PHY 564
Advanced Accelerator Physics
Lecture 13
Parameterization and Action-angle variables

Vladimir N. Litvinenko
Yichao Jing
Gang Wang

CENTER for ACCELERATOR SCIENCE AND EDUCATION
Department of Physics & Astronomy, Stony Brook University
Collider-Accelerator Department, Brookhaven National Laboratory

We had considered parameterization of stable particles motion in periodic Hamiltonian system using eigen vectors of round trip matrix. A quick walk through our findings:

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s)\cdot X,\ \mathbf{H}(s+C) = \mathbf{H}(s); \tag{1}$$

$$\mathbf{T}(s) = \mathbf{M}\left(s\middle|s+C\right) \tag{2}$$

$$\det\left[\mathbf{T} - \lambda_i \cdot \mathbf{I}\right] = 0 \tag{3}$$

$$\mathbf{T}\cdot Y_k = \lambda_k \cdot Y_k; \qquad \lambda_k = e^{i\mu_k}; \qquad k = 1,2....n \tag{4}$$

$$X = \sum_{i=1}^{2n} a_i Y_i \equiv \mathbf{U}\cdot A, \quad \mathbf{U} = [Y_1......Y_{2n}], \qquad A^T = [a_1......a_{2n}]. \tag{5}$$

$$\mathbf{T}\cdot\mathbf{U} = \mathbf{U}\cdot\Lambda,\ \Lambda = \begin{bmatrix} \lambda_1 & ... & 0 \\ ... & ... & .... \\ 0 & .... & \lambda_{2n} \end{bmatrix} \tag{6}$$

$$\mathbf{U}^{-1}\cdot\mathbf{T}\cdot\mathbf{U} = \Lambda,\ or\ \ \mathbf{T} = \mathbf{U}\cdot\Lambda\cdot\mathbf{U}^{-1} \tag{7}$$

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_{j\neq k} = 0; \quad Y_k^T\cdot\mathbf{S}\cdot Y_j = 0;\ . \tag{8}$$

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_k = 2i, \tag{9}$$

$$\mathbf{U}^T\cdot\mathbf{S}\cdot\mathbf{U} \equiv \tilde{\mathbf{U}}^T\cdot\mathbf{S}\cdot\tilde{\mathbf{U}} = -2i\mathbf{S},\ \mathbf{U}^{-1} = \frac{1}{2i}\mathbf{S}\cdot\mathbf{U}^T\cdot\mathbf{S}. \tag{10}$$

$$\tilde{Y}_k(s_1) = \mathbf{M}\left(s\middle|s_1\right)\tilde{Y}_k(s) \Leftrightarrow \frac{d}{ds}\tilde{Y}_k = \mathbf{D}\left(s\right)\cdot\tilde{Y}_k \tag{11}$$

$$\tilde{Y}_k(s) = Y_k(s)e^{\psi_k(s)};\quad Y_k(s+C) = Y_k(s);\qquad \psi_k\left(s+C\right) = \psi_k\left(s\right) + \mu_k \tag{12}$$

$$\tilde{\mathbf{U}}(s_1) = \mathbf{M}\left(s\middle|s_1\right)\tilde{\mathbf{U}}(s) \Leftrightarrow \frac{d}{ds}\tilde{\mathbf{U}} = \mathbf{D}\left(s\right)\cdot\tilde{\mathbf{U}} \tag{13}$$

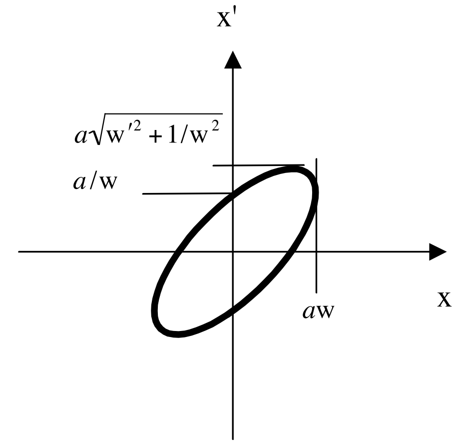
$$\tilde{\mathbf{U}}(s) = \mathbf{U}(s)\cdot\Psi(s),\ \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \ldots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix} \tag{14}$$

$$X_o = \sum_{i=1}^{2n} a_i Y_i \Rightarrow X(s) = \frac{1}{2}\sum_{k=1}^{n}\left(a_k\tilde{Y}_k + a_k^*\tilde{Y}_k^*\right) \equiv \mathrm{Re}\sum_{k=1}^{n} a_k Y_k e^{i\psi_k} \equiv \frac{1}{2}\tilde{\mathbf{U}}\cdot A = \frac{1}{2}\mathbf{U}\cdot\Psi\cdot A = \frac{1}{2}\mathbf{U}\cdot\tilde{A} \tag{15}$$

$$\begin{gathered} a_i = \frac{1}{2i}Y_i^{*T}SX;\ \tilde{a}_i \equiv a_i e^{i\Psi_i} = \frac{1}{2i}Y_i^{*T}SX; \\ A = 2\tilde{\mathbf{U}}^{-1}\cdot X = -i\Psi^{-1}\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X;\ \tilde{A} = \Psi A = -i\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X. \end{gathered} \tag{16}$$

1D

$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \psi' = \frac{1}{\mathrm{w}^2}; \tilde{Y} = Ye^{i\psi} \tag{17}$$



The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(ae^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$

$$\begin{aligned} x &= a \cdot \mathrm{w(s)} \cdot \cos(\psi(s) + \varphi) \\ x' &= a \cdot (\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) - sin(\psi(s) + \varphi)/\mathrm{w(s)}) \end{aligned} \tag{18}$$

$$\beta \equiv \mathrm{w}^2 \;\Rightarrow\; \psi' = 1/\beta. \tag{19}$$

$$\alpha \equiv -\beta'/2 \equiv -\mathrm{w}\,\mathrm{w}',\; \gamma \equiv \frac{1+\alpha^2}{\beta}. \tag{20}$$

$$\begin{aligned} x &= a \cdot \sqrt{\beta(\mathrm{s})} \cdot \cos(\psi(s) + \varphi) \\ x' &= -\frac{a}{\sqrt{\beta(\mathrm{s})}} \cdot (\alpha(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi)) \end{aligned} \tag{21}$$

$$\mathbf{T} = \mathbf{U}\Lambda\mathbf{U}^{-1} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu; \quad \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \mathbf{J}^2 = -\mathbf{I} \tag{22}$$

2D

$$X=\begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix}=\operatorname{Re}\tilde{a}_1 Y_1+\operatorname{Re}\tilde{a}_2 Y_2=\operatorname{Re} a_1 \tilde{Y}_1+\operatorname{Re}\tilde{a}_2 \tilde{Y}_2 \tag{23}$$

$$Y_k=R_k+iQ_k;\quad \tilde{Y}_k=\begin{bmatrix} w_{kx}e^{i\psi_{kx}} \\ (u_{kx}+iv_{kx})e^{i\psi_{kx}} \\ w_{ky}e^{i\psi_{ky}} \\ (u_{ky}+iv_{ky})e^{i\psi_{ky}} \end{bmatrix};\ \psi_{kx}(s+C)=\psi_{kx}(s)+\mu_k;\psi_{ky}(s+C)=\psi_{ky}(s)+\mu_k;$$

$$w_{kx}v_{kx}+w_{ky}v_{ky}=1; \tag{24}$$

Conditions: there are

$$Y_k^{*T}SY_k=2i;\quad Y_1^{*T}SY_2=0;\ Y_1^T SY_2=0;\quad \theta_k=\psi_{kx}-\psi_{ky}$$

$$a)\ \ w_{1x}v_{1x}=w_{2y}v_{2y}=1-q\qquad \Rightarrow v_{1x}=\frac{1-q}{w_{1x}};\ \ v_{2y}=\frac{1-q}{w_{2y}}$$

$$b)\ \ w_{1y}v_{1y}=w_{2x}v_{2x}=q\quad \Rightarrow v_{2x}=\frac{q}{w_{2x}};\ \ w_{1y}=\frac{q}{w_{1y}}$$

$$c)\ \ c=w_{1x}w_{1y}\sin\theta_1=-w_{2x}w_{2y}\sin\theta_2$$

$$d)\ \ d=w_{1x}\left(u_{1y}\sin\theta_1-v_{1y}\cos\theta_1\right)=-w_{2x}\left(u_{2y}\sin\theta_2-v_{2y}\cos\theta_2\right)$$

$$e)\ \ e=w_{1y}\left(u_{1x}\sin\theta_1+v_{1x}\cos\theta_1\right)=-w_{2y}\left(u_{2x}\sin\theta_2+v_{2x}\cos\theta_2\right) \tag{25}$$

$$Y_1=\begin{bmatrix} w_{1x}e^{i\varphi_{1x}} \\ \left(u_{1x}+i\dfrac{q}{w_{1x}}\right)e^{i\varphi_{1x}} \\ w_{1y}e^{i\varphi_{1y}} \\ \left(u_{1y}+i\dfrac{1-q}{w_{1y}}\right)e^{i\varphi_{1y}} \end{bmatrix};\quad Y_2=\begin{bmatrix} w_{2x}e^{i\varphi_{2x}} \\ \left(u_{2x}+i\dfrac{1-q}{w_{2x}}\right)e^{i\varphi_{2x}} \\ w_{2y}e^{i\varphi_{2y}} \\ \left(u_{2y}+i\dfrac{q}{w_{2y}}\right)e^{i\varphi_{2y}} \end{bmatrix} \tag{26}$$

3D

$$\left|\lambda_k\right|=1;\ \lambda_k=e^{i\mu_k};\ \mu_k=2\pi Q_k;\ k=1,2,3 \tag{27}$$

$$X=\begin{bmatrix} x \\ P_x \\ y \\ P_y \\ \tau \\ P_\tau \end{bmatrix}=\operatorname{Re}\tilde{a}_1Y_1+\operatorname{Re}\tilde{a}_2Y_2+\operatorname{Re}\tilde{a}_3Y_3=\operatorname{Re}a_1\tilde{Y}_1+\operatorname{Re}a_2\tilde{Y}_2+\operatorname{Re}a_3\tilde{Y}_3 \tag{23}$$

$$Y_k(s)=\begin{bmatrix} \mathrm{w}_{kx}e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx}+i\frac{q_{kx}}{\mathrm{w}_{kx}}\right)e^{i\chi_{kx}} \\ \mathrm{w}_{ky}e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky}+i\frac{q_{ky}}{\mathrm{w}_{ky}}\right)e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau}e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau}+i\frac{q_{k\tau}}{\mathrm{w}_{k\tau}}\right)e^{i\chi_{k\tau}} \end{bmatrix};Y_k(s+C)=Y_k(s);T(s)Y_k(s)=e^{i\mu_k}Y_k(s);k=1,2,3 \tag{25}$$

$$Y_k^{\ T}SY_j=0;\quad Y^{*\ T}_{\ j}SY_k=2i\delta_{kj}; \tag{26}$$

15 relations on the component of the eigen vectors, with the simples being:

$$q_{kx}+q_{ky}+q_{k\tau}=1;\ k=1,2,3 \tag{27}$$

Parameterization using real (non-complex) parameters. Since for a stable system eigen vectors are uni-modular complex numbers, eigen vectors are also complex and satisfy purely imaginary symplectic orthogonally conditions (9). Naturally matrix T can not be diagonalized using real matrices, but it can be brought to a block-diagonal form comprising simple 2x2 rotation matrices using following considerations:

$$Y_k = R_k + iQ_k; Y_k^* = R_k - iQ_k; \mathbf{T}\cdot Y_k = e^{i\mu_k} Y_k;\ \mathbf{T}\cdot Y_k^* = e^{-i\mu_k} Y_k; \\ \mathbf{T}\cdot R_k = R_k \cdot \cos\mu_k - Q_k \cdot \sin\mu_k; \mathbf{T}\cdot Q_k = Q_k \cdot \cos\mu_k + R_k \cdot \sin\mu_k; \tag{28}$$

which is equivalent to

$$\mathbf{Q} = \left(R_1, Q_1, \ldots R_n, Q_n\right) \to \mathbf{T}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{O} \to \mathbf{T} = \mathbf{Q}\cdot\mathbf{O}\cdot\mathbf{Q}^{-1}; \\ \mathbf{O} = \begin{pmatrix} O_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & O_n \end{pmatrix}; O_k = \begin{pmatrix} \cos\mu_k & \sin\mu_k \\ -\sin\mu_k & \cos\mu_k \end{pmatrix}; \mathbf{O}^T = \mathbf{O}^{-1} \tag{29}$$

where by construction matrix **Q** is real. We can use a symbolic form of expressing block diagonal shape of **O** by writing

$$\mathbf{O} = \begin{pmatrix} O_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & O_n \end{pmatrix} = \sum_{k\oplus} O_k; \tag{29}$$

It is also symplectic, which is result of simple observation that follows from symplectic orthogonally of R_k, Q_k pairs:

$$Y_k^{*T}\mathbf{S}Y_{m\neq k} = 0 \rightarrow R_k^T\mathbf{S}R_m = 0; R_k^T\mathbf{S}Q_{m\neq k} = 0; Q_k^T\mathbf{S}Q_m = 0;$$

$$Y_k^{T}SY_k = \left(R_k - iQ_k\right)^T\mathbf{S}\left(R_k + iQ_k\right) = \left(-iQ_k\right)iR_k^T\mathbf{S}Q_k - iQ_k^T\mathbf{S}R_k = 2iR_k^T\mathbf{S}Q_k = 2i;$$

$$R_k^T\mathbf{S}Q_k = -Q_k^T\mathbf{S}R_k = 1;$$

$$\mathbf{Q}^T\mathbf{S}\mathbf{Q} = \left(...R_k, Q_k...\right)^T\mathbf{S}\left(...R_k, Q_k...\right)\left(...R_k Q_k...\right)^T\left(...\mathbf{S}R_k, \mathbf{S}Q_k...\right) =$$

$$\begin{pmatrix} \begin{pmatrix} R_1^T\mathbf{S}R_1 & R_1^T\mathbf{S}Q_1 \\ Q_1^T\mathbf{S}R_1 & Q_1^T\mathbf{S}Q_1 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} R_n^T\mathbf{S}R_n & R_n^T\mathbf{S}Q_n \\ Q_n^T\mathbf{S}R_n & Q_n^T\mathbf{S}Q_n \end{pmatrix} \end{pmatrix} = \tag{30}$$

$$= \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} = \mathbf{S}\#$$

There is one to one connection between real matrix **Q** and complex matrix **U**

$$\mathbf{U}=\mathbf{Q}\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \end{pmatrix};\mathbf{Q}=\frac{\mathbf{U}}{2}\begin{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \end{pmatrix} \tag{31}$$

which means that putting matrix **Q** in motion is

$$\tilde{\mathbf{Q}}(s_1)=\mathbf{M}(s|s_1)\mathbf{Q}(s)=\mathbf{Q}(s_1)\cdot\tilde{\mathbf{O}}(s|s_1);$$
$$\tilde{\mathbf{O}}(s|s_1)=\sum_{\oplus}\begin{pmatrix} \cos\left(\psi_k(s_1)-\psi_k(s)\right) & \sin\left(\psi_k(s_1)-\psi_k(s)\right) \\ -\sin\left(\psi_k(s_1)-\psi_k(s)\right) & \cos\left(\psi_k(s_1)-\psi_k(s)\right) \end{pmatrix} \tag{32}$$

Again, it gives us connection between transport matrices and parametrization:

$$\mathbf{M}(s|s_1)=\mathbf{Q}(s_1)\cdot\tilde{\mathbf{O}}(s|s_1)\mathbf{Q}(s)^{-1}=-\mathbf{Q}(s_1)\cdot\tilde{\mathbf{O}}(s|s_1)\mathbf{S}\mathbf{Q}^T(s)\mathbf{S} \tag{33}$$

Probably the most interesting is application of this expression for full period matrix (either from eq. (33) or eq. (29)):

$$\mathbf{T}=\mathbf{Q}\cdot\mathbf{O}\cdot\mathbf{Q}^{-1}=\sum_{k\oplus}\mathbf{Q}\cdot\left[O_k\right]\cdot\mathbf{Q}^{-1};\ \left[O_k\right]=\begin{pmatrix} 0 & & & & 0 \\ & \cdots & 0 & \cdots & \\ \cdots & 0 & \sigma & 0 & \cdots \\ & \cdots & 0 & \cdots & \\ 0 & & \cdots\cdot & & 0 \end{pmatrix} \tag{34}$$

$$O_k=\begin{pmatrix} \cos\mu_k & \sin\mu_k \\ -\sin\mu_k & \cos\mu_k \end{pmatrix}=\cos\mu_k I_k+\sin\mu_k\sigma_k;\sigma=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where I_k,σ_k are block diagonal 2x2 matrices with non-zero block in k-position on the diagonal. Now we will extract constants and expand one-turn transport matrix though eigen matrices:

$$\begin{gathered}\mathbf{T}=\sum_{k\oplus}\left(\mathbf{E}_k\cos\mu_k+\mathbf{J}_k\sin\mu_k\right);\\ \mathbf{E}_k=\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1};\mathbf{J}_k=\mathbf{Q}\cdot\left[\sigma_k\right]\cdot\mathbf{Q}^{-1};\ \mathbf{Q}^{-1}=-\mathbf{S}\mathbf{Q}^T\mathbf{S}.\end{gathered} \tag{35}$$

This matrices have very nice features of n mutually orthogonal pair of *1 and i*:

$$\begin{gathered}\left[I_k\right]^2=\left[I_k\right]\rightarrow\mathbf{E}_k^2=\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1}\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1}=\mathbf{Q}\cdot\left[I_k\right]\cdot\mathbf{Q}^{-1}=\mathbf{E}_k;\\ \left[\sigma_k\right]^2=-\left[I_k\right]\rightarrow\mathbf{J}_k^2=\mathbf{Q}\cdot\left[\sigma_k\right]\cdot\mathbf{Q}^{-1}\cdot\mathbf{Q}\cdot\left[\sigma_k\right]\cdot\mathbf{Q}^{-1}=-\mathbf{E}_k\\ \left[I_k\right]\left[\sigma_k\right]=\left[I_k\right]\left[\sigma_k\right]=\left[\sigma_k\right]\rightarrow\mathbf{E}_k\mathbf{J}_k=\mathbf{J}_k\mathbf{E}_k=\mathbf{J}_k;\\ \left[I_k\right]\left[I_{m\neq k}\right]=\left[\sigma_k\right]\left[I_{m\neq k}\right]=\left[\sigma_k\right]\left[\sigma_{m\neq k}\right]\equiv 0\rightarrow\mathbf{E}_k\mathbf{E}_{m\neq k}=\mathbf{E}_k\mathbf{J}_{m\neq k}=\mathbf{J}_k\mathbf{J}_{m\neq k}\equiv 0\end{gathered} \tag{36}$$

which result in trivial adding phase advance in equation (35):

$$\mathbf{T}^n=\sum_{k\oplus}\left(E_k\cos n\mu_k+J_k\sin n\mu_k\right). \tag{37}$$

This expression is especially beautiful for 1D case when because matrix is just a 2x2 block itself:

$$[I_k] = \mathbf{I}; [\sigma_k] = \mathbf{S}; [\sigma_k]$$

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{E} = \mathbf{Q}\cdot\mathbf{I}\cdot\mathbf{Q}^{-1} = \mathbf{I}; \mathbf{J} = -\mathbf{Q}\cdot\mathbf{S}\cdot\mathbf{S}\mathbf{Q}^T\mathbf{S} = \mathbf{Q}\cdot\mathbf{Q}^T\cdot\mathbf{S}$$

where we can use specific expression for $\mathbf{Q}$

$$\mathbf{Q} = [\operatorname{Re}Y, \operatorname{Im}Y] = \begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & \dfrac{1}{\mathrm{w}} \end{bmatrix}; \mathbf{Q}\cdot\mathbf{Q}^T = \begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & \dfrac{1}{\mathrm{w}} \end{bmatrix}\begin{bmatrix} \mathrm{w} & \mathrm{w}' \\ 0 & \dfrac{1}{\mathrm{w}} \end{bmatrix} = \begin{bmatrix} \mathrm{w}^2 & \mathrm{ww}' \\ \mathrm{ww}' & \dfrac{1}{\mathrm{w}^2} + \mathrm{w}'^2 \end{bmatrix}$$

$$\mathbf{J} = \mathbf{Q}\cdot\mathbf{Q}^T\cdot\mathbf{S} = \begin{bmatrix} \mathrm{w}^2 & \mathrm{ww}' \\ \mathrm{ww}' & \dfrac{1}{\mathrm{w}^2} + \mathrm{w}'^2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\mathrm{ww}' & \mathrm{w}^2 \\ -\left(\dfrac{1}{\mathrm{w}^2} + \mathrm{w}'^2\right) & \mathrm{ww}' \end{bmatrix}$$

and you can directly show that $\mathbf{J}^2 = -\mathbf{I}$. Using traditional definitions of α,β,γ functions introduced by Courant and Snider we can rewrite (38) in form you would find in standard accelerator books:

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;\ \mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}; \mathbf{J}^2 = -\mathbf{I}; \tag{38}$$

$$\beta = \mathrm{w}^2;\ \alpha = -\mathrm{ww}' = -\beta'/2; \gamma = \mathrm{w}'^2 + \mathrm{w}^{-2} = \frac{1+\alpha^2}{\beta}.$$

Now we are ready to make use of our parameterization:

$$X_o = \mathrm{Re}\sum_{k=1}^{n} a_k Y_k(s) e^{i\psi_k(s)} \equiv$$

$$\sum_{k=1}^{n} |a_k| \left(R_k(s)\cos\left(\psi_k(s)+\varphi_k\right) - Q_k(s)\sin\left(\psi_k(s)+\varphi_k\right)\right); \;\; a_k = |a_k| e^{i\varphi_k}; \tag{39}$$

with 2n constants of motion coming in pairs of amplitude and phased of ocialLator $\{a_k, \varphi_k\}, k = 1,...,n$. Starting from this point we will use real amplitudes $a_k \to |a_k|$ and separate phase explicitly: $a_k \to a_k e^{i\varphi_k}$.

Symplectic transformation is a Canonical transformation. Let us now, again, demonstrate that symplectic transformation $X(s) \Rightarrow \tilde{X}(s)$

$$X(s) = \mathbf{V}(s)\tilde{X}, \quad \mathbf{V}'(s) = \mathbf{SH}(s)\mathbf{V}(s) \leftrightarrow \mathbf{V}^T\mathbf{SV} = \mathbf{S};. \tag{40}$$

is Canonical. Beginning from a Hamiltonian composed of two parts, a linear part and an arbitrary one

$$\mathcal{H} = \frac{1}{2} X^T \mathbf{H}(s) X + \mathcal{H}_1(X,s). \tag{41}$$

The equation of motion

$$\frac{dX}{ds} = \mathbf{S} \cdot \frac{\partial \mathcal{H}}{\partial X} = \mathbf{SH}(s) \cdot X + \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X}. \tag{42}$$

becomes with substitution (40)

$$\left(\mathbf{V}\tilde{X}\right)' = \mathbf{SHV} \cdot \tilde{X} + \mathbf{V}\tilde{X}' = \mathbf{SH}(s) \cdot \mathbf{V}\tilde{X} + \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X} \Rightarrow \mathbf{V}\tilde{X}' = \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X}. \tag{43}$$

equivalent to the equations of motion with the new Hamiltonian: $\mathcal{H}_1(\mathbf{V}\tilde{X},s)$

$$\tilde{X}' = \mathbf{V}^{-1}\mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial X}; \frac{\partial}{\partial X} = \mathbf{V}^{-1T} \frac{\partial}{\partial \tilde{X}} \Rightarrow \tilde{X}' = \left(\mathbf{V}^{-1}\mathbf{S}\mathbf{V}^{-1T}\right) \cdot \frac{\partial \mathcal{H}_1}{\partial \tilde{X}} \Rightarrow \tilde{X}' = \mathbf{S} \cdot \frac{\partial \mathcal{H}_1}{\partial \tilde{X}}. \tag{44}$$

Action-angle variables. A very important transformation (not-only!) in accelerator physics is the transformation to the action-angle variables $\left\{\varphi_k, I_k = \frac{a_k^{\ 2}}{2}\right\}$. Usually this requires two steps: The first is to Canonically transfer to Canonical conjugate oscillators (you may remember them from quantum mechanics?):

$$\left\{\tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}\right\}. \tag{45}$$

$$X^T \equiv \{..q_k, p_k...\} \Leftrightarrow A_{qo}^{\ \ T} \equiv \left\{\tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}\right\};$$
$$X^T = \mathbf{V}A_{qo}; \quad \mathbf{V} = \frac{1}{\sqrt{2}}\left[Y_1, iY^*_{\ 1}......\right] \Rightarrow \mathbf{V}^T\mathbf{S}\mathbf{V} = \mathbf{S} \ \# \tag{46}$$

The second step is very simple since it is well known from classical theory of harmonic oscillators. A generation function transformation making this Canonical transformation happening is very simple to construct:

$$\left\{q_k = \varphi_k; p_k \equiv I_k = \frac{a^2}{2}\right\} \Leftrightarrow \left\{\tilde{q}_k = \frac{a_k e^{i\varphi_k}}{\sqrt{2}}, \tilde{p}_k = i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}\right\}$$
$$F(q,\tilde{q}) = -\sum_{k=1}^{n} i\frac{\tilde{q}_k^{\ 2}}{2}e^{-2i\varphi_k}; \frac{\partial F}{\partial s} = 0 \rightarrow \tilde{H} = H \tag{47}$$
$$I_k = \frac{\partial F}{\partial q_k} \equiv \frac{\partial F}{\partial \varphi_k} = \tilde{q}_k^2 e^{-2i\varphi_k} = \frac{a_k^{\ 2}}{2}; \ \tilde{p}_k = -\frac{\partial F}{\partial \tilde{q}_k} = i\tilde{q}_k e^{-2i\varphi_k} i\frac{a_k e^{-i\varphi_k}}{\sqrt{2}}.$$

Similarly, we can make transformation for pairs of real oscillator components:

$$\left\{ \tilde{q}_k = a_k \cos\varphi_k, \tilde{p}_k = -a_k \sin\varphi_k \right\}. \tag{48}$$

with obvious symplectic transformation

$$\begin{gathered} A^T{}_{osc} = \left\{ \ldots q_k, p_k \ldots \right\} = \left\{ \ldots a_k \cos\varphi_k, -a_k \sin\varphi_k \ldots \right\} \\ X = \mathbf{Q} \cdot A_{osc} \to A_{osc} = \mathbf{Q}^{-1} X;\ \mathbf{Q}^{-1T} \mathbf{S} \mathbf{Q}^{-1} = \mathbf{S}. \end{gathered} \tag{49}$$

Again, the generation function transformation making this Canonical transformation happening is very simple to construct:

$$\left\{ q_k = \varphi_k; p_k \equiv I_k = \frac{a_k{}^2}{2} \right\} \Leftrightarrow \left\{ \tilde{q}_k = a_k \cos\varphi_k, \tilde{p}_k = -a_k \sin\varphi_k \right\}$$

$$F(q, \tilde{q}) = \sum_{k=1}^{n} \frac{\tilde{q}_k{}^2}{2} \tan\varphi_k; \frac{\partial F}{\partial s} = 0 \to \tilde{H} = H \tag{50}$$

$$I_k = \frac{\partial F}{\partial q_k} \equiv \frac{\partial F}{\partial \varphi_k} = \frac{\tilde{q}_k^2}{2\cos^2\varphi} = \frac{a_k{}^2}{2};\ \tilde{p}_k = -\frac{\partial F}{\partial \tilde{q}_k} = -\tilde{q}_k \tan\varphi_k = -a_k \sin\varphi_k.$$

This result (even though expected) has long-lasting consequences – the trivial (linear) part in the Hamiltonian can be removed from equations of motion, so allowing one to use this in perturbation theory or at least to focus only on non-trivial part of the motion.

Finally, we know that for any canonical transformation:

But by design for a linear Hamiltonian system,

$$H_L = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s)\cdot X \qquad (51)$$

A^T = consts. It means that

$$\frac{\partial F(q,\tilde{q},s)}{\partial s} = -H_L \qquad (52)$$

It means that equation of motion for a linear *s*-dependent Hamiltonian system are reduced to a set of constant: amplitudes and phases of oscillations:

$$\varphi_k = const; I_k = \frac{a_k{}^2}{2} = const;\ k = 1,2...,n \qquad (53)$$

What it important to note that I_k is an adiabatic invariant of an oscillator, e.g. is the phase space area of the covered by oscillator divided by π. We can call it emittance of the k-th mode.

Thus, if we are applying transformation of the action-angle Canonical variables of an arbitrary (in general case, nonlinear) Hamiltonian system

$$H(X,s) = H_L(X,s) + H_1(X,s) \tag{53}$$

we will come to the reduced equations of motion with the Hamiltonian:

$$\begin{gathered}\tilde{H} = H + \frac{\partial F}{\partial s} = H - H_L = H_1(X,s);\\ \tilde{H}(A,s) = H_1(X(A,s),s).\end{gathered} \tag{54}$$

where we eliminated "boring" oscillating part of the motion.

Since next step of transformation to the action-angle variables (41) does not change the Hamiltonian, we finally get:

$$\begin{gathered}\tilde{H}(\varphi_k, I_k, s) = H_1(X(\varphi_k, I_k, s), s);\\ \frac{d\varphi_k}{ds} = \frac{\partial \tilde{H}}{\partial I_k}; \frac{dI_k}{ds} = -\frac{\partial \tilde{H}}{\partial \varphi_k}.\end{gathered} \tag{55}$$

These "reduced" equations of motion can be very useful when H_1 can be treated as perturbation or in studies of a non-linear map. We will return to them again and again through the course.

What we learned today

- We expanded parameterization of linear motion from complex notation real number notation – naturally the resulting motion is the same
- We proved that symplectic transformation is equivalent to a Canonical transformation
- If transformation matrix is a solution of linear Hamiltonian system $X^T\mathbf{H}X/2$, than this Canonical transform removes the $X^T\mathbf{H}X/2$ from the Hamiltonian
- We defined two sets of oscillator coordinates and momenta and showed that this transformation is Canonical
- Than we made transformation to action-angle variables, which comprise Canonical pairs $\left\{ q_k = \varphi_k; p_k \equiv I_k = \frac{a^2}{2} \right\}$
- Using variables will allow us to study a number of phenomena using perturbation methods – next class will be devoted to this