PHY 554. Homework 2.

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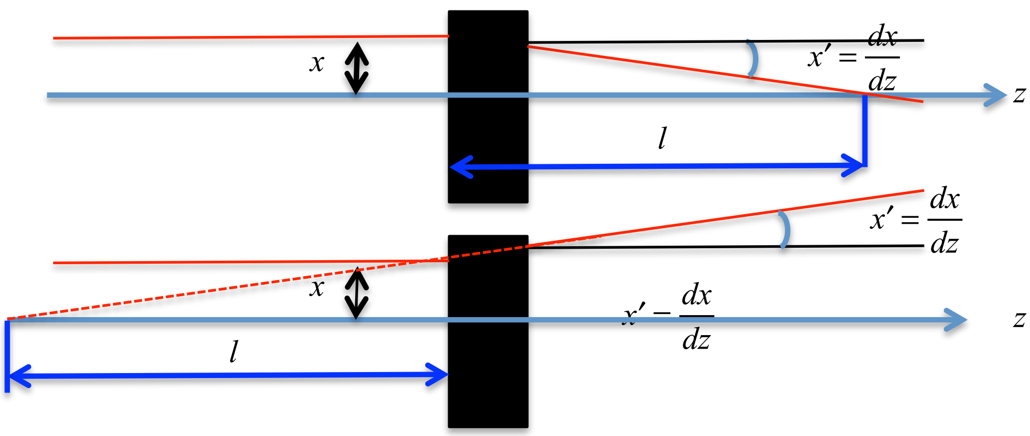
HW 1 (5 point): Let's first determine an effective focal length, *F*, of the of a paraxial (e.g. small angles!) focusing object (a black-box) as ratio between a parallel displacement of trajectory at its entrance to corresponding change of the angle at its exit (see figure below):

$$F = -\frac{x}{x'}; x' \equiv \frac{dx}{dz}$$

see figure below for



For completeness, the distance from the entrance to the object to the trajectory crossing the axis, *l*, in general is not equal to the focal length. In beam optics this is frequently, but not correctly, referred as astigmatism – in contrast, the astigmatism is defined as dependence of the focal strengths on the direction of propagation of the ray (particle).
Let consider a doublet of two thin lenses: a focusing (*F*) and defocusing (*D*) lenses with equal but opposite in sign focal length F with center separated by distance L as in Fig. 1.

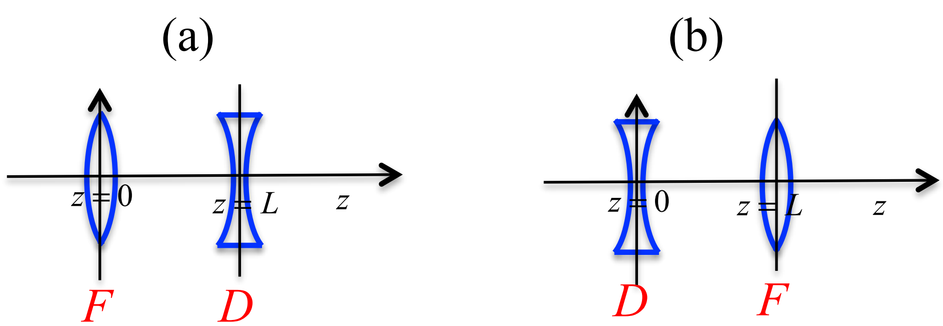


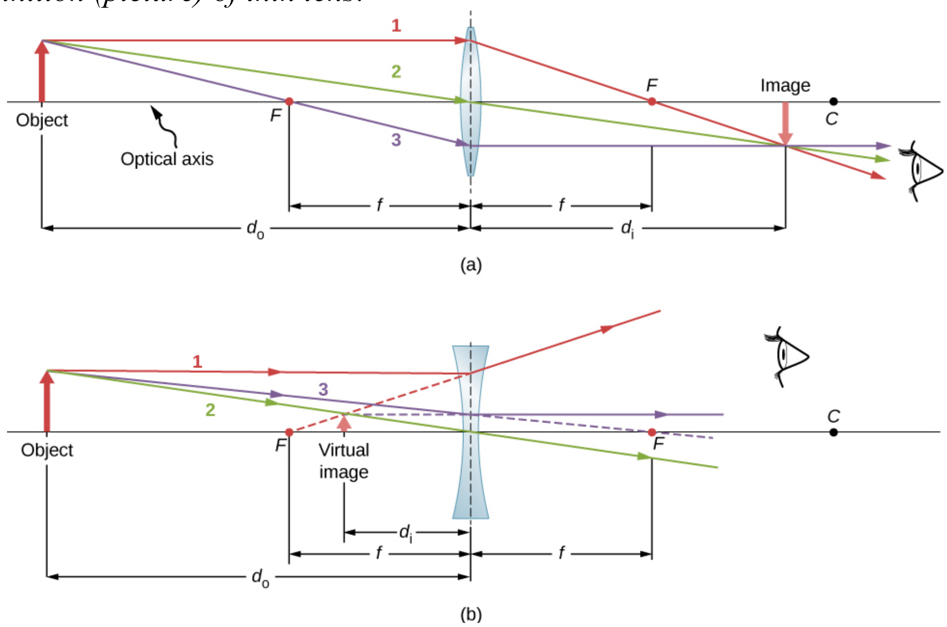
Fig.1. Two combinations of a doublet: *FD* and *DF*.

1. (3 points) Show through a calculation of the ray trajectory that the focal lengths of *FD* and *DF* doublets are equal and given by following expression:

$$F_{eff} = \frac{F^2}{L}$$

2. (2 points) Determine location of the ray crossing the axis and find their difference between *FD* and *DF* doublets – this indeed would be an astigmatism of doublet built from two quadruples.

P.S. Definition (picture) of thin lens:



Solution: In both cases we start from initial conditions

$$x = x_o; x' = 0;$$

and apply following transformations:

$$F\,lens: x_{out} = x_{in}; x'_{out} = x'_{in} - \frac{x_{in}}{F};$$

$$D\,lens: x_{out} = x_{in}; x'_{out} = x'_{in} + \frac{x_{in}}{F};$$

$$Drift: x_{out} = x_{in} + Lx'_{in}; x'_{out} = x'_{in};$$

For FD case is gives us

$$x_1 = x_0; x'_1 = -\frac{x_0}{F} \to x_2 = x_0 - L\frac{x_0}{F}; x'_2 = -\frac{x_0}{F} \to$$

$$x_3 = x_0 - L\frac{x_0}{F}; x'_3 = -\frac{x_0}{F} + \frac{1}{F}\left(x_0 - L\frac{x_0}{F}\right) = L\frac{x_0}{F^2}; \qquad (1)$$

and for DF case

$$x_1 = x_0; x_1' = +\frac{x_0}{F} \rightarrow x_2 = x_0 + L\frac{x_0}{F}; x_2' = +\frac{x_0}{F} \rightarrow$$
$$x_3 = x_0 + L\frac{x_0}{F}; x_3' = +\frac{x_0}{F} - \frac{1}{F}\left(x_0 + L\frac{x_0}{F}\right) = -L\frac{x_0}{F^2}; \tag{2}$$

with x_3', x_3' being the position an the angle at the exit of the :black box". The answer for the first question is coming from $x_3' = -L\frac{x_0}{F^2}$ for both FD and DF cases.

The location of the ray crossing the z-axis coming from dividing the position at the exit of the second lens by the angle and adding L (distance from the starting point):

$$F: \ Z = L - \frac{x_3}{x_3'} = L + \frac{F^2}{L}\left(1 - \frac{L}{F}\right) = L - F + \frac{F^2}{L}$$
$$D: \ Z = L - \frac{x_3}{x_3'} = L + \frac{F^2}{L}\left(1 + \frac{L}{F}\right) = L + F + \frac{F^2}{L} \tag{3}$$

Hence, the astigmatism of FD set is equal to 2F.

HW 2 (2 points): Spectral brightness (sometimes called brilliance) of a light source is defined as

$$B = \frac{dN_{ph}}{dt d\Omega dA\left(d\lambda / \lambda\right)} = \frac{dN_{ph}}{dt d\Omega dA\left(d\omega / \omega\right)};$$

where $\frac{dN_{ph}}{dt}$ is the number of photons per second with the spectral bandwidth $d\omega / \omega$ radiated from an area dA into the solid angle $d\Omega$. The units used for brightness are expressed in photons per second

$$\left[B\right] = \frac{photons}{\sec \cdot mm^2 \cdot mrad^2 \left(10^{-3} d\lambda / \lambda\right)}$$

As an exercise, calculate spectral brightness of NdYAG laser with average power of 10 W, wavelength of λ=1.064 μm, Bandwidth of $\Delta\omega = 700\ GHz$ and with diffraction limited spot size and angular spread:

$$\Delta x \cdot \Delta\theta_x = \frac{\lambda}{4\pi}; \Delta y \cdot \Delta\theta_y = \frac{\lambda}{4\pi}.$$

Solution: First, lets calculate the frequency ω, the bandwidth and the photon energy $\hbar\omega$

$$\omega = 2\pi \frac{c}{\lambda} = 1.77 \cdot 10^{15}\ Hz; \hbar = 1.05459 \cdot 10^{-34};$$

$$\frac{\Delta\omega}{\omega} = 3.95 \cdot 10^{-4} = 0.395 \cdot 10^{-3}; \quad E_{ph} = \hbar\omega = 1.87 \cdot 10^{-19}\ J;$$

Then the number of photons per second:

$$\frac{dN_{ph}}{dt} = \frac{P_{laser}}{\hbar\omega} = 5.36 \cdot 10^{19}\ \frac{photons}{\sec}$$

The product of the area and he angular spread can be calculated for the diffraction limited laser beam as

$$A \cdot \Omega = \Delta x \cdot \Delta y \cdot \Delta\theta_y \cdot \Delta\theta_x = \left(\frac{\lambda}{4\pi}\right)^2 = 7.17 \cdot 10^{-15} m^2 rad^2;$$

$$A \cdot \Omega = 7.17 \cdot 10^{-3} mm^2 mrad^2$$

Hence, using three red numbers, the spectral brightness of this laser is

B= 1.89 10^{22} ph/sec/mm^2/mrad^2/0.1%BW.

HW 3 (3 points): In a fixed Cartesian coordinates for a trajectory with $\frac{dz}{dt} \neq 0$ of a particle moving in magnetic field $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$ equation for its trajectory can be written in terms of z as independent variable:

$$\frac{d^2x}{dz^2} = \frac{e}{p}\sqrt{1+x'^2+y'^2}\left(y'B_z - (1+x'^2)B_y + x'y'B_x\right);$$
$$\frac{d^2y}{dz^2} = -\frac{e}{p}\sqrt{1+x'^2+y'^2}\left(x'B_z - (1+y'^2)B_x + x'y'B_y\right);$$
$$x' \equiv \frac{dx}{dz}; y' \equiv \frac{dy}{dz};$$

where *e* is the particle's charge and $p = \gamma m \mathrm{v}$ is its relativistic momentum.

Hint: consider constants of motion in a magnetic field.

Solution: Equation of motion with time as independent variable are:

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\left[\vec{\mathrm{v}} \times \vec{B}\right]; \vec{p} = \gamma m \vec{\mathrm{v}},$$

which with addition of the fact that energy is a integral of motion in magnetic field $E = \gamma m c^2 = const$ yields:

$$\frac{d\vec{\mathrm{v}}}{dt} = \frac{e}{\gamma mc}\left[\vec{\mathrm{v}} \times \vec{B}\right]; \vec{\mathrm{v}} = \hat{\mathrm{x}}\frac{d\mathrm{x}}{dt} + \hat{y}\frac{d\mathrm{y}}{dt} + \hat{z}\frac{d\mathrm{z}}{dt} = \hat{\mathrm{x}}\dot{x} + \hat{y}\dot{y} + \hat{z}\dot{z}$$
$$\vec{\mathrm{v}}^2 \equiv \mathrm{v}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = const;$$
$$\frac{d^2\mathrm{x}}{dt^2} = \frac{e}{\gamma mc}\left(B_z\frac{d\mathrm{y}}{dt} - B_y\frac{d\mathrm{z}}{dt}\right); \frac{d^2\mathrm{y}}{dt^2} = \frac{e}{\gamma mc}\left(B_x\frac{d\mathrm{z}}{dt} - B_z\frac{d\mathrm{x}}{dt}\right); \tag{1}$$
$$\frac{d^2\mathrm{z}}{dt^2} = \frac{e}{\gamma mc}\left(B_y\frac{d\mathrm{x}}{dt} - B_x\frac{d\mathrm{y}}{dt}\right);$$

which we need to transfer to equation of motion with z as independent coordinate. We shall start from expressing *dt* ins term of *dz*:

$$\mathrm{v}^2 dt^2 = dx^2 + dy^2 + dz^2 = \left(1 + \left(\frac{d\mathrm{x}}{dz}\right)^2 + \left(\frac{d\mathrm{y}}{dz}\right)^2\right)dz^2;$$
$$dt = \sqrt{1+x'^2+y'^2}\,\frac{dz}{\mathrm{v}}; x' \equiv \frac{d\mathrm{x}}{dz}; y' \equiv \frac{d\mathrm{y}}{dz}; z' \equiv \frac{d\mathrm{z}}{dz} = 1.$$

where we are using traditional for accelerator definition of dimensionless derivatives x', y'. Establishing rules for transformation for derivatives

$$\dot{f} \equiv \frac{df}{dt} = \frac{\mathrm{v}}{\sqrt{1+x'^2+y'^2}}\frac{df}{dz} \equiv \mathrm{v}\frac{f'}{\sqrt{1+x'^2+y'^2}}$$

$$\ddot{f} \equiv \frac{d^2 f}{dt^2} \equiv \frac{\mathrm{v}^2}{\sqrt{1+x'^2+y'^2}}\frac{d}{dz}\frac{f'}{\sqrt{1+x'^2+y'^2}};$$

$$= \frac{\mathrm{v}^2}{1+x'^2+y'^2} f'' - \frac{\mathrm{v}^2 f'\left(x'x''+y'y''\right)}{\left(1+x'^2+y'^2\right)^2} =$$

$$\mathrm{v}^2 \frac{f''\left(1+x'^2+y'^2\right) - f'\left(x'x''+y'y''\right)}{\left(1+x'^2+y'^2\right)^2} =$$

we can rewrite (1) as

$$x''\left(1+x'^2+y'^2\right) - x'\left(x'x''+y'y''\right) = \frac{e}{pc}\left(B_z y' - B_y\right)\left(1+x'^2+y'^2\right)^{3/2};$$

$$y''\left(1+x'^2+y'^2\right) - y'\left(x'x''+y'y''\right) = \frac{e}{pc}\left(B_x - B_z x'\right)\left(1+x'^2+y'^2\right)^{3/2}; \qquad (2)$$

$$-\left(x'x''+y'y''\right) = \frac{e}{pc}\left(B_y x' - B_x y'\right)\left(1+x'^2+y'^2\right)^{3/2};$$

with important note that because to the absolute value of the velocity is constant, one of these three equation is redundant! We can easily resolve first two equations with respect to x'', y'':

$$x''\left(1+y'^2\right) - y''\cdot x'y' = \frac{e}{pc}\left(B_z y' - B_y\right)\left(1+x'^2+y'^2\right)^{3/2};$$

$$y''\left(1+x'^2\right) - x''\cdot x'y' = \frac{e}{pc}\left(B_z - B_z x'\right)\left(1+x'^2+y'^2\right)^{3/2}; \qquad (3)$$

$$\begin{bmatrix} 1+y'^2 & -x'y' \\ -x'y' & 1+x'^2 \end{bmatrix}\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \frac{e}{pc}\left(1+x'^2+y'^2\right)^{3/2}\begin{bmatrix} B_z y' - B_y \\ B_x - B_z x' \end{bmatrix};$$

I suggest that you check for yourself this simple 2x2 matrix manipulations:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to M^{-1} = \frac{1}{\det M}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}; \det\begin{bmatrix} 1+y'^2 & -x'y' \\ -x'y' & 1+x'^2 \end{bmatrix} == 1+x'^2+y'^2$$

$$\begin{bmatrix} 1+y'^2 & -x'y' \\ -x'y' & 1+x'^2 \end{bmatrix}^{-1} = \frac{1}{1+x'^2+y'^2}\begin{bmatrix} 1+x'^2 & x'y' \\ x'y' & 1+y'^2 \end{bmatrix} \qquad (3)$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \frac{e}{pc}\sqrt{1+x'^2+y'^2}\begin{bmatrix} 1+x'^2 & x'y' \\ x'y' & 1+y'^2 \end{bmatrix}\begin{bmatrix} B_z y' - B_y \\ B_x - B_z x' \end{bmatrix};$$

to get

$$x'' = \frac{e}{pc}\sqrt{1+x'^2+y'^2}\left\{\left(1+x'^2\right)\left(B_z y' - B_y\right) + x'y'\left(B_x - B_z x'\right)\right\};$$
$$y'' = \frac{e}{pc}\sqrt{1+x'^2+y'^2}\left\{x'y'\left(B_z y' - B_y\right) + \left(1+y'^2\right)\left(B_x - B_z x'\right)\right\};$$

which gives the final result

$$\begin{aligned} x'' &= \frac{e}{pc}\sqrt{1+x'^2+y'^2}\left\{x'y'B_x - \left(1+x'^2\right)B_y + y'B_z\right\}; \\ y'' &= \frac{e}{pc}\sqrt{1+x'^2+y'^2}\left\{\left(1+y'^2\right)B_x - x'y'B_y - x'B_z\right\}; \end{aligned} \tag{5}$$

which differs from one given in the problem only by ordering of terms in the brackets and using SGS units – hence extra *c* in the denominator.

Just as a sanity check, we check that third equation in (2) we had dropped is indeed redundant< Using expressions for x'', y'' from (5)

$$\frac{x'x'' + y'y''}{\frac{e}{pc}\sqrt{1+x'^2+y'^2}} =$$
$$x'\left(x'y'B_x - \left(1+x'^2\right)B_y + y'B_z\right) - y'\left(\left(1+y'^2\right)B_x - x'y'B_y - x'B_z\right) =$$
$$\left(1+y'^2+x'^2\right)\left(B_x y' - x'B_y\right)$$

which is identical to the third equation in (2). No surprise here – it is a consequence of the constant velocity, momentum and energy.