

## Home Work PHY 554 #9

**HW 1 (2 points):** Calculate relations between three dimensionless infinitesimal parameters:

$$\frac{dE}{E} \equiv \frac{d\gamma}{\gamma}; \frac{dp}{p} \equiv \frac{d(\beta\gamma)}{\beta\gamma}; \frac{dv}{v} \equiv \frac{d\beta}{\beta}$$

where E is energy, p is momentum and v is velocity of a particle. Hint: use relativistic relations between  $\beta$  and  $\gamma$ .

Solution: The easiest way is to use the following

$$\begin{aligned} \beta^2 &= 1 - \gamma^{-2} \rightarrow \beta d\beta = \gamma^{-3} d\gamma \\ \frac{dv}{v} &= \frac{d\beta}{\beta} = \frac{1}{\beta^2 \gamma^2} \frac{d\gamma}{\gamma} = \frac{1}{\beta^2 \gamma^2} \frac{dE}{E}; \\ E^2 &= p^2 c^2 + m^2 c^4 \rightarrow E dE = c^2 p dp; \\ \frac{dp}{p} &= \frac{E^2}{p^2 c^2} \frac{dE}{E} \equiv \frac{1}{\beta^2} \frac{dE}{E}; \\ \frac{dv}{v} &= \frac{1}{\gamma^2} \frac{dp}{p}. \end{aligned}$$

**HW 2 (5 points):** In class we introduced the map of longitudinal motion in a storage ring

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}}{\beta^2 E_o} (\sin \phi_n - \sin \phi_s); \quad (1)$$

$$\phi_{n+1} = \phi_n + 2\pi h \eta \cdot \delta_{n+1},$$

1. For small oscillation variations of the RF phase about the synchronous phase

$$\varphi = \phi - \phi_s; |\varphi| \ll 1$$

linearize the map (1) by keeping only first order on  $\varphi$  and find one turn transport matrix  $M$  for longitudinal motion:

$$\begin{pmatrix} \varphi \\ \delta \end{pmatrix}_{n+1} = M \begin{pmatrix} \varphi \\ \delta \end{pmatrix}_n$$

**Solution:** linearization is straight forward

$$\sin \phi_n = \sin \phi_s + \varphi_n \cos \phi_s + O(\varphi_n^2)$$

$$\delta_{n+1} \equiv \delta_n + \frac{eV_{rf} \cos \phi_s}{\beta^2 E_o} \varphi_n; \quad (1)$$

$$\varphi_{n+1} = \varphi_n + 2\pi h \eta \cdot \delta_{n+1} = 2\pi h \eta \left( \delta_n + \frac{eV_{rf} \cos \phi_s}{\beta^2 E_o} \varphi_n \right);$$

and giving us desirable one turn matrix

$$\begin{pmatrix} \varphi \\ \delta \end{pmatrix}_{n+1} = \begin{pmatrix} 1 + 2\pi h\eta \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o} & 2\pi h\eta \\ \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o} & 1 \end{pmatrix} \begin{pmatrix} \varphi \\ \delta \end{pmatrix}_n; \quad (2)$$

$$M = \begin{pmatrix} 1 + 2\pi h\eta \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o} & 2\pi h\eta \\ \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o} & 1 \end{pmatrix}$$

2. Using Courant-Snyder parametrization we used for transverse motion find value of  $\cos\mu_s, \beta_s, \alpha_s$  in parametric form (e.g. using  $\sin\mu_s = \sqrt{1 - \cos^2\mu_s}$ ,  $\mu_s = 2\pi Q_s = \cos^{-1}(\cos\mu_s)$ ).

**Solution:**

$$M = \begin{pmatrix} 1 + 2\pi h\eta \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o} & 2\pi h\eta \\ \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o} & 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_s + \alpha_s \sin\mu_s & \beta_s \sin\mu_s \\ \gamma_s \sin\mu_s & \cos\mu_s - \alpha_s \sin\mu_s \end{pmatrix};$$

$$A = \pi h\eta \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o}; \cos\mu_s = \frac{1}{2} \text{Trace} M = 1 + A; \quad (2)$$

$$\sin\mu_s = \sqrt{-2A - A^2}; \alpha_s = \frac{A}{\sqrt{-2A - A^2}}; \beta_s = \frac{2\pi h\eta}{\sqrt{-2A - A^2}}$$

Motion is stable when  $A(2+A) < 0$ ;  $-2 < A < 0$ . We derived first  $A < 0$  condition in our class. Second condition  $A > -2$ ; protest from never observed over-focusing by RF cavity...

3. Assuming that  $\mu_s \ll 1$ , find analytical expression for synchrotron tune and compare it with that we found in Lecture 12:

**Solution:** for  $\mu_s \ll 1$

$$\cos\mu_s = 1 - \frac{\mu_s^2}{2} + O(\mu_s^4) = 1 + 2\pi h\eta \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o};$$

$$\mu_s = \sqrt{-2\pi h\eta \frac{eV_{rf} \cos\phi_s}{\beta^2 E_o}}; Q_s = \frac{\mu_s}{2\pi} = \sqrt{-\frac{h\eta eV_{rf} \cos\phi_s}{2\pi \beta^2 E_o}}$$

with synchrotron tune, naturally, identical to that derived in Lecture 12 when  $\eta \cos\phi_s < 0$ .

**HW 3 (3 points):** For our example in lecture 12, find the synchrotron tunes for 100 GeV and 15 GeV protons in a storage ring for the following parameters (similar to RHIC collider at BNL):

RF voltage,

$$V = 500 \text{ kV}$$

Depending on the sign of the slip factor the synchronous phase is zero or 180 degrees,

$$\phi_s = 0, \pi - \text{it is also called zero crossing}$$

Harmonic number,

$$h = 360$$

Compaction factor,

$$\alpha_c = 0.002$$

**Solution:** for zero crossing  $\sin\phi_s = 0$  and proper sign  $\eta \cos\phi_s < 0$  we have

$$Q_s = \sqrt{\frac{h|\eta| eV_{rf}}{2\pi \beta^2 E_0}}$$

where we should add relation between compaction and slip factors:

$$\eta = \alpha_c - \frac{1}{\gamma^2}$$

The rest requires to know the rest mass of proton which is 0.9382720 GeV. The rest is just crunching the numbers:

Energy, GeV	15	100
$\gamma$	15.987	106.579
$\beta$	0.9980	1.0000
V, GeV	0.0005	0.0005
h	360	360
$\alpha_c$	0.002	0.002
$\eta$	-0.00191	0.00191
Qs	0.00191	0.00074