A Possible Proof of $\det M = 1$

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We will prove that the determinant of a symplectic matrix has to be 1[1]. Since

$$M^T S M = S, \tag{b1}$$

taking determinant of eq. (b1) yields

$$\det M = \pm 1. \tag{b2}$$

Assuming λ is an eigenvalue of M, one has

$$\det(M - \lambda I) = 0. \tag{b3}$$

Considering the following relation

$$S^{-1}\left(M^T - \lambda I\right)S = M^{-1} - \lambda I = -\lambda M^{-1} M + \lambda \lambda^{-1} M^{-1} = -\lambda M^{-1}\left(M - \lambda^{-1} I\right), \tag{b4}$$

and taking the determinant of eq. (b4) leads to

$$\det\left(M^T - \lambda I\right) = -\det(\lambda)\det\left(M^{-1}\right)\det\left(M - \lambda^{-1} I\right). \tag{b5}$$

Since for a general matrix A with non-zero determinant, we have the following relations:

$$\det A^T = \det A \tag{b6}$$

and

$$\det A^{-1} = \frac{1}{\det A}. \tag{b7}$$

Applying eq. (b2), (b6) and (b7) to eq. (b5) generates

$$\det(M - \lambda I) = \pm \lambda^{2n} \det\left(M - \lambda^{-1} I\right). \tag{b8}$$

Since eq. (b2) requires that

$$\lambda \neq 0, \tag{b9}$$

satisfying eq. (b3) and eq. (b8) simultaneously necessarily requires

$$\det\left(M - \lambda^{-1} I\right) = 0, \tag{b10}$$

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[1] This part of proof is based on 'Accelerator Physics' by S. Y. Lee, edition 2, page 67

i.e. if λ is an eigenvalue of M , λ^{-1} is also an eigenvalue of M . In addition, if we define the form of polynomial as

$$P(\lambda) = \det(M - \lambda I), \tag{b11}$$

the corresponding polynomial for λ^{-1} has identical form, i.e. $P(\lambda^{-1})$. As a result, if one can factorize $P(\lambda)$ as

$$P(\lambda) = \prod_i (\lambda - \lambda_i)^{n_i} , \tag{b12}$$

it necessarily follows

$$P(\lambda^{-1}) = \prod_i (\lambda^{-1} - \lambda_i)^{n_i} . \tag{b13}$$

Eq. (b12) and (b13) suggest that the multiplicity for a root λ_i is the same as that of λ_i^{-1}. Consequently, the eigenvalues of a symplectic matrix of $2n$ dimention are composed of n pairs of reciprocal values and hence

$$\det M = \prod_i \lambda_i^{n_i} \lambda_i^{-n_i} = 1 . \tag{b14}$$