau \left(\frac{mc}{p_o}\right)^2 + C^T S\left(\tilde{Z} + \tilde{\pi}_\tau \eta^T\right); \qquad (21\text{-}19)$$

$$\left(\eta^T S\tilde{Z}\right)' = \left(\eta^T D^T{}_{4x4} + C^T\right)S\tilde{Z} + \eta^T SD^T{}_{4x4}\tilde{Z} = C^T S\tilde{Z}; \ \to \frac{d\tilde{\tau}}{ds} = \tilde{\pi}_\tau \left(\left(\frac{mc}{p_o}\right)^3 + C^T S\eta^T \right)$$

with $C^T = \begin{bmatrix} 0 & -g_x & 0 & -g_y \end{bmatrix}$ and we used obvious

$$\eta^T D^T{}_{4x4} S\tilde{Z} + \eta^T SD_{4x4}\tilde{Z} = \eta^T H_{4x4}\tilde{Z} - \eta^T H_{4x4}\tilde{Z} = 0\,.$$

We can re-write (21019) for $\tilde{\tau}'$ explicitly as

$$\frac{d\tilde{\tau}}{ds} = \tilde{\pi}_\tau \left(\left(\frac{mc}{p_o} \right)^2 + \eta_x g_x + \eta_y g_y \right); \tag{21-20}$$

and the new Hamiltonian as

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta;$$

$$\mathcal{H}_\beta = \frac{\pi_{x\beta}{}^2 + \pi_{y\beta}{}^2}{2} + \frac{F}{p_o c}\frac{x_\beta{}^2}{2} + \frac{N}{p_o c} x_\beta y_\beta + \frac{G}{p_o c}\frac{y_\beta{}^2}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$

$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o} \right)^2 + g_x \eta_x + g_y \eta_y \right) \frac{\pi_\tau{}^2}{2} = c_\tau \frac{\pi_\tau{}^2}{2} \tag{20-23}$$

$$\frac{dX_\beta}{ds} = S \frac{\partial \mathcal{H}_\beta}{\partial X_\beta} = D_\beta X_\beta; \quad \frac{d}{ds}\begin{bmatrix} \tilde{\tau} \\ \pi_\tau \end{bmatrix} = \begin{bmatrix} c_\tau \\ 0 \end{bmatrix} \pi_\tau ; \rightarrow \tilde{\tau} = \pi_\tau \int_o^s c_\tau(\xi) d\xi$$

where I dropped tilde for compactness. It is important to remember that in this new variables

$$\tilde{\tau} = \frac{\partial F}{\partial \tilde{\pi}_\tau} = \tau - \eta_x \pi_{x\beta} + \eta_{px} x_\beta - \eta_y \tilde{\pi}_y + \eta_{py} y_\beta;$$

$$\tau = c\left(t_o(s) - t\right). \tag{20-24}$$

It means that arrival time dependence on transverse oscillations is well hidden in

$$t = t_o(s) - \frac{\tau}{c} = t_o(s) - \frac{\tilde{\tau}}{c} - \frac{\eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta}{c} \tag{20-25}$$

Since, without time dependent components in the Hamiltonian, the betatron and longitudinal motions are fully decoupled. It also means that in new variables our eigen vectors become:

$$Y_{k\beta} = \begin{bmatrix} Y_{k\beta} \\ 0 \\ 0 \end{bmatrix}; \; Y_{\tilde{\tau}} = \begin{bmatrix} O \\ 1 \\ 0 \end{bmatrix}; Y_{\tilde{\delta}} = \begin{bmatrix} O \\ 0 \\ 1 \end{bmatrix}; \tag{21-26}$$

The reason for disappearance of the transverse component in $Y_{\tilde{\delta}}$: it is caused by measuring the transverse orbit from the closed orbit for deviated energy. Similarly, disappearance time component in betatron eigen vectors is caused by its explicit inclusion into the new time variable.

Adding RF fields

Finally we are ready to move to synchrotron oscillations. Let's consider that we adding alternating (AC) longitudinal electric field on the beam axis

$$\frac{d\mathrm{E}}{ds} = -eE_s(s,t) \tag{21-27}$$

For a moment we do not need to pick any specific form of this field, as far it does supports our assumption that the reference particle's trajectory in time, space and momentum exist. Specifically it means that we request that

$$\frac{d\mathrm{E}_o}{ds} = -eE_s\left(s,t_o(s)\right) = 0 \tag{21-28}$$

e.g. that alternating electric field crosses zero at the time of the passing of the reference particle. For a storage ring (a periodic system) the accelerating field has to be is periodic. The AC field (called RF field in the accelerators) has to satisfy the same condition.

$$\begin{gathered} E_s(s+C,t) = E_s(s+C,t);\ E_s\left(s+nC,t_o(s+nC)\right) = 0; \\ t_o(s+nC) = t_o(s) + nT_o;\ T_o = \frac{C}{\mathrm{v}_o}. \end{gathered} \tag{21-29}$$

where T_o is called revolution period in the storage ring.

In practice, the alternating EM fields are generated in resonant cavities and have a sine-wave time dependence:

$$E_s(s,t) = \sum_n \operatorname{Re} E_n(s) e^{i\omega_n t} = \sum_n |E_n(s)| \sin(-\omega_n t + \phi_n(s)); \tag{21-30}$$

where we simply numerated various RF frequencies ω_n, which frequently can be just a single frequency. In combination with (21-29) it yields requirement that all RF frequencies have to be harmonic of the revolution frequency:

$$\begin{gathered} T_o\omega_n = 2\pi h_n; h_n - inger; \;\; \omega_n = h_n\omega_o \\ f_{RFn} = \frac{\omega_n}{2\pi} = h_n f_{rev}; \;\; f_{rev} = \frac{\omega_o}{2\pi} = \frac{1}{T_o} = \frac{\mathrm{v}_o}{C}. \end{gathered} \tag{21-31}$$

with the field on axis of:

$$\begin{gathered} E_s(s,t) = \sum_n \operatorname{Re} E_n(s) e^{i\omega_n t} = e\sum_n |E_n(s)| \sin(h_n\omega_o(t - t_o(s)) - \phi_n(s)); \\ \sum_n |E_n(s)| \sin(\phi_n(s)) = 0. \end{gathered} \tag{21-32}$$

For a single harmonic RF (21-32) becomes $\phi_n(s) = \pm\pi$ and

$$E_s(s,t) = eE_{rf}(s)\sin(h_{rf}\omega_o(t - t_o(s))) \tag{21-33}$$

Note that the sign of E_{rf} depends on what node of the sin-wave we choose. We can add the term corresponding to this longitudinal field using our full accelerator Hamiltonian (which doable but not necessary), or by noticing that

$$\frac{d\pi_\tau}{ds} = \frac{1}{p_o c}\frac{d\left(\mathrm{E}\text{-}\mathrm{E}_o\right)}{ds} = \frac{e}{p_o c}\sum_n \left|E_n(s)\right| \sin\left(h_n \omega_o t - \phi_n(s)\right) \tag{21-32}$$

corresponds to a term in Hamiltonian of

$$\delta H = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right| \cos\left(h_n k_o\left(\tilde{\tau} + \tau_{add}\right) + \phi_n\right)}{h_n k_o};\ k_o = \frac{\omega_o}{c} = \frac{2\pi}{C}\frac{\mathrm{v}_o}{c};$$

$$\tau_{add} = \eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta; \tag{21-33}$$

$$\frac{d\pi_\tau}{ds} = -\frac{\partial(\delta H)}{\partial \tilde{\tau}} = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right| \sin\left(h_n k_o\left(\tilde{\tau} + \tau_{add}\right) + \phi_n\right)}{h_n k_o};$$

Thus, we can write a generic Hamiltonian without expansion in time domain:

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta + \delta\mathcal{H}$$

$$\mathcal{H}_\beta = \frac{mc}{p_o}\cdot\frac{\pi_{x\beta}^{\ 2} + \pi_{y\beta}^{\ 2}}{2} + \frac{F}{p_o}\frac{x_\beta^{\ 2}}{2} + \frac{N}{mc}x_\beta y_\beta + \frac{G}{p_o}\frac{y_\beta^{\ 2}}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$

$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o}\right)^2 + g_x \eta_x + g_y \eta_y\right)\frac{\pi_\tau^{\ 2}}{2} = c_\tau \frac{\pi_\tau^{\ 2}}{2} \tag{21-34}$$

$$\delta\mathcal{H} = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right| \cos\left(h_n k_o\left(\tilde{\tau} + \tau_{add}\right) + \phi_n\right)}{h_n k_o}$$

Linearized part of the additional Hamiltonian term is

$$\delta \mathcal{H}_{\tau} = -\frac{e\left(\tilde{\tau}+\tau_{add}\right)2}{p_o c}\sum_n h_n k_o \left|E_n\right| \cos\left(\phi_n\right). \tag{21-35}$$

While looking simpler than original Hamiltonian, adding RF fields made the Hamiltonian fully 3D coupled through τ_{add}. Hence, next step – let's consider case without betatron oscillations $\tilde{\tau}=\tau$:

$$\mathcal{H}_s = \left(\left(\frac{mc}{p_o}\right)^3 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} + \frac{e}{p_o c}\sum_n \frac{\left|E_n\right|\cos\left(h_n k_o \tau + \phi_n\right)}{h_n k_o} \tag{21-37}$$

or in linear case

$$\mathcal{H}_{sL} = \left(\left(\frac{mc}{p_o}\right)^3 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} - \frac{\tau^2}{2}\frac{e}{p_o c}\sum_n h_n k_o \left|E_n\right|\cos\left(\phi_n\right) \tag{21-38}$$

Coefficients in both Hamiltonians are s-dependent and the Hamiltonians are not constants of motion. Naturally the (21-38) linear system, when stable, can be solved using 1D parameterization

$$\begin{gathered}\tau = a_s \mathrm{w}_s(s)\cos\left(\psi_s(s)+\varphi_s\right);\ \ \psi_s' = \frac{1}{\mathrm{w}_s};\\ \pi_\tau \frac{mc}{p_0} = \left\{a_s \mathrm{w}_s'(s)\cos\left(\psi_s(s)+\varphi_s\right) - \frac{1}{\mathrm{w}_s(s)}\sin\left(\psi_s(s)+\varphi_s\right)\right\}.\end{gathered} \tag{21-39}$$

This said, in majority of the storage rings, synchrotron oscillations are very slow and it takes from hundreds to tens of thousands turns to complete a single synchrotron oscillations. In this case small variations during one pass around a ring can be averaged. The easiest way is just to average the Hamiltonians (21-37) and (21-38): Beware, this is an approximation which brakes if synchrotron tune is relatively larger (let's say ~ 0.1). Still, it is easy way to get something useful – lets' do it:

$$\left\langle \mathcal{H}_s \right\rangle = \left(\eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{eU_{RF}(\tilde{\tau})}{p_o c} \right); U'_{RF}(0) = 0;$$

$$U_{RF}(\tilde{\tau}) = \frac{1}{C} \cdot \frac{e}{mc} \sum_n \frac{V_n}{h_n k_o} \cos\left(h_n k_o \tilde{\tau} + \bar{\phi}_n\right);\ \eta_\tau = \left(\frac{mc}{p_o}\right)^3 + \left\langle g_x \eta_x + g_y \eta_y \right\rangle; \qquad (21\text{-}40)$$

$$V_n \cos\left(\theta + \bar{\phi}_n\right) = \frac{\cos\theta}{h_n k_o} \int_o^C \left|E_n(s)\right| \cos\phi_n(s) - \frac{\sin\theta}{h_n k_o} \int_o^C \left|E_n(s)\right| \sin\phi_n(s).$$

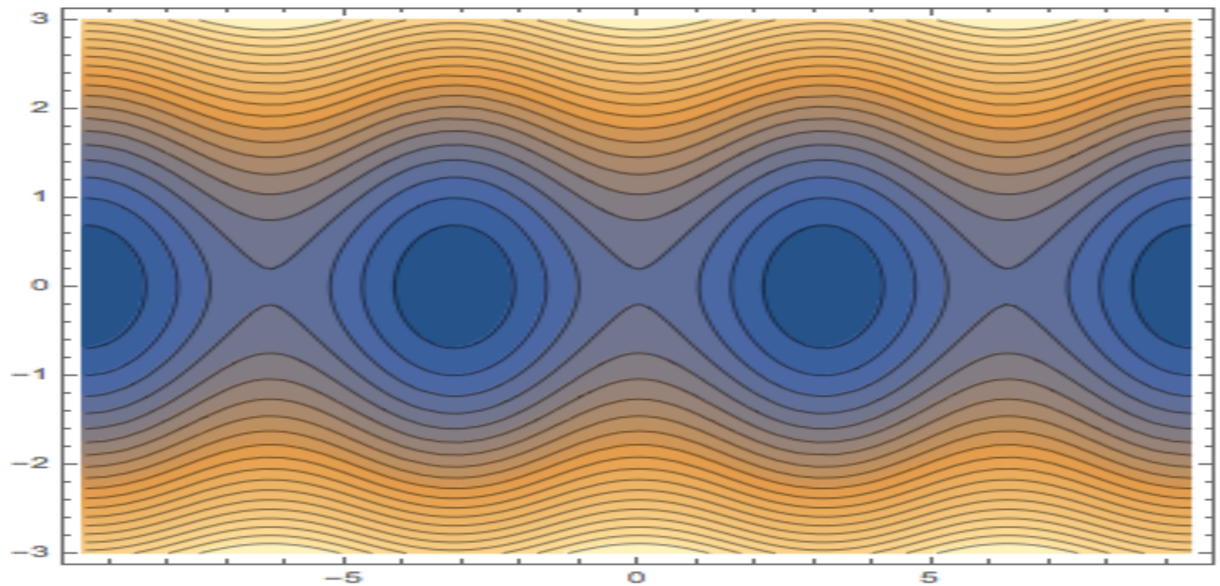
The averaged Hamiltonian does not depend on s and is invariant of motion. Thus we can say that

$$\left\langle \mathcal{H}_s \right\rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{eU_{RF}(\tilde{\tau})}{mc} = \mathcal{H}_o; \qquad (21\text{-}41)$$

are equivalent to trajectories in the phase space of τ, π .

Let's consider a single frequency RF – a traditional single frequency RF – well know pendulum equation:

$$\left\langle \mathcal{H}_s \right\rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}} = \mathcal{H}_o;$$
$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \ \ \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin\left(k_o h_{rf}\tau\right); \qquad (21\text{-}42)$$



Plot of the equipotential for Hamiltonian (21-42) – stable motion occurs around the zero or 180 degrees, depending on the relative sign of $eU_{RF}\eta_\tau$.

Stationary points are

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau = 0; \rightarrow \pi_\tau = 0; \frac{d\pi_\tau}{ds} = 0; \rightarrow \phi_o = k_o h_{rf}\tau = N\pi; \qquad (21\text{-}43)$$

Expanding around the stationary point we

$$\left\langle \mathcal{H}_s \right\rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} - \frac{1}{C}\frac{eV_{RF}}{p_o c} k_o h_{rf} \frac{\cos(\tau)}{k_o h_{rf}} \frac{\tau^2}{2} \cos(\phi_o);$$

$$\Omega_s^{\ 2} = -\eta_\tau k_o h_{rf} \frac{eV_{RF}}{mc}\cos(\phi_o);\ \cos(\phi_o) = \pm 1;\ Ck_o = 2\pi\beta_o$$

$$\Omega_s = \sqrt{\left|\frac{\eta_\tau k_o h_{rf}}{C}\frac{eV_{RF}}{p_o c}\right|} = \sqrt{\left|\left\langle \left(\frac{mc}{p_o}\right)^3 + g_x\eta_x + g_y\eta_y \right\rangle k_o h_{rf} \frac{eV_{RF}}{mc}\right|}; \tag{21-44}$$

$$\Omega_s = \sqrt{\left|\eta_\tau k_o h_{rf} \frac{eV_{RF}}{mc}\right|}; \mu_s = \Omega_s C = \sqrt{\left|2\pi\eta_{\tau o} h_{rf} \frac{eV_{RF}}{E_o}\right|};$$

$$Q_s = \frac{\mu_s}{2\pi} = \sqrt{\left|\frac{\eta_{\tau o} h_{rf}}{2\pi}\frac{eV_{RF}}{E_o}\right|}$$

Stable points are

$$-\eta_\tau eV_{RF}\cos(\phi_o) > 0; \Rightarrow \ \phi_s = 2N\pi; \qquad \eta_\tau eV_{RF} < 0;$$

$$\phi_s = (2N+1)\pi;\ \ \eta_\tau eV_{RF} > 0;$$

As we discussed during last class η_τ determines the sign on the longitudinal mass. When it is negative, not minima but maxima of the potential correspond to stable points.

Finally, let's look what happens with transverse motion when RF fields are present? Using perturbation theory we have.

$$\delta\phi_k = \frac{1}{2}\int^{s} \tilde{Y}_k^{*T}(\zeta)\Delta\mathbf{H}(\zeta)\tilde{Y}_k(\zeta)d\zeta;$$
$$\Delta\mathbf{H}_{55} = -\frac{e}{p_o c^2}\frac{\partial E_{rf}}{\partial t};\ \delta Q_k = -\frac{1}{4\pi}\frac{e}{p_o c}\int_0^C \frac{\partial E_{rf}}{\partial ct}\left|Y_{k5}\right|^2 ds \qquad (21\text{-}45)$$

It is possible to show that $\delta Q_k \sim \delta Q_s^{\ 2}$