Synchrotron Motion

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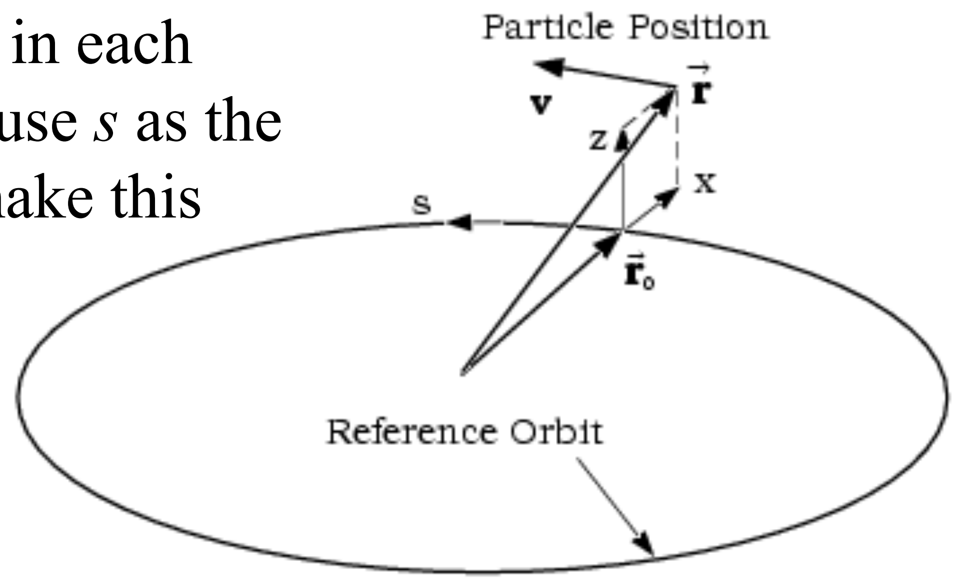
Longitudinal motion is hidden in the Hamiltonian:

$$H = e\Phi + c\left[m^2c^2 + \frac{(p_s - eA_s)^2}{(1+x/\rho)^2} + (p_x - eA_x)^2 + (p_z - eA_z)^2 \right]^{1/2}$$

$$\dot{s} = \frac{\partial H}{\partial p_s}, \dot{p}_s = -\frac{\partial H}{\partial s}; \quad \dot{x} = \frac{\partial H}{\partial p_x}, \dot{p}_x = -\frac{\partial H}{\partial x}; \quad \dot{z} = \frac{\partial H}{\partial p_z}, \dot{p}_z = -\frac{\partial H}{\partial z}.$$

The phase space coordinates are (x,s,z) with independent coordinate t. In one revolution, the time advances T_0, called the orbital period. In one orbital period, the particle orbit is equal to the circumference C.

All accelerator components repeat in each orbital period. It would be nice to use s as the independent coordinate. How to make this coordinate transfer?



$$x' = \frac{dx}{ds} = \frac{\dot{x}}{\dot{s}} = \left(\frac{\partial H}{\partial p_x}\right)\left(\frac{\partial H}{\partial p_s}\right)^{-1} = \frac{\partial(-p_s)}{\partial p_x},$$

$$dH = (\partial H/\partial p_x)dp_x + (\partial H/\partial p_s)dp_s = 0$$

$$t' = \frac{\partial p_s}{\partial H}, H' = -\frac{\partial p_s}{\partial t}; \quad x' = -\frac{\partial p_s}{\partial p_x}, p_x' = \frac{\partial p_s}{\partial x}; \quad z' = -\frac{\partial p_s}{\partial p_z}, p_z' = \frac{\partial p_s}{\partial z}.$$

These equations indicate that $-p_s$ becomes the new Hamiltonian with the $(x,p_x,z,p_z,\mathbf{t},\mathbf{-H})$ and *s* as the independent coordinate.

$$\tilde{H} = -\left(1+\frac{x}{\rho}\right)\left[\frac{(H-e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2\right]^{1/2} - eA_s,$$

$$\widetilde{H} \approx -p(1+\frac{x}{\rho}) + \frac{1+x/\rho}{2p}[(p_x - eA_x)^2 + (p_z - eA_z)^2] - eA_s$$

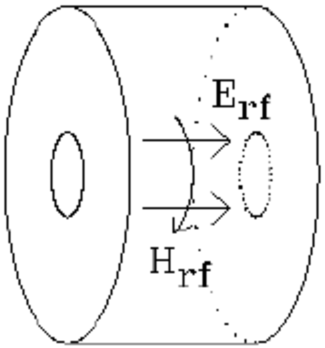
$$x' = \frac{\partial \widetilde{H}}{\partial p_x},\ p_x' = -\frac{\partial \widetilde{H}}{\partial x}, \quad z' = \frac{\partial \widetilde{H}}{\partial p_z},\ p_z' = -\frac{\partial \widetilde{H}}{\partial z}, \quad \boxed{t' = \frac{\partial \widetilde{H}}{\partial(-H)},\quad (-H)' = -\frac{\partial \widetilde{H}}{\partial t}.}$$

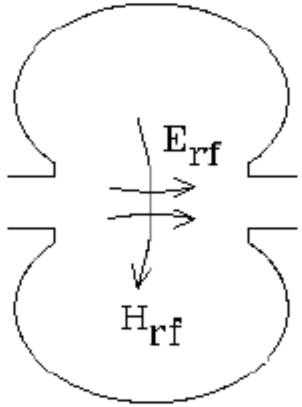
$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho}$$

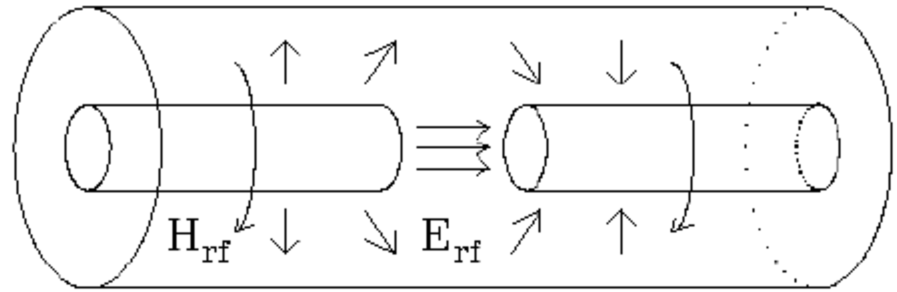
Hill's equation

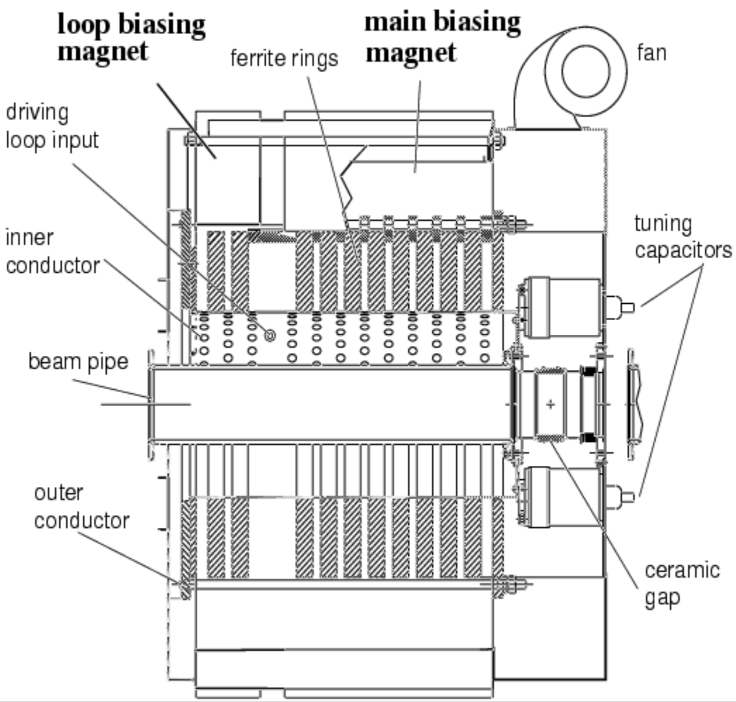
$$\boxed{\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s), \quad \phi_{n+1} = \phi_n + \frac{2\pi\eta}{\beta^2 E}\Delta E_{n+1}}$$

Synchrotron motion

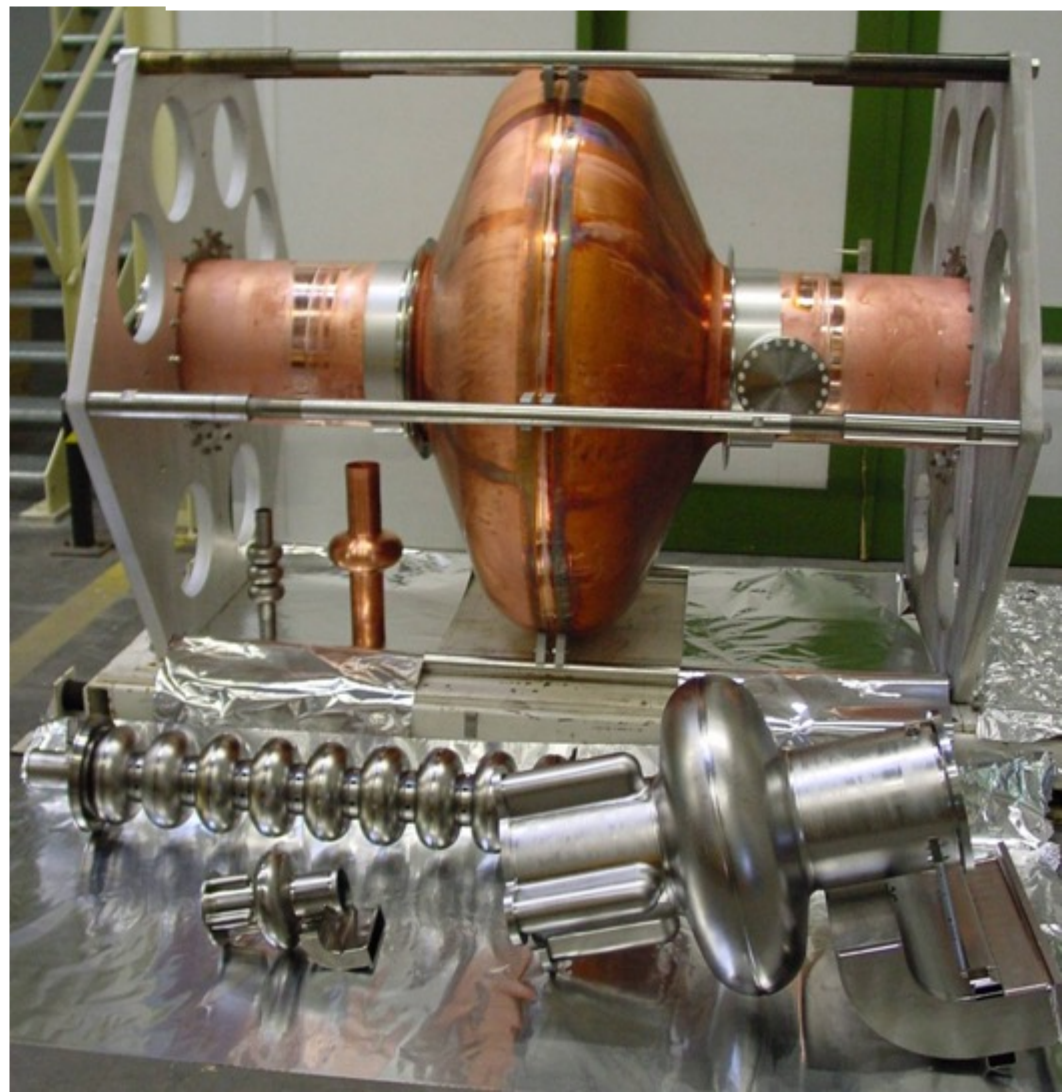








A collection of SRF cavities developed at Cornell University with frequencies spanning 200 MHz to 3 GHz



The longitudinal electric field at an rf gap

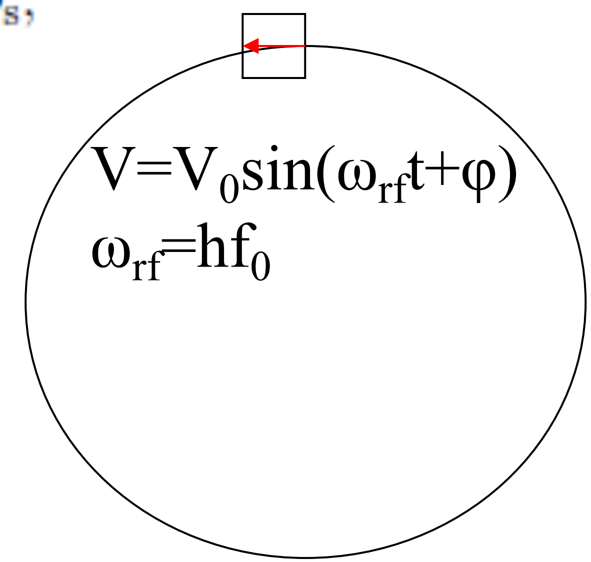
$$\mathcal{E} = \mathcal{E}_0 \sin(\phi_{\rm rf}(t) + \phi_{\rm s}), \quad \phi_{\rm rf} = h\omega_0 t$$

where $\omega_0 = \beta_0 c/R_0$ is the angular revolution frequency of a reference (synchronous) particle, ε_0 is the amplitude of the electric field, $\beta_0 c$ and R_0 are respectively the speed and the average radius of the reference orbiting particle, *h* is the harmonic number, and ϕ_s is the phase angle for a synchronous particle.

We assume that the reference particle passes through the cavity gap in time $t \in nT_0 + (-g/2\beta c,\ g/2\beta c)$ (n = integer), where g is the rf cavity gap width. The energy gain for the reference particle per passage is

$$\Delta E = e\mathcal{E}_0 \beta c \int_{-g/2\beta_0 c}^{g/2\beta_0 c} \sin(h\omega_0 t + \phi_{\rm s}) dt = e\mathcal{E}_0 g T \sin\phi_{\rm s},$$

$$T = \frac{\sin(hg/2R_0)}{(hg/2R_0)}$$



The effective voltage seen by the orbiting particle is $V = \varepsilon_0 gT$. The acceleration rate for a synchronous particle is

$$\dot{E}_0 = \frac{\omega_0}{2\pi} eV \sin\phi_{\rm s},$$

We consider a non-synchronous particle with small deviations of rf parameters from the synchronous particle, i.e.

$$\begin{cases} \omega = \omega_0 + \Delta\omega, \quad \phi = \phi_s + \Delta\phi, \quad \theta = \theta_s + \Delta\theta, \\ p = p_0 + \Delta p, \quad E = E_0 + \Delta E. \end{cases}$$

Here ϕ_s, θ_s, ω_0, p_0, E_0 are respectively the rf phase angle, azimuthal orbital angle, angular revolution frequency, momentum, and energy of a synchronous particle, and ϕ, θ, ω, p, E are the corresponding parameters for an off-momentum particle. The phase coordinate is related to the orbital angle by $\Delta\phi=\phi-\phi_s = -h\Delta\theta$, or

$$\Delta\omega = \frac{d}{dt}\Delta\theta = -\frac{1}{h}\frac{d}{dt}\Delta\phi = -\frac{1}{h}\frac{d\phi}{dt}.$$

The energy gain per revolution for this non-synchronous particle is eV sin φ, where φ is the rf phase angle. Thus the acceleration rate of a non-synchronous particle is

$$\dot{E} = \frac{\omega}{2\pi}eV\sin\phi$$

The equation of motion for the energy difference is $\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi}eV(\sin\phi - \sin\phi_s)$

$$\frac{1}{\omega}\dot{E} - \frac{1}{\omega_0}\dot{E}_0 = \frac{1}{\omega_0}\dot{\Delta E} - \dot{E}\frac{\Delta\omega}{\omega_0^2} \approx \frac{1}{\omega_0}\dot{\Delta E} + \left[\dot{E}\frac{\Delta(1/\omega_0)}{\Delta E}\right]\Delta E + \cdots = \frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right)$$

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi} eV(\sin\phi - \sin\phi_s) \qquad \delta = \frac{\Delta p}{p_0} = \frac{\omega_0}{\beta^2 E}\frac{\Delta E}{\omega_0}$$

$$\dot{\delta} = \frac{\omega_0}{2\pi\beta^2 E} eV(\sin\phi - \sin\phi_s)$$

The time evolution of the phase angle variable ϕ is

$$\dot{\phi} = -h(\omega - \omega_0) = -h\Delta\omega \qquad \Delta\omega = \frac{d}{dt}\Delta\theta = -\frac{1}{h}\frac{d}{dt}\Delta\phi = -\frac{1}{h}\frac{d\phi}{dt}.$$

$$\omega R/\omega_0 R_0 = \beta/\beta_0 \qquad \frac{\Delta\omega}{\omega_0} = \frac{\beta R_0}{\beta_0 R} - 1.$$

$$R = R_0(1 + \alpha_0\delta + \alpha_1\delta^2 + \alpha_2\delta^3 + \cdots),$$

$$\alpha_c = \frac{1}{R_0}\frac{dR}{d\delta} = \alpha_0 + 2\alpha_1\delta + 3\alpha_2\delta^2 + \cdots \equiv \frac{1}{\gamma_T^2}$$

Let $p = mc\beta\gamma = p_0 + p$ be the momentum of a non-synchronous particle. The fractional off-momentum coordinate δ is

$$\delta = \frac{\Delta p}{p_0} = \frac{\beta\gamma}{\beta_0\gamma_0} - 1.$$

Expressing β and γ in terms of the off-momentum coordinate δ, we obtain

$$\frac{\gamma}{\gamma_0} = \sqrt{1 + 2\beta_0^2\delta + \beta_0^2\delta^2}$$

$$\frac{\beta}{\beta_0} = \frac{1+\delta}{\sqrt{1 + 2\beta_0^2\delta + \beta_0^2\delta^2}} = 1 + \frac{1}{\gamma_0^2}\delta - \frac{3\beta_0^2}{2\gamma_0^2}\delta^2 + \frac{\beta_0^2(5\beta_0^2 - 1)}{2\gamma_0^2}\delta^3 + \cdots.$$

$$\frac{\Delta\omega}{\omega_0} = -\eta(\delta)\delta = -(\eta_0 + \eta_1\delta + \eta_2\delta^2 + \cdots)\delta,$$

$$\eta_0 = (\alpha_0 - \frac{1}{\gamma_0^2}), \quad \eta_1 = \frac{3\beta_0^2}{2\gamma_0^2} + \alpha_1 - \alpha_0\eta_0,$$

$$\eta_2 = -\frac{\beta_0^2(5\beta_0^2 - 1)}{2\gamma_0^2} + \alpha_2 - 2\alpha_0\alpha_1 + \frac{\alpha_1}{\gamma_0^2} + \alpha_0^2\eta_0 - \frac{3\beta_0^2\alpha_0}{2\gamma_0^2}.$$

$$\dot{\phi} = h\omega_0\eta\delta = \frac{h\omega_0^2\eta}{\beta^2 E}\left(\frac{\Delta E}{\omega_0}\right)$$

where $(\phi, E/\omega_0)$ or equivalently (ϕ, δ) are **conjugate** phase-space coordinates.

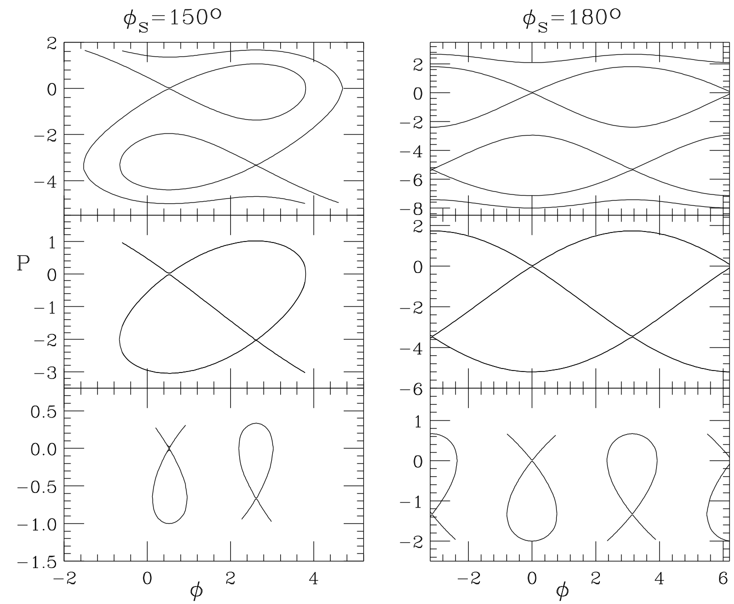
$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta, \qquad \eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}.$$

$$R = R_0(1 + \alpha_0\delta + \alpha_1\delta^2 + \alpha_2\delta^3 + \ldots)$$

$$\alpha_c = \frac{dR}{R_0 d\delta} = \alpha_0 + 2\alpha_1\delta + 3\alpha_2\delta^2 + \ldots \equiv \frac{1}{\gamma_T^2}$$

$$\frac{\Delta\omega}{\omega_0} = -\eta(\delta)\delta = -(\eta_0 + \eta_1\delta + \eta_2\delta^2 + \ldots)\delta$$

$$\eta_0 = (\alpha_0 - \frac{1}{\gamma_0^2}), \quad \eta_1 = \frac{3\beta_0^2}{2\gamma_0^2} + \alpha_1 - \alpha_0\eta_0, \ \ldots\ldots$$



For the moment, we neglect all nonlinear effects in the phase slip factor. The equation of motion is given by

$$\frac{d}{dt}(\phi - \phi_s) = -h\Delta\omega = h\omega_0\frac{\Delta T}{T_0} = h\eta\omega_0\frac{\Delta p}{p_0} = \frac{\eta h\omega_0^2}{\beta^2 E_0}\frac{\Delta E}{\omega_0}.$$

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi}eV_0(\sin\phi - \sin\phi_s),$$

$$\dot{\delta} = \frac{\omega_0}{2\pi\beta^2 E}eV(\sin\phi - \sin\phi_s)$$

The Synchrotron Hamiltonian

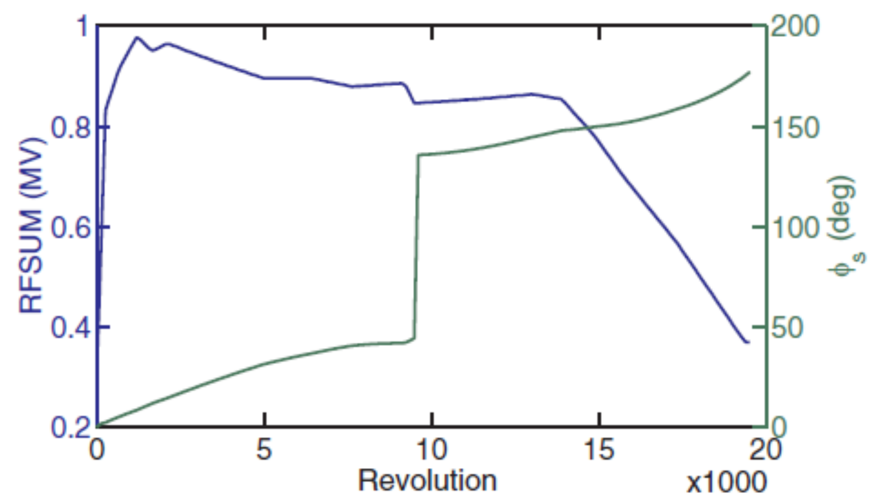
$$H = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2 E}\left(\frac{\Delta E}{\omega_0}\right)^2 + \frac{eV}{2\pi}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

$$= \frac{1}{2}h\omega_0\eta_0\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

$$\dot{\phi} = \frac{\partial H}{\partial \delta} = h\eta\omega_0\delta \qquad \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{\omega_0 eV_0}{2\pi\beta^2 E}(\sin\phi - \sin\phi_s) \approx \frac{\omega_0 eV_0\cos\phi_s}{2\pi\beta^2 E}(\phi - \phi_s)$$

$$\frac{d^2}{dt^2}(\phi - \phi_s) = \frac{h\omega_0^2 eV\eta_0\cos\phi_s}{2\pi\beta^2 E}(\phi - \phi_s)$$

The stability condition for synchrotron oscillation is $\eta_0 \cos\varphi_s < 0$, discovered by McMillan and Veksler. Below the transition energy, with $\gamma < \gamma_T$ or $\eta_0 < 0$, the synchronous phase angle should be $0 < \varphi_s < \pi/2$. Similarly the synchronous phase angle should be shifted to $\pi - \varphi_s$ above the transition energy.



$$\frac{d^2}{dt^2}(\phi - \phi_s) = \frac{h\omega_0^2 eV \eta_0 \cos\phi_s}{2\pi\beta^2 E}(\phi - \phi_s)$$

The angular synchrotron frequency is $\omega_s = \omega_0 \sqrt{\frac{heV|\eta_0 \cos\phi_s|}{2\pi\beta^2 E}} = \frac{c}{R}\sqrt{\frac{heV|\eta \cos\phi_s|}{2\pi E}}$

where c is the speed of light and R is the average radius of the synchrotron. The synchrotron tune, defined as the number of synchrotron oscillations per revolution, is

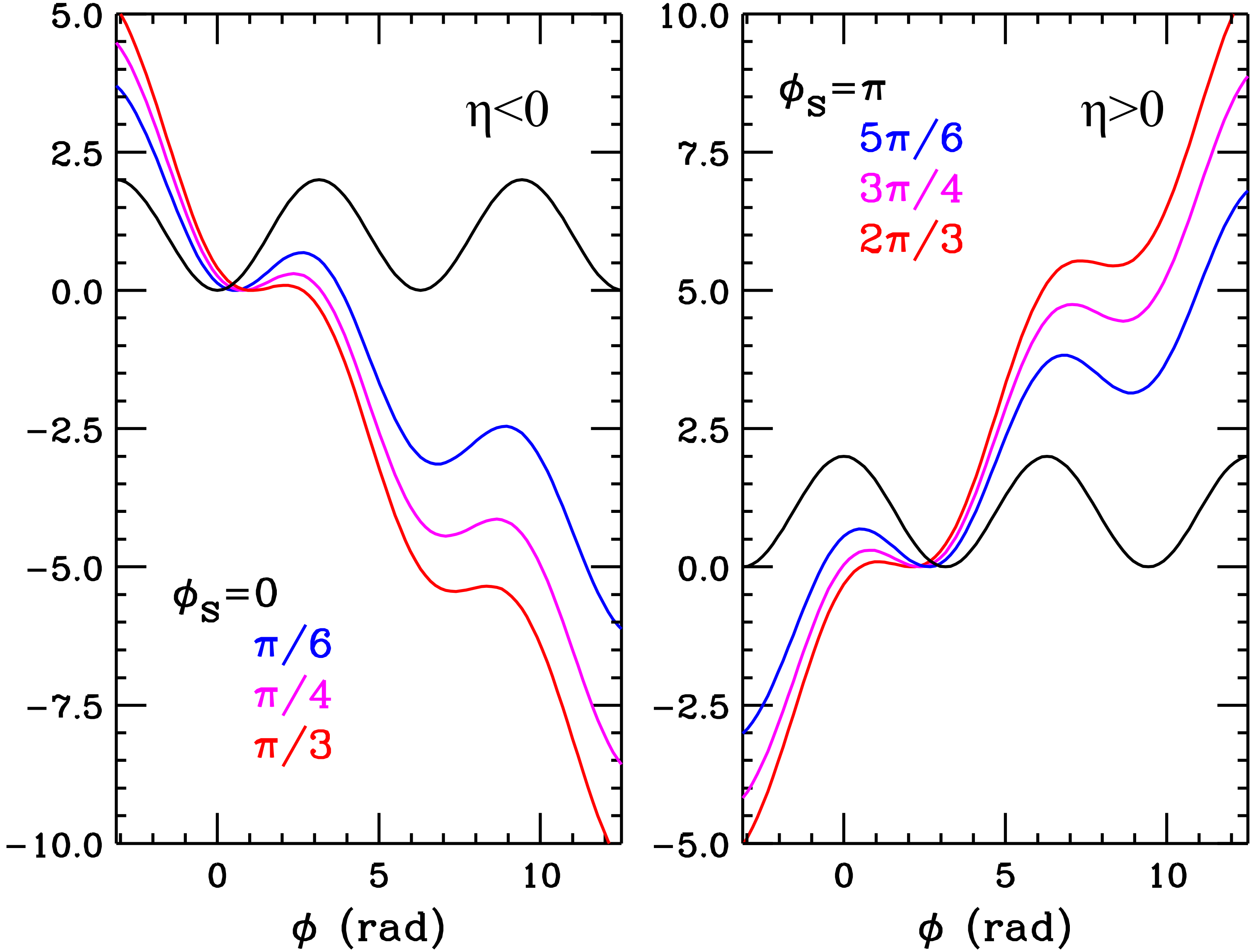
$$Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{heV|\eta_0 \cos\phi_s|}{2\pi\beta^2 E}}$$

Typically the synchrotron tune is of the order of $\leq 10^{-3}$ for proton synchrotrons and 10^{-1} for electron storage rings.

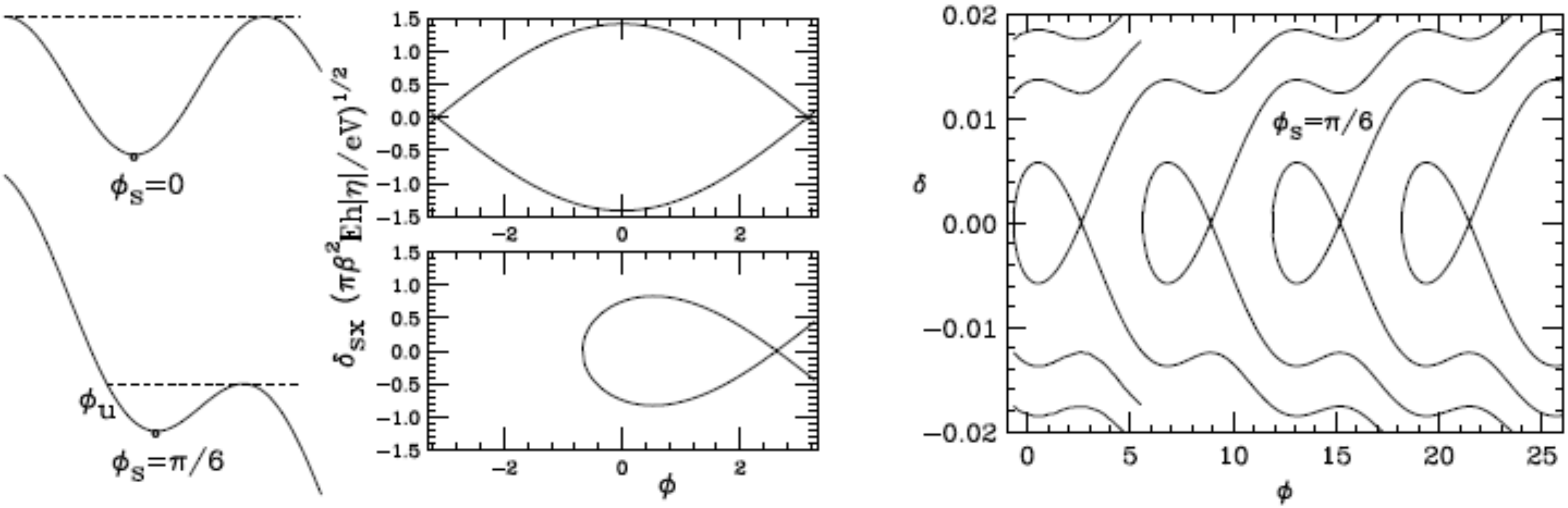
P-synchrotron	AGS	RHIC	FNAL-MI	FNAL-BST	SSC	Cooler
K.E. [GeV/u]	0.2	28	8	0.2	2000	0.045
V_{rf} [MV]	0.3	0.3	2	0.95	10	0.0001
h	12	342	588	84	17424	1
γ_T	8.5	24.5	21.8	5.446	140	4.6
C [m]	807.12	3833.84	3319.4	474.2	87120	86.8
ν_s	3.23E-02	5.89E-04	1.37E-02	1.50E-01	8.39E-04	3.95E-04

At 0.4 GeV injection, the synchrotron tune becomes 0.0924 for FNAL-Booster

$$H = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2 E}\left(\frac{\Delta E}{\omega_0}\right)^2 + \frac{eV}{2\pi}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

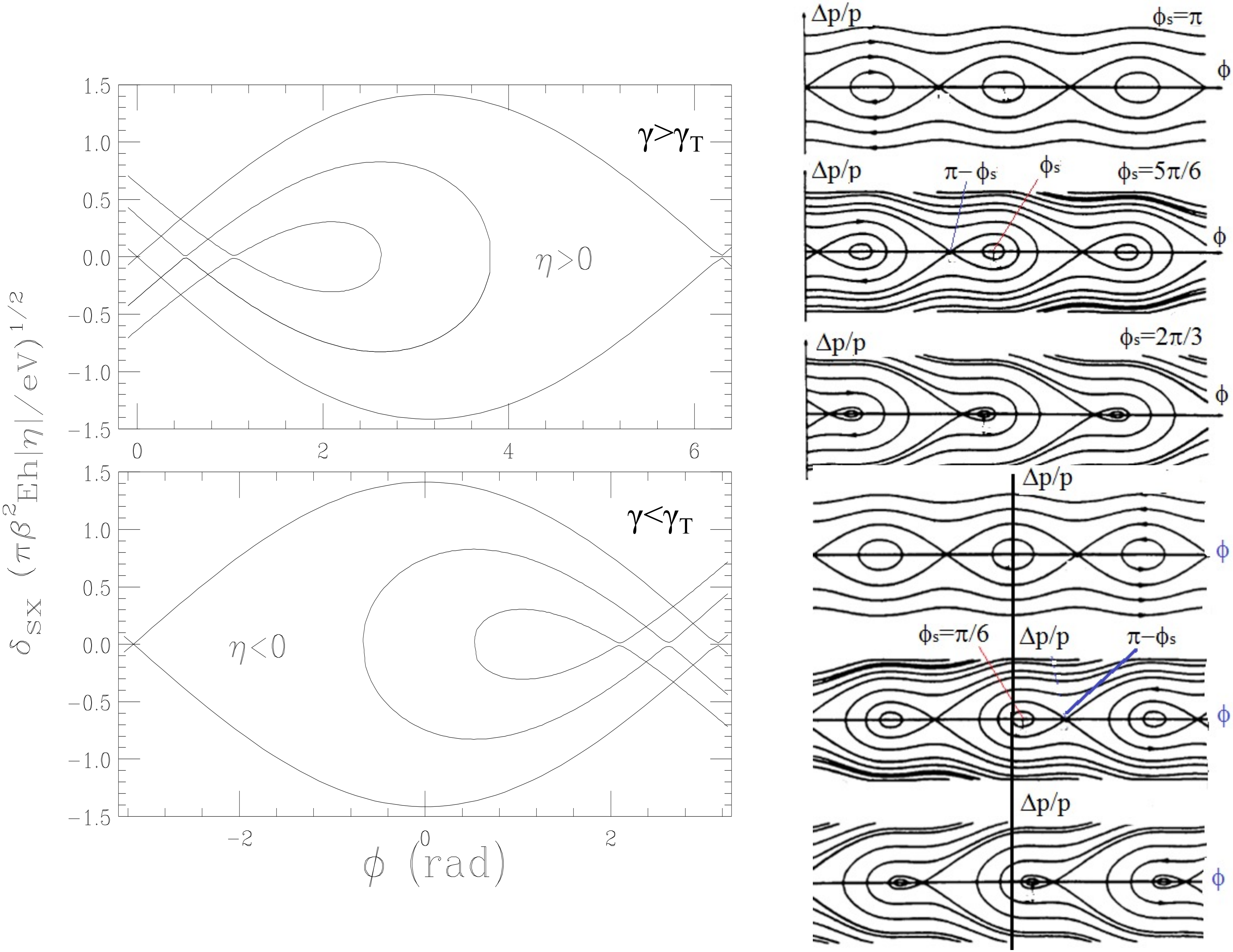


Synchrotron bucket area, separatrix



Left: schematic drawing of the rf potentials for ϕ_s=0 and π/6. The dashed line shows the maximum "energy" for stable synchrotron motion. Middle: the corresponding separatrix orbits in $(\pi h|\eta|/eV\beta^2E)^{1/2}E_{sx}$ vs ϕ. The phase ϕ_u is the turning point of the separatrix orbit. Right: an example of stable rf buckets, called fish diagram, with $\phi_s=\pi/6$.


Synchrotron
motion
Ice forming at
Kamchatka



γ>γT
η>0
γ<γT
η<0
δsx (πβ²Eh|η|/eV)^1/2
φ (rad)
Δp/p
φs=π
φ
π−φs
φs
φs=5π/6
φs=2π/3
φs=π/6

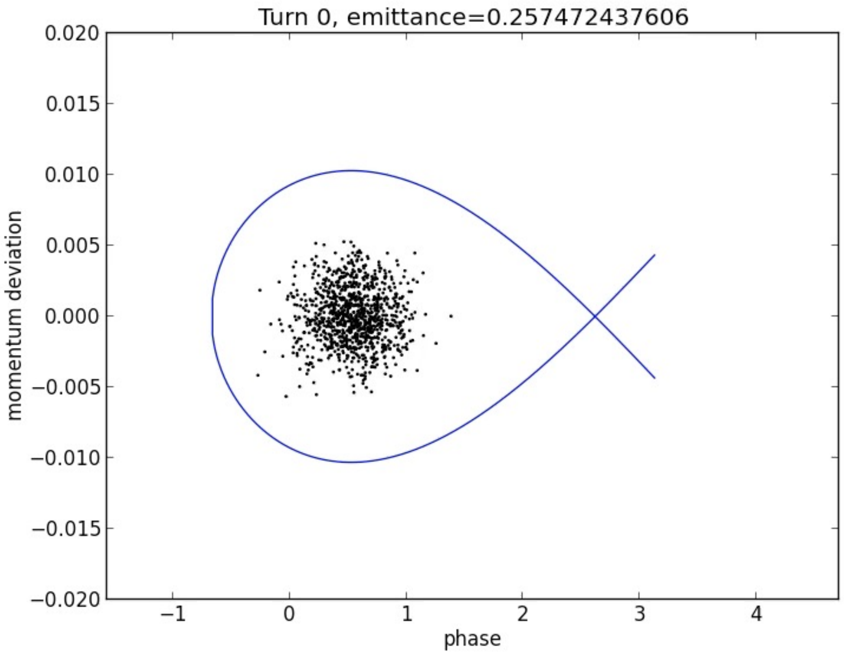
In Hamiltonian formalism, the rf electric field is considered to be uniformly distributed in an accelerator. In reality, rf cavities are localized in a short section of a synchrotron, and therefore synchrotron motion is more realistically described by the symplectic mapping equation

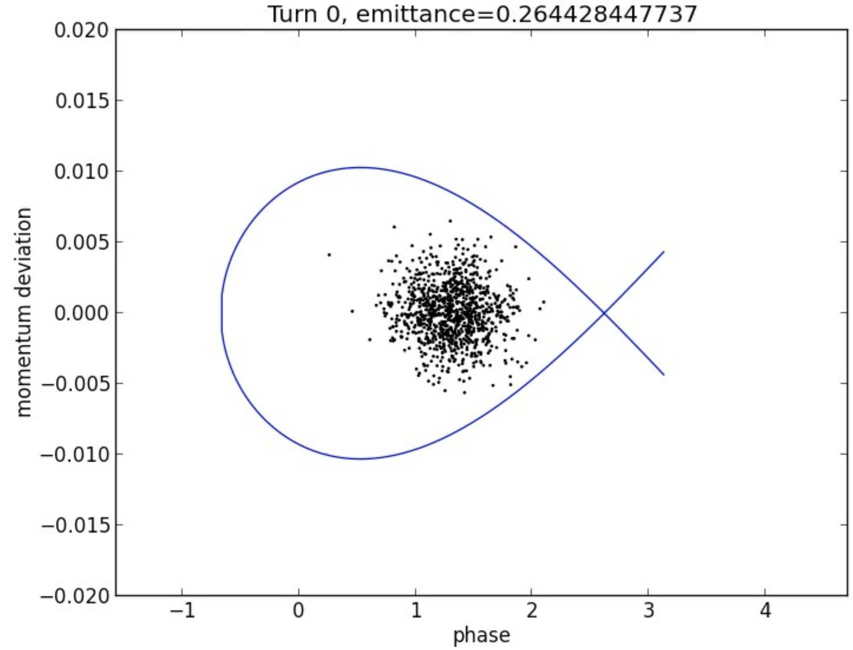
$$\begin{cases} \delta_{n+1} = \delta_n + \dfrac{eV}{\beta^2 E}(\sin\phi_n - \sin\phi_s), \\ \phi_{n+1} = \phi_n + 2\pi h\eta(\delta_{n+1})\delta_{n+1}. \end{cases}$$

The physics of the mapping equation can be visualized as follows. First, the particle gains or loses energy at its nth passage through the rf cavity, then the rf phase ϕ_{n+1} depends on the new off-momentum coordinate δ_{n+1}. It is no surprise that satisfies the symplectic condition:

$$\text{Jacobian} = \frac{\partial(\delta_{n+1}, \phi_{n+1})}{\partial(\delta_n, \phi_n)} = 1.$$

The mapping from (ϕ_n, δ_n) to $(\phi_{n+1}, \delta_{n+1})$ preserves phase-space area. The phase-space area enclosed by a trajectory (φ, δ) obtained from the above mapping equation is independent of energy. It can not be used in tracking simulations of beam acceleration. During beam acceleration, the phase-space area in $(\phi, E/\omega_0)$ is invariant. The phase-space mapping equation for phase-space coordinates $(\phi, E/\omega_0)$ should be used. The adiabatic damping of phase-space area can be obtained by transforming phase-space coordinates $(\phi, E/\omega_0)$ to (ϕ, δ).


Turn 0, emittance=0.257472437606
momentum deviation
phase
0.020
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.020
−1
0
1
2
3
4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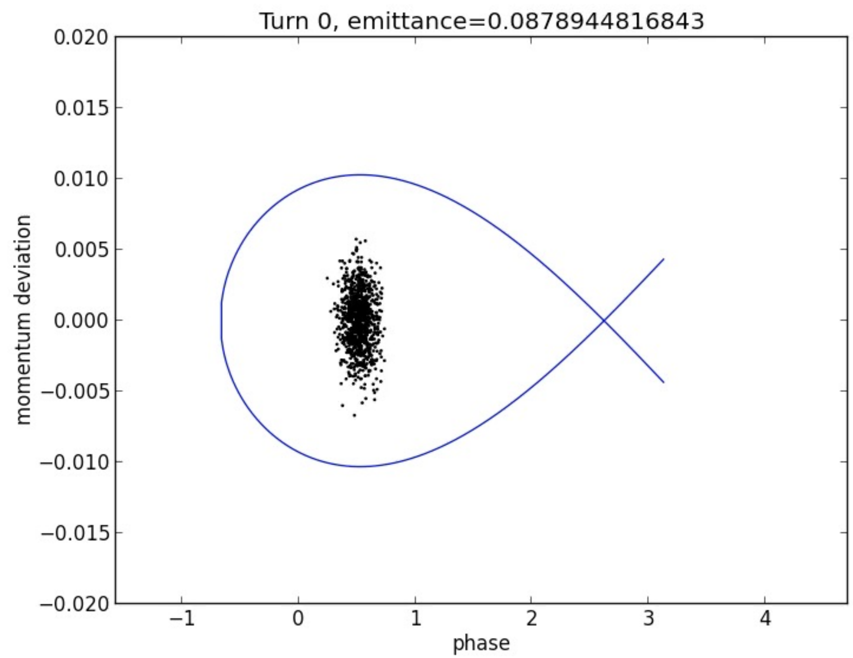
momentum deviation
phase
0.020
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.020
−1
0
1
2
3
4

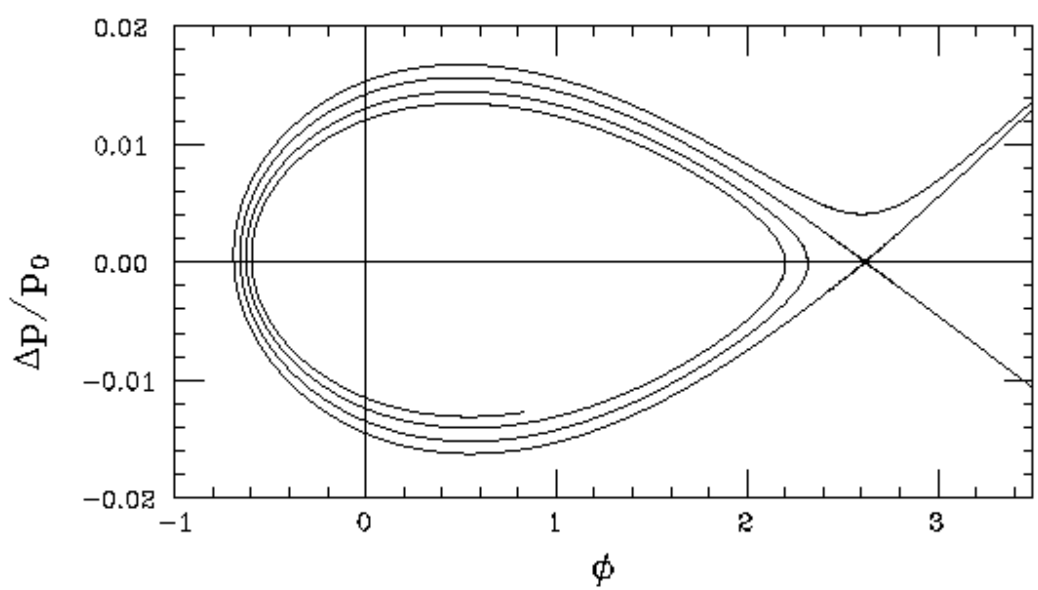

Turn 0, emittance=0.0878944816843
momentum deviation
phase
0.020
0.015
0.010
0.005
0.000
−0.005
−0.010
−0.015
−0.020
−1
0
1
2
3
4

When the acceleration rate is high, tori of the synchrotron mapping equations are not closed curves. The mapping equations for synchrotron phase-space coordinates (φ,ΔE). Figure below shows two tori in phase-space coordinates (φ, E/β^2E) with parameters V=100 kV, h=1, α_c=0.04340, φs = 30º at 45 MeV proton kinetic energy. Note that the actual attainable rf voltage V is about 200-1000 V in a low energy proton synchrotron. Since the separatrix is not a closed curve, the phase-space tori change from a fish-like to a golf-club-like shape. This is equivalent to the *adiabatic damping* of phase-space area. Since the acceleration rate for proton (ion) beams is normally low, the separatrix torus is a good approximation. When the acceleration rate is high, e.g. in many electron accelerators, the tori near the separatrix may resemble that of picture below.

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s),$$

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_{n+1},$$

Two tori in phase-space coordinates (φ, E/β^2E) obtained from mapping equations with parameters V = 100 kV, h = 1, α_c=0.04340, and φs = 30º at 45MeV proton kinetic energy. IUCF Cooler Ring has typical rf voltage at about 1–2 kV.

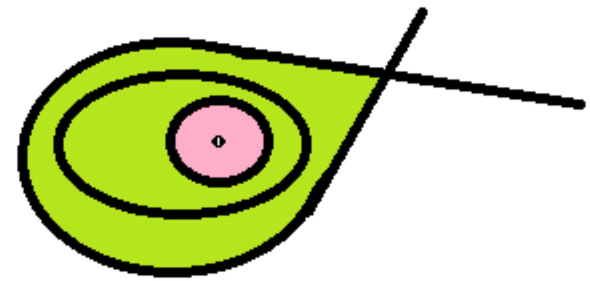


During beam acceleration, the synchrotron Hamiltonian depends on time. However, if the acceleration rate is low, the Hamiltonian can be considered as quasistatic. This corresponds to adiabatic synchrotron motion, where parameters in the synchrotron Hamiltonian change slowly so that the particle orbit is a torus of constant Hamiltonian value. The condition for **adiabatic synchrotron motion** is

$$\alpha_{\mathrm{ad}} = \left| \frac{1}{\omega_{\mathrm{s}}^2} \frac{d\omega_{\mathrm{s}}}{dt} \right| = \frac{1}{2\pi} \left| \frac{dT_{\mathrm{s}}}{dt} \right| \ll 1$$

where ω_s is the angular synchrotron frequency and α_{ad} is called the adiabaticity coefficient. Typically, when $\alpha_{ad} \leq 0.05$, the time variation of synchrotron period is small and the trajectories of particle motion can be approximately described by tori of constant Hamiltonian values.

Fixed points of a Hamiltonian are located at phase space points with zero velocity field:



$$H(q,p,t): \quad \dot{q}_i = \frac{\partial H}{\partial p_i} = 0, \;\; \dot{p}_i = -\frac{\partial H}{\partial q_i} = 0.$$

$$H = \frac{1}{2} h\omega_0\eta_0\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E}[\cos\phi - \cos\phi_{\rm s} + (\phi - \phi_{\rm s})\sin\phi_{\rm s}]$$

$$\dot{\phi} = \frac{\partial H}{\partial \delta} = h\eta\omega_0\delta \qquad \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{\omega_0 eV_0}{2\pi\beta^2 E}(\sin\phi - \sin\phi_{\rm s})$$

The fixed points of the synchrotron Hamiltonian are at phase space points: $(\phi_s,0)$ and $(\pi-\phi_s,0)$. Small amplitude motion around the stable fixed point $(\phi_s,0)$ is elliptical with synchrotron tune Q_s. Motion near the UFP is hyperbolical.

The Hamiltonian torus that passes through the unstable fixed point is called the separatrix.

$$H(\phi,\delta) = H_{\mathrm{sx}} = H(\pi-\phi_{\mathrm{s}},0) = \frac{\omega_0 eV_0}{2\pi\beta^2 E}[-2\cos\phi_{\mathrm{s}} + (\pi-2\phi_{\mathrm{s}})\sin\phi_{\mathrm{s}}]$$

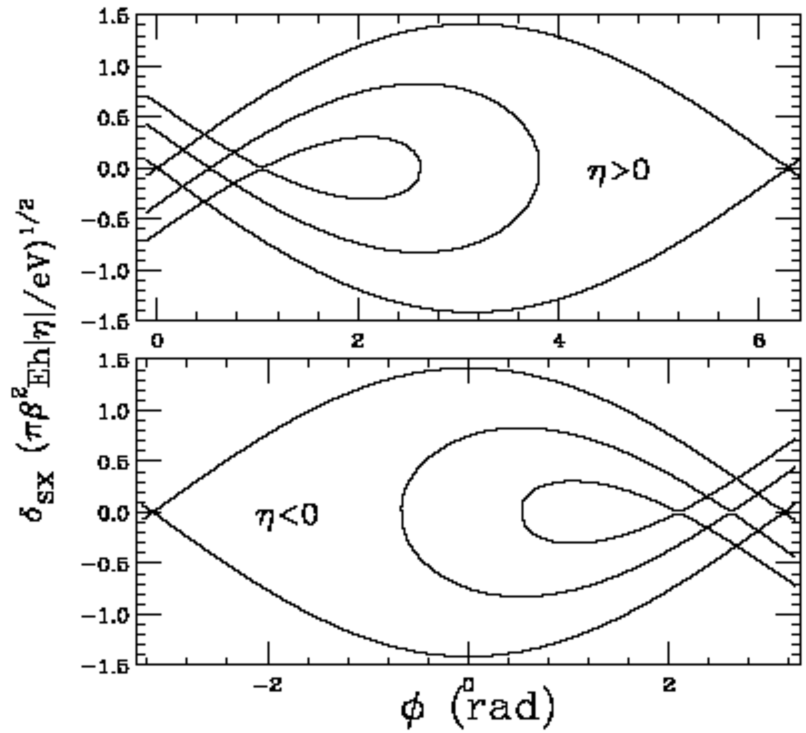
$$\delta_{\mathrm{sx}}^2 + \frac{eV_0}{\pi\beta^2 Eh\eta}[\cos\phi + \cos\phi_{\mathrm{s}} - (\pi-\phi-\phi_{\mathrm{s}})\sin\phi_{\mathrm{s}}] = 0$$

The separatrix has two turning points, φ_u and $\pi - \varphi_s$, where φ_u is

$$\cos\phi_{\mathrm{u}} + \phi_{\mathrm{u}}\sin\phi_{\mathrm{s}} = -\cos\phi_{\mathrm{s}} + (\pi-\phi_{\mathrm{s}})\sin\phi_{\mathrm{s}}.$$

The phase space area enclosed by the separatrix is called the bucket, where particle motion around the stable fixed point is elliptical. The motion around the unstable fixed point is hyperbolical.

The bucket length is $|(\pi - \varphi_s) - \varphi_u|$.



The phase-space area enclosed by the separatrix orbit is called the bucket area.

$$\tilde{A}_{\mathrm{B}} = \oint \delta_{\mathrm{sx}}(\phi)d\phi = 16\sqrt{\frac{eV_0}{2\pi\beta^2 Eh|\eta|}}\alpha_{\mathrm{b}}(\phi_{\mathrm{s}}) = \frac{16\,\nu_{\mathrm{s}}}{h|\eta|}\alpha_{\mathrm{b}}(\phi_{\mathrm{s}})$$

$$\alpha_{\mathrm{b}}(\phi_{\mathrm{s}}) = \frac{1}{4\sqrt{2}}\int_{\phi_{\mathrm{u}}}^{\pi-\phi_{\mathrm{s}}}\left[-\frac{|\eta|}{\eta}\left[\cos\phi + \cos\phi_{\mathrm{s}} - (\pi - \phi - \phi_{\mathrm{s}})\sin\phi_{\mathrm{s}}\right]\right]^{1/2} d\phi.$$

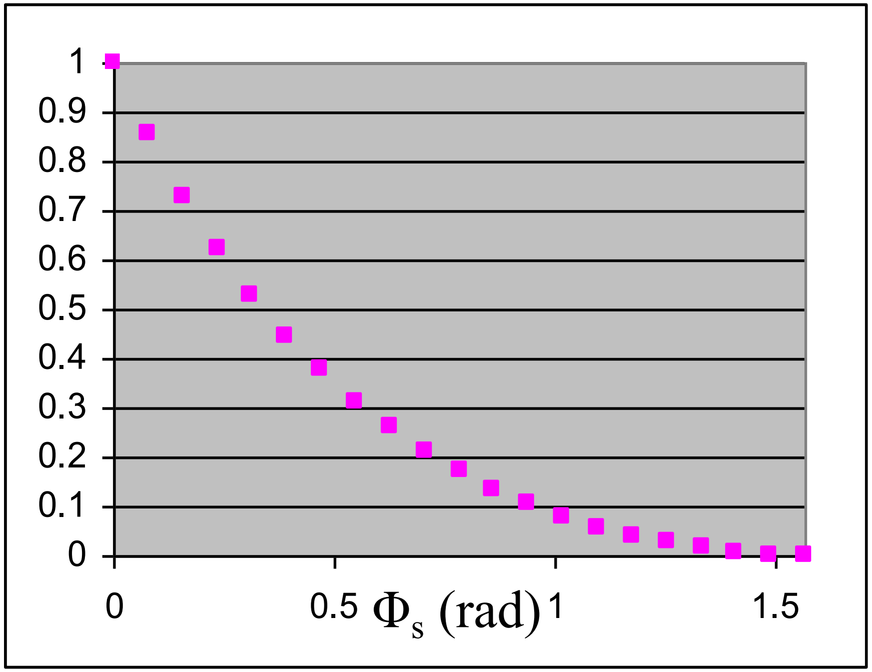
Here α_{b} is the moving bucket factor.

The maximum momentum deviation of the separatrix orbit is called the bucket height.

$$\delta_{\mathrm{B}} = \left(\frac{2eV}{\pi\beta^2 Eh|\eta|}\right)^{1/2} Y(\phi_{\mathrm{s}}) = \frac{2Q_{\mathrm{s}}}{h|\eta|}\,\tilde{Y}(\phi_{\mathrm{s}}).$$

$$Y(\phi_{\mathrm{s}}) = \left|\cos\phi_{\mathrm{s}} - \frac{\pi - 2\phi_{\mathrm{s}}}{2}\sin\phi_{\mathrm{s}}\right|^{1/2}$$

$$\tilde{Y}(\phi_{\mathrm{s}}) = \left|1 - \frac{\pi - 2\phi_{\mathrm{s}}}{2}\tan\phi_{\mathrm{s}}\right|^{1/2}$$



$\sin\phi_s$	ϕ_u	$\pi - \phi_s$	$Y(\phi_s)$	$\alpha_b(\phi_s)$	$\frac{1-\sin\phi_s}{1+\sin\phi_s}$
0.00	−180.00	180.00	1.0000	1.0000	1.0000
0.05	−136.47	177.13	0.9606	0.8888	0.9048
0.10	−118.90	174.26	0.9208	0.8041	0.8182
0.15	−105.32	171.37	0.8807	0.7294	0.7391
0.20	−93.71	168.46	0.8402	0.6611	0.6667
0.25	−83.26	165.52	0.7992	0.5980	0.6000
0.30	−73.59	162.54	0.7577	0.5388	0.5385
0.35	−64.45	159.51	0.7156	0.4832	0.4815
0.40	−55.66	156.42	0.6729	0.4305	0.4286
0.45	−47.11	153.26	0.6295	0.3806	0.3793
0.50	−38.69	150.00	0.5852	0.3333	0.3333
0.55	−30.31	146.63	0.5399	0.2885	0.2903
0.60	−21.88	143.13	0.4936	0.2460	0.2500
0.65	−13.31	139.46	0.4459	0.2058	0.2121
0.70	−4.48	135.57	0.3967	0.1679	0.1765
0.75	4.75	131.41	0.3455	0.1323	0.1429
0.80	14.59	126.87	0.2919	0.0991	0.1111
0.85	25.38	121.79	0.2349	0.0685	0.0811
0.90	37.77	115.84	0.1731	0.0408	0.0526
0.95	53.42	108.19	0.1028	0.0170	0.0256
1.00	90.00	90.00	0.	0.	0.

The bucket area in phase space $(\phi, \Delta E/\omega_0)$ is $A_B = \frac{\beta^2 E}{\omega_0} \tilde{A}_B = h \Delta t \Delta E$

The phase space area measures the time-width, and energy-spread of the bunch distribution. Thus the dimension of the phase space area is eV-sec. For example, a beam bunch with 100 ns bunch length and 1 MeV energy spread have a bunch area of 0.1 eV-sec. A beam with 1 MeV energy spread with 1 GeV energy has a fractional energy spread of 10^{-3}.

	$(\phi, \frac{\Delta E}{\omega_0})$	(ϕ, δ)	$(\phi, \frac{h\lvert\eta\rvert}{\nu_s}\delta)$
Bucket Area	$16\left(\frac{\beta^2 EeV}{2\pi\omega_0^2 h\lvert\eta\rvert}\right)^{1/2} \alpha_b(\phi_s)$	$16\left(\frac{eV}{2\pi\beta^2 Eh\lvert\eta\rvert}\right)^{1/2} \alpha_b(\phi_s)$	$16\,\alpha_b(\phi_s)$
Bucket Height	$2\left(\frac{\beta^2 EeV}{2\pi\omega_0^2 h\lvert\eta\rvert}\right)^{1/2} Y(\phi_s)$	$2\left(\frac{eV}{2\pi\beta^2 Eh\lvert\eta\rvert}\right)^{1/2} Y(\phi_s)$	$2\,Y(\phi_s)$
Bucket area	$\frac{\beta^2 E}{\omega_0}\frac{16\nu_s}{h\lvert\eta\rvert}\alpha_b(\phi_s)$	$\frac{16\nu_s}{h\lvert\eta\rvert}\alpha_b(\phi_s)$	
Bucket height	$\frac{\beta^2 E}{\omega_0}\frac{2\nu_s}{h\lvert\eta\rvert}Y(\phi_s)$	$\frac{2\nu_s}{h\lvert\eta\rvert}Y(\phi_s)$	