Homework 10

This is first STAR problem. The rules are as follow:

(a) if you find a standard solution (for example from a good Analytical mechanics book or on a Website) – you got nominal number of points. I rely on your comment at the end of the derivation.

(b) If you derived your own original proof – you have 5-fold increase in the score.

Problem 1. 15 points. Prove that

(a) A canonical transformation is equivalent to a symplectic map (locality symplectic linear transformation);

(b) a symplectic map (locality symplectic linear transformation) is a Canonical transformation,

Suggestion: you may ask for additional hint during next class.

Solution:

(a) Prove that Canonical transformation is equivalent to a symplectic map.

In other words, we have to show that it is locally symplectic. First, let's write all four forms of generation functions of the same canonical transformation:

$$F(q,\tilde{q},t) \qquad \Rightarrow dF = P_i dq_i - \tilde{P}_i d\tilde{q}_i + (H' - H)dt;\ P_i = \frac{\partial F}{\partial q_i}; \tilde{P}_i = -\frac{\partial F}{\partial \tilde{q}_i}; H' = H + \frac{\partial F}{\partial t}.$$

$$\Phi(q,\tilde{P},t) = F + \tilde{q}_i\tilde{P}_i \ \Rightarrow d\Phi = P_i dq_i + \tilde{q}_i d\tilde{P}_i + (H' - H)dt; P_i = \frac{\partial \Phi}{\partial q_i}; \tilde{q}_i = \frac{\partial \Phi}{\partial \tilde{P}_i}; H' = H + \frac{\partial \Phi}{\partial t};$$

$$\Omega(P,\tilde{q},t) = F - P_i q_i \ \Rightarrow d\Omega = -q_i dP_i - \tilde{P}_i d\tilde{q}_i + (H' - H)dt; q_i = -\frac{\partial \Omega}{\partial P_i}; \tilde{P}_i = -\frac{\partial \Omega}{\partial \tilde{q}_i}; H' = H + \frac{\partial \Omega}{\partial t};$$

$$\Lambda(P,\tilde{P},t) = \Phi - P_i q_i \Rightarrow d\Lambda = \tilde{q}_i d\tilde{P}_i - q_i dP_i + (H' - H)dt; q_i = -\frac{\partial \Lambda}{\partial P_i}; \tilde{q}_i = \frac{\partial \Lambda}{\partial \tilde{P}_i}; H' = H + \frac{\partial \Lambda}{\partial t};$$

with

$$\tilde{X} = \tilde{X}(X,s); X = \{q,P\}; \tilde{X} = \{\tilde{q},\tilde{P}\}$$

Then for local transformation we can write

$$\delta\tilde{X} = M(s)\delta X \Leftrightarrow \ \delta X = M^{-1}(s)\delta\tilde{X}$$

We need to show that

$$M^T S M = S \Leftrightarrow \ M^{-1} = -S M^T S$$

or in details

$$M^{-1}_{ik} = S_{ij} M_{lj} S_{lk}; \quad S_{ij} = \left\{ \begin{array}{c} 1;\ i = 2m-1, j = 2m \\ -1;\ j = 2m-1, i = 2m \\ 0, \mathit{otherwise} \end{array} \right\}$$

specifically

$$M^{-1}_{2m-1,2k-1} = M_{2k,2m};\ M^{-1}_{2m-1,2n} = -M_{2k-1,2m}$$
$$M^{-1}_{2m,2k-1} = -M_{2k,2m-1};\ M^{-1}_{2m,2k} = M_{2k-1,2m-1}$$
$$k, m = \{1, ..., n\}$$

4x4 example:

$$M = \begin{pmatrix} \mathtt{m11} & \mathtt{m12} & \mathtt{m13} & \mathtt{m14} \\ \mathtt{m21} & \mathtt{m22} & \mathtt{m23} & \mathtt{m24} \\ \mathtt{m31} & \mathtt{m32} & \mathtt{m33} & \mathtt{m34} \\ \mathtt{m41} & \mathtt{m42} & \mathtt{m43} & \mathtt{m44} \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} \mathtt{m22} & -\mathtt{m12} & \mathtt{m42} & -\mathtt{m32} \\ -\mathtt{m21} & \mathtt{m11} & -\mathtt{m41} & \mathtt{m31} \\ \mathtt{m24} & -\mathtt{m14} & \mathtt{m44} & -\mathtt{m34} \\ -\mathtt{m23} & \mathtt{m13} & -\mathtt{m43} & \mathtt{m33} \end{pmatrix}$$

Important reminder:

$$x_{2k-1} \equiv q_k;\ \ x_{2k} \equiv P_k;\ \tilde{x}_{2k-1} \equiv \tilde{q}_k;\ \tilde{x}_{2k} \equiv \tilde{P}_k.$$

Let consider all four cases: $M_{2k-1,2m-1}$ vs $M^{-1}_{2m,2k}$. It is easy to prove that they equal

$$P_m = \frac{\partial \Phi}{\partial q_m}; \tilde{q}_k = \frac{\partial \Phi}{\partial \tilde{P}_k};$$
$$M^{-1}_{2m,2k} = \frac{\partial P_m}{\partial \tilde{P}_k} = \frac{\partial}{\partial \tilde{P}_k}\frac{\partial \Phi}{\partial q_m} = \frac{\partial^2 \Phi}{\partial \tilde{P}_k \partial q_m} = \frac{\partial^2 \Phi}{\partial q_m \partial \tilde{P}_k}$$
$$M_{2k-1,2m-1} = \frac{\partial \tilde{q}_k}{\partial q_m} = \frac{\partial}{\partial q_m}\frac{\partial \Phi}{\partial \tilde{P}_k} = \frac{\partial^2 \Phi}{\partial q_m \partial \tilde{P}_k} = M^{-1}_{2m,2k}$$

Let consider all four cases: $M_{2m,2k}$ vs $M^{-1}_{2k-1,2m-1}$. It is easy to prove that they equal

$$q_k = -\frac{\partial \Omega}{\partial P_k}; \tilde{P}_m = -\frac{\partial \Omega}{\partial \tilde{q}_m};$$
$$M_{2m,2k} = \frac{\partial \tilde{P}_m}{\partial P_k} = -\frac{\partial}{\partial P_k}\left(\frac{\partial \Omega}{\partial \tilde{q}_m}\right) = -\frac{\partial^2 \Omega}{\partial P_k \partial \tilde{q}_m} = -\frac{\partial^2 \Phi}{\partial \tilde{q}_m \partial P_k}$$
$$M^{-1}_{2k-1,2m-1} = \frac{\partial q_k}{\partial \tilde{q}_m} = -\frac{\partial}{\partial \tilde{q}_m}\left(\frac{\partial \Omega}{\partial P_k}\right) = -\frac{\partial^2 \Phi}{\partial \tilde{q}_m \partial P_k} = M_{2m,2k}$$

Let consider all four cases: $M_{2k-1,2m}$ vs $M^{-1}_{2m-1,2k}$. It is easy to prove that they equal

$$\tilde{q}_k = \frac{\partial \Lambda}{\partial \tilde{P}_k}; q_m = -\frac{\partial \Lambda}{\partial P_m};$$

$$M_{2k-1,2m} = \frac{\partial \tilde{q}_k}{\partial P_m} = \frac{\partial}{\partial P_m}\left(\frac{\partial \Lambda}{\partial \tilde{P}_k}\right) = \frac{\partial^2 \Lambda}{\partial P_m \partial \tilde{P}_k}$$

$$M^{-1}_{2m-1,2k} = \frac{\partial q_m}{\partial \tilde{P}_k} = -\frac{\partial}{\partial \tilde{P}_k}\left(\frac{\partial \Lambda}{\partial P_m}\right) = -\frac{\partial^2 \Lambda}{\partial P_m \partial \tilde{P}_k} = -M_{2k-1,2m}$$

And using the same drill for $M_{2k,2m-1}$ vs $M^{-1}_{2m,2k-1}$

$$P_m = \frac{\partial F}{\partial q_m}; \tilde{P}_k = -\frac{\partial F}{\partial \tilde{q}_k}$$

$$M_{2k,2m-1} = \frac{\partial \tilde{P}_k}{\partial q_m} = -\frac{\partial}{\partial q_m}\left(\frac{\partial F}{\partial \tilde{q}_k}\right) = -\frac{\partial^2 F}{\partial q_m \partial \tilde{q}_k}$$

$$M^{-1}_{2m,2k-1} = \frac{\partial P_m}{\partial \tilde{q}_k} = \frac{\partial}{\partial \tilde{q}_k}\left(\frac{\partial F}{\partial q_m}\right) = \frac{\partial^2 F}{\partial \tilde{q}_k \partial q_m} = -M_{2k,2m-1}$$

Note, that we used all four forms of otherwise identical Canonical transformation to prove that every Canonical transformation is a symplectic map.

(b). Let’s now show that a symplectic map (e.g. generating symplectic matrices) is equivalent to a Canonical transformation.
Let start from a generic transformation in the phase space:

$$X = F(\tilde{X}, s) \tag{1}$$

where with $\tilde{X}$ is point in the phase space.

$$M_{ij} = \frac{\partial X_i}{\partial \tilde{X}_j} \equiv \frac{\partial F_i(\tilde{X}, s)}{\partial \tilde{X}_j}; M = \left[\frac{\partial F}{\partial \tilde{X}}\right]; \;\; M^T S M = S$$

Since $\tilde{X}$ is initial condition

$$\frac{d\tilde{X}}{ds} = 0$$

and we need to show that

$$\frac{dX}{ds} = S\frac{\partial H}{\partial X}$$

where H is some Hamiltonian in phase space X.

$$\frac{dX}{ds} = \frac{dF(\tilde{X}, s)}{ds} = \frac{\partial F(\tilde{X}, s)}{\partial s} + \frac{\partial F(\tilde{X}, s)}{\partial \tilde{X}}\frac{d\tilde{X}}{ds} = \frac{\partial F(\tilde{X}, s)}{\partial s};$$

There is a number of steps to prove this. First, lets expand about the trajectory starting at $\tilde{X} = \tilde{X}_o$:

$$\tilde{X} = \tilde{X}_o + \delta\tilde{X};$$

$$X = X_o(s) + \delta X;\ \ \delta X(s) = M(s)\delta\tilde{X}(s)$$
$$X_o(s) = F(\tilde{X}_o, s);$$

e.g. we determined the local symplectic transformation in the entire phase space as function of $(s, \tilde{X}_o)$. It allows us to rewrite the transformation (1) function as

$$\frac{d}{ds}\delta X = \frac{dM}{ds}\delta\tilde{X} = \left(\frac{dM}{ds}M^{-1}\right)\delta X; \tag{2}$$

where we used $\delta\tilde{X} = M^{-1}\delta X$. Let's try to find Hamiltonian in following form with symmetric Hamiltonian matrix A:

$$\begin{gathered} H(X,s) = -X^T S\frac{dX_o(s)}{ds} + \frac{1}{2}\delta X^T \cdot A(s) \cdot \delta X; A^T = A;\ \ \delta X = X - X_o(s) \\ \frac{dX}{ds} = S\frac{\partial H}{\partial X} \equiv S\frac{\partial H}{\partial(\delta X)} = -S^2\frac{dX_o(s)}{ds} + S \cdot A \cdot \delta X = \frac{dX_o(s)}{ds} + \left(\frac{dM}{ds}M^{-1}\right)\delta X \end{gathered} \tag{3}$$

whom where we find first term is correct and for system (2) to be a Hamiltonian the matrix $S \cdot \frac{dM}{ds}M^{-1}$ must be symmetric:

$$-S^2 = I \Rightarrow \frac{dX}{ds} = \frac{dX_o(s)}{ds} + S \cdot A \cdot \delta X = \frac{dX_o(s)}{ds} + \left(\frac{dM}{ds}M^{-1}\right)\delta X$$
$$A \overset{?}{=} -S \cdot \frac{dM}{ds}M^{-1}$$

in other words, for motion described by equation (2), matrix $S \cdot \frac{dM}{ds}M^{-1}$ must be symmetric

$$\left(S \cdot \frac{dM}{ds}M^{-1}\right)^T = \left(S \cdot \frac{dM}{ds}M^{-1}\right) \tag{4}$$

He we must use symplecticity of matrix M:

$$MSM^T = S \Rightarrow M^{-1} = -SM^T S\ \ and$$
$$\frac{d}{ds}\left(MSM^T\right) = \frac{dS}{ds} = 0 \Rightarrow \frac{dM}{ds}SM^T = -MS\frac{dM^T}{ds}$$
$$U = -S \cdot \frac{dM}{ds}M^{-1} = S \cdot \frac{dM}{ds}SM^T S;\ \ S^T = -S;$$
$$U^T = \left(S \cdot \frac{dM}{ds}SM^T S\right)^T = -S \cdot MS\frac{dM^T}{ds}S = S \cdot \frac{dM}{ds}SM^T S.$$

Thus, infinitesimal motion around any trajectory correspond to a Hamiltonian system if map is locally symplectic. Note, that condition (4) is not satisfied for arbitrary matrices. We proved that any symplectic map corresponds to local Hamiltonian system (3), which generates local canonical transformation from $\tilde{X}_o + \delta\tilde{X} \to X_o(s) + \delta X(s)$. Since map is symplectic everywhere, we can expand canonical transformation this to the entire phase space $\tilde{X} \to X(s)$.