PHY 554

Fundamentals of Accelerator Physics

Lecture 10: RF accelerators, fundamentals

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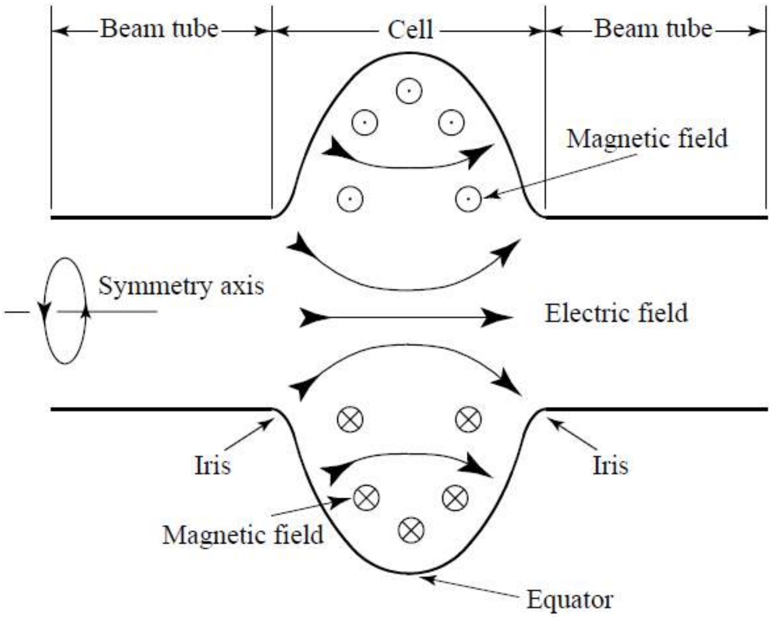


Realistic RF cavity (linac)
Figures of merit

- Final conductivity of the surfaces
  - Approximation of the boundary conditions
  - Surface impedance, losses in the surface
- Main RF cavity characteristics
  - Accelerating voltage, peak electric and magnetic field
    - Q factor: internal, external, total
    - Geometrical factor, G
    - Shunt impedance R_{sh}, R_{sh}/Q
    - Coupling coefficient, *ONE MORE β!*

This part is usually related to more "engineering" factors measured in ohms, watts, etc.… – hence, for a change, we are using SI system…

Again, the main idea of this course: you are learning accelerator lingo and basis behind it

Maxwell Equations in vacuum

- Plane waves and oscillating fields
- This is the simplest way of getting into the waveguides and cavities

SI

$$\vec{\nabla}\cdot\vec{\mathbf{E}} \equiv div\vec{\mathbf{E}} = 0; \qquad \vec{\nabla}\cdot\vec{\mathbf{B}} \equiv div\vec{\mathbf{B}} = 0;$$

$$\vec{\nabla}\times\vec{\mathbf{E}} \equiv curl\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \qquad \vec{\nabla}\times\vec{\mathbf{B}} \equiv curl\vec{\mathbf{B}} = \varepsilon_o\mu_o\frac{\partial\vec{\mathbf{E}}}{\partial t}$$

SGS

$$\vec{\nabla}\cdot\vec{\mathbf{E}} \equiv div\vec{\mathbf{E}} = 0; \qquad \vec{\nabla}\cdot\vec{\mathbf{B}} \equiv div\vec{\mathbf{B}} = 0;$$

$$\vec{\nabla}\times\vec{\mathbf{E}} \equiv curl\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{c\partial t} \qquad \vec{\nabla}\times\vec{\mathbf{B}} \equiv curl\vec{\mathbf{B}} = \frac{\partial\vec{\mathbf{E}}}{c\partial t}$$

- By simple manipulation they reduced to plane waves

$$\vec{\nabla}^2\cdot\vec{\mathbf{E}} - \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = 0; \qquad \vec{\nabla}^2\cdot\vec{\mathbf{B}} - \frac{1}{c^2}\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2} = 0$$

$$\Rightarrow$$

$$u = u\left(ct - \vec{n}\cdot\vec{r}\right); |\vec{n}| = 1$$

$$\vec{\nabla}^2\cdot u - \frac{1}{c^2}\frac{\partial u}{\partial t} = \left(|\vec{n}|^2 - 1\right)u \equiv 0\,\#$$

- With most interesting for us oscillating solutions

$$\vec{\mathbf{E}} = \mathrm{Re}\,\vec{\mathbf{E}}_o e^{i(\vec{k}\vec{r}-\omega t)}; \ |\vec{k}| = \frac{\omega}{c}; \qquad \mathbf{E}_{x,y,z} = \mathbf{E}_{ox,y,z}\cos(\vec{k}\vec{r} - \omega t + \varphi_{x,y,z})$$

$$\vec{\mathbf{B}} = \mathrm{Re}\,\vec{\mathbf{B}}_o e^{i(\vec{k}\vec{r}-\omega t)}; \ \vec{k} = \frac{\omega}{c}\vec{n}; \qquad \mathbf{B}_{x,y,z} = \mathbf{B}_{ox,y,z}\cos(\vec{k}\vec{r} - \omega t + \phi_{x,y,z})$$

$$+$$

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = 0 \rightarrow \vec{k}\cdot\vec{\mathbf{E}} = 0; \ \vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \rightarrow \vec{k}\cdot\vec{\mathbf{B}} = 0;$$

$$\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{1}{c}\frac{\partial\vec{\mathbf{B}}}{\partial t}_o \rightarrow \vec{k}\times\vec{\mathbf{E}} = i\omega\vec{\mathbf{B}} \Rightarrow |\vec{\mathbf{E}}| = |\vec{\mathbf{B}}|$$

$$\vec{\mathbf{E}} \perp \vec{\mathbf{B}} \perp \vec{n}$$

- So called transverse electromagnetic waves

The key for oscillating EM waves in vacuum

$$k = \frac{2\pi}{\lambda};\ \omega = 2\pi f$$

$$\vec{k}^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

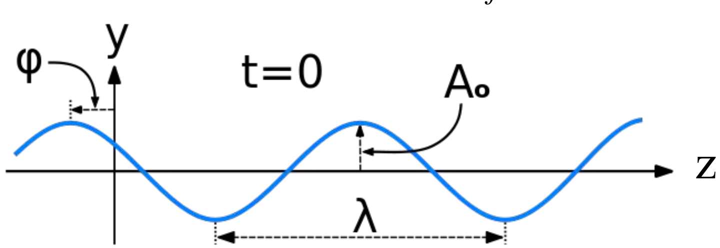
Simple things to remember

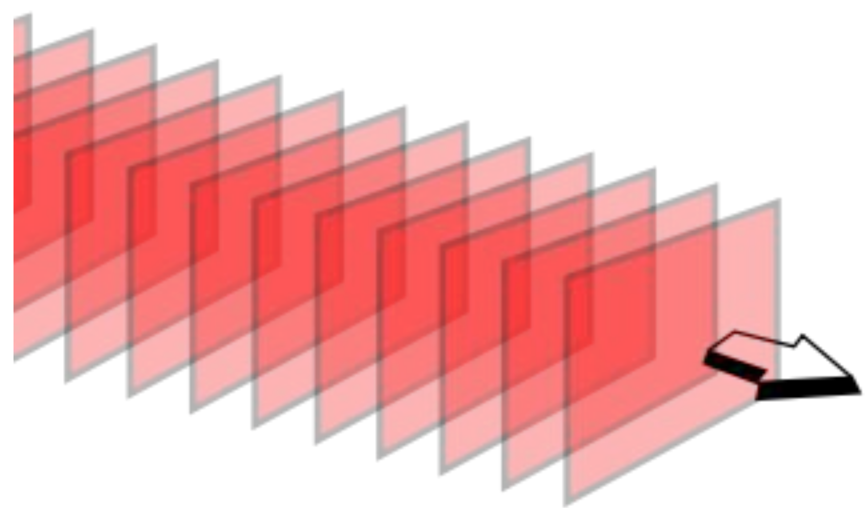
- Superposition principle: if pairs $(\vec{\mathbf{E}}_1,\vec{\mathbf{B}}_1)$ and $(\vec{\mathbf{E}}_2,\vec{\mathbf{B}}_2)$ are solutions of Maxwell equations, their linear combination $(\alpha\vec{\mathbf{E}}_1+\beta\vec{\mathbf{E}}_2,\alpha\vec{\mathbf{B}}_1+\beta\vec{\mathbf{B}}_2)$ is also a solution of Maxwell equations
- Plane transverse electro-magnetic (TEM) wave have an oscillation frequency ω and direction of propagation
- The electric and magnetic fields are perpendicular to each other and to direction of propagation
- Each component of the field oscillates as a sine-wave
- Components of the field and their phases determine wave's polarization
- Any plane wave can be presented as superposition of two waves with linear polarization

$$\vec{\mathbf{E}} = \boxed{\hat{x}\cdot\mathbf{E}_x\cos(kz-\omega t+\varphi_x)} + \boxed{\hat{y}\mathbf{E}_y\cos(kz-\omega t+\varphi_y);}$$

$$\vec{\mathbf{B}} = \boxed{\hat{y}\cdot\frac{\mathbf{E}_x}{c}\sin(kz-\omega t+\varphi_x)} - \boxed{\hat{x}\frac{\mathbf{E}_y}{c}\sin(kz-\omega t+\varphi_y);}$$

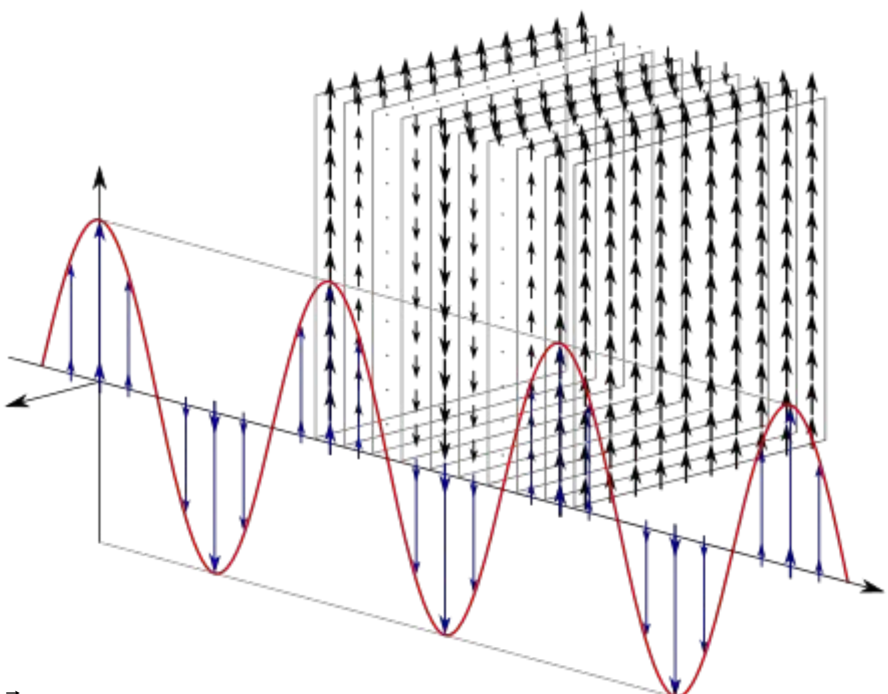
$$\vec{n} = \hat{z};\;\; \lambda \equiv \frac{2\pi}{k} \equiv \frac{2\pi c}{\omega} \equiv \frac{c}{f}$$





Linearly polarized and circularly polarized plane TEM waves

Linear

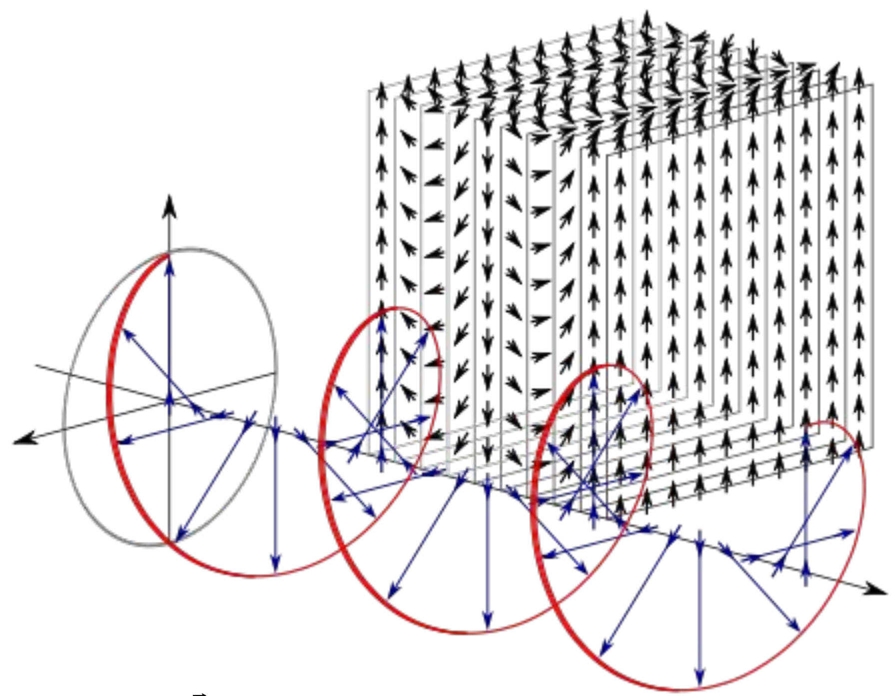


$$\vec{\mathbf{E}} = \hat{x}\cdot\mathbf{E}_x\cos(kz-\omega t+\varphi_x)+\hat{y}\mathbf{E}_y\cos(kz-\omega t+\varphi_y);$$

$$\vec{\mathbf{B}} = \hat{y}\cdot\frac{\mathbf{E}_x}{c}\sin(kz-\omega t+\varphi_x)-\hat{x}\frac{\mathbf{E}_y}{c}\sin(kz-\omega t+\varphi_y);$$

$$\varphi_y = \varphi_y$$

Circular



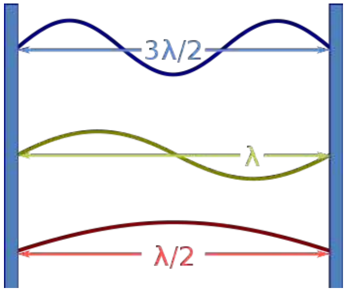
$$\vec{\mathbf{E}} = \hat{x}\cdot\mathbf{E}_x\cos(kz-\omega t+\varphi_x)+\hat{y}\mathbf{E}_y\cos(kz-\omega t+\varphi_y);$$

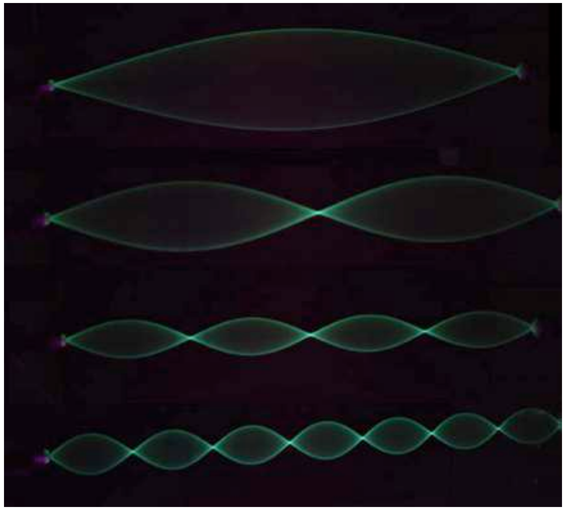
$$\vec{\mathbf{B}} = \hat{y}\cdot\frac{\mathbf{E}_x}{c}\sin(kz-\omega t+\varphi_x)-\hat{x}\frac{\mathbf{E}_y}{c}\sin(kz-\omega t+\varphi_y);$$

$$\varphi_y = \varphi_y + \frac{\pi}{2}$$

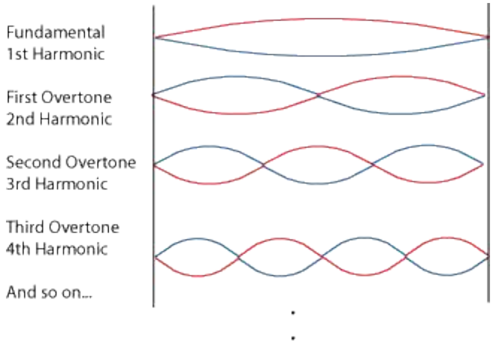
Standing waves

Sting vibrations



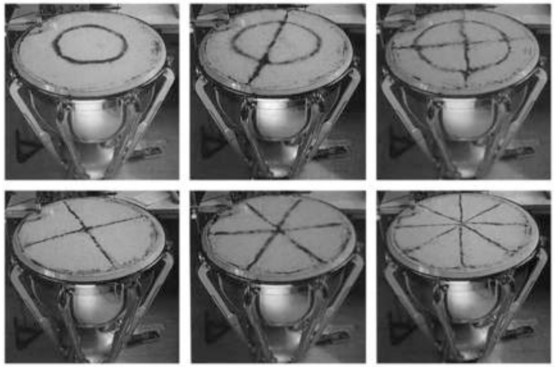


$$\boxed{a=N\lambda/2}$$



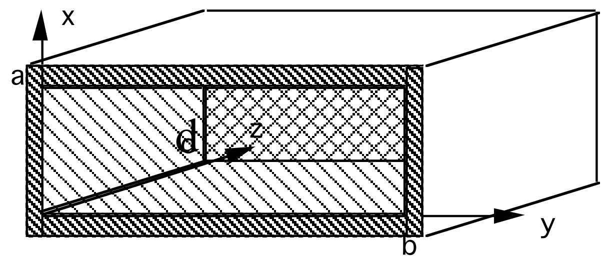
Acoustic resonators

Water waves



Building a match-box cavity

- Since we are constructing it for accelerating we will need to use an TM mode which has non-zero electric field along z-axis
- Expression is a bit cumbersome, but still a simple combination of plane waves



$$\mathbf{E}_z = -E_z \cdot \sin(k_x x)\sin(k_y y)\left\{\sin(k_z z - \omega t + \varphi) - \sin(k_z z + \omega t + \varphi)\right\}/2 = E_z \cdot \sin(k_x x)\sin(k_y y)\sin(k_z z + \varphi)\sin(\omega t)$$

$$\mathbf{E}_x = -\frac{k_x k_z}{k_x^2 + k_y^2} E_z \cos(k_x x)\sin(k_y y)\left\{\cos(k_z z - \omega t + \varphi) + \cos(k_z z + \omega t + \varphi)\right\}/2 = -\frac{k_x k_z}{k_x^2 + k_y^2} E_z \cos(k_x x)\sin(k_y y)\cos(k_z z + \varphi)\cos(\omega t)$$

$$\mathbf{E}_y = -\frac{k_y k_z}{k_x^2 + k_y^2} E_z \sin(k_x x)\cos(k_y y)\left\{\cos(k_z z - \omega t + \varphi) + \cos(k_z z + \omega t + \varphi)\right\}/2 = -\frac{k_y k_z}{k_x^2 + k_y^2} E_z \sin(k_x x)\cos(k_y y)\cos(k_z z + \varphi)\cos(\omega t)$$

- Solution is straight forward and in addition to the solution for boundary conditions give use the resonant frequency of the TM_{MNK} mode

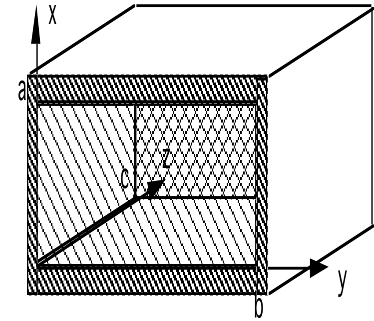
$$k_z = K\frac{\pi}{d}; K = 0,1,2...$$

$$\omega_{res} = c\sqrt{\left(\frac{M}{a}\right)^2 + \left(\frac{N}{b}\right)^2 + \left(\frac{K}{d}\right)^2}$$

- Since we are interested in the fastest way of acceleration, K=0 gives us the best case scenario – constant amplitude of the accelerating field
- It also turns transverse components of the electric field into zero! since $k_z = 0$
- Let also select a=b and M=N=1

$$\boxed{\omega = \sqrt{2}\,\frac{c}{a}}$$

Matchbox cavity a=b



EM Energy

$$\mathbf{E}_z = E_z \cdot \cos\left(\pi \frac{x}{a}\right)\cos\left(\pi \frac{y}{a}\right)\sin(\omega t);$$

$$E = \int\left(\varepsilon_o \frac{\vec{\mathbf{E}}^2}{2} + \frac{\vec{\mathbf{B}}^2}{2\mu_o}\right)dV = \frac{\varepsilon_o}{2}\int\left(\vec{\mathbf{E}}^2 + c^2\vec{\mathbf{B}}^2\right)dV = \varepsilon_o E_z^2 \cdot \frac{a^2 d}{8}$$

$$\mathbf{B}_x = \frac{1}{\sqrt{2}c} E_z \cdot \cos\left(\pi \frac{x}{a}\right)\sin\left(\pi \frac{y}{a}\right)\cos(\omega t)$$

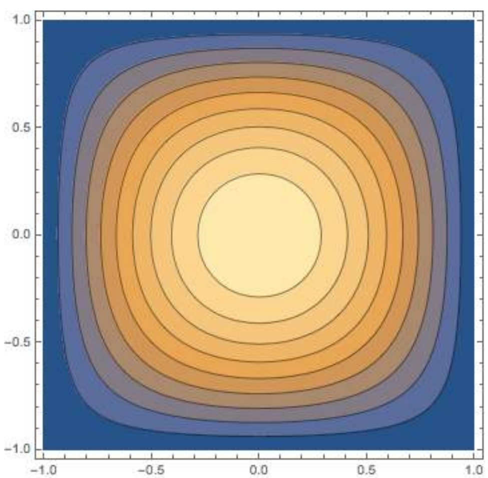
$$\mathbf{B}_y = \frac{1}{\sqrt{2}c} E_z \sin\left(\pi \frac{x}{a}\right)\cos\left(\pi \frac{y}{a}\right)\cos(\omega t)$$

Oscillates between electric field magnetic field

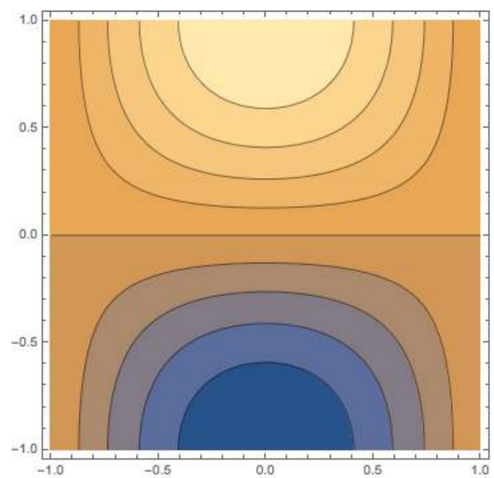
$$E_E = \int \vec{\mathbf{E}}^2\, dV = \frac{E_z^2 a^2 d}{4}\sin^2(\omega t)$$

$$E_M = c^2 \int \vec{\mathbf{B}}^2\, dV = \frac{E_z^2 a^2 d}{4}\cos^2(\omega t)$$

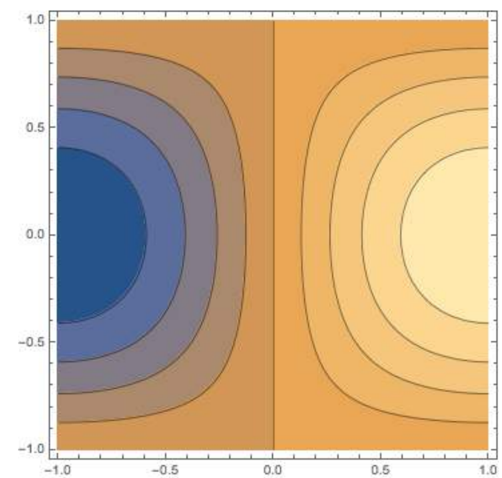
$\mathbf{E}_z$ $\sin(\omega t) = 1$



$\mathbf{B}_x$ $\cos(\omega t) = 1$

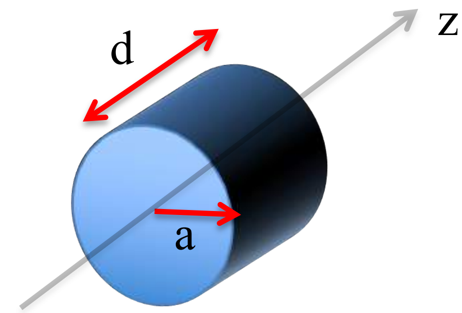


$\mathbf{B}_y$ $\cos(\omega t) = 1$



SI

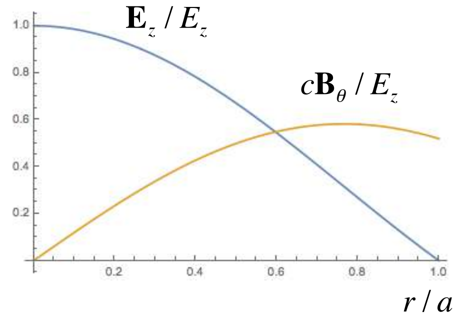
Pillbox Cavity



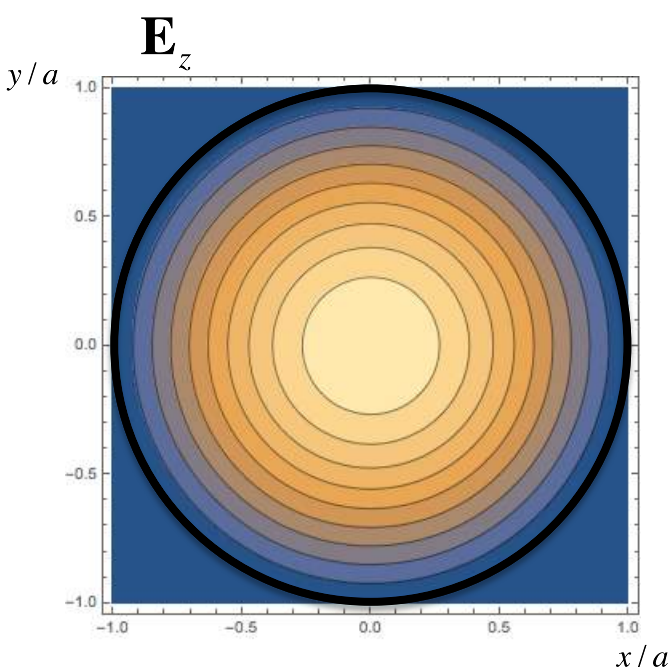
- Similarly to a previous exercise, we need to pick TM mode to have non-zero E_z component
- We also select TEM01 waveguide mode and $k_z=0$

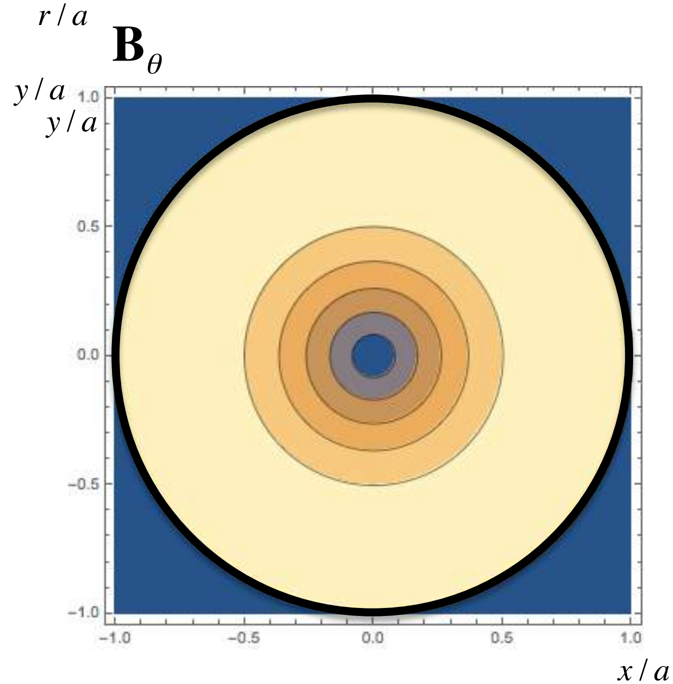
$$\mathbf{E}_z = E_z \cdot J_o\left(2.405\frac{x}{a}\right)\sin(\omega t);$$

$$\mathbf{B}_\theta = \frac{1}{c}E_z \cdot J_1\left(2.405\frac{x}{a}\right)\cos(\omega t)$$



EM energy oscillates between electric field magnetic field: Peaking at the same value: total energy

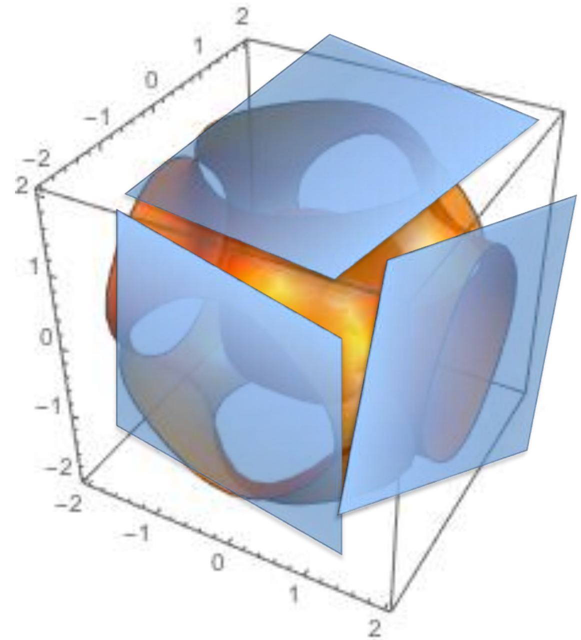




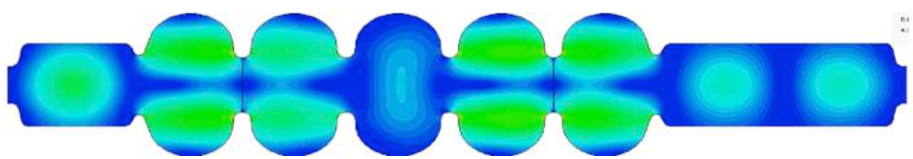
SI

Arbitrary Shape Cavity

- Each **closed** cavity has countable, but infinite number of modes
- Each mode has its own resonant frequency – the EM field having this structure oscillates with this frequency – it can not oscillate at any other frequency
- The energy is bouncing back and forth between the electric and magnetic fields
- It possible to show that average energy stored in magnetic and electric fields are equal



$$\boxed{\begin{aligned} \vec{\mathbf{E}} &= \vec{\mathbf{E}}_o(\vec{r})\cos\left(\omega t + \varphi(\vec{r})\right) \\ \vec{\mathbf{B}} &= \vec{\mathbf{B}}_o(\vec{r})\sin\left(\omega t + \psi(\vec{r})\right) \\ \int \vec{\mathbf{E}}_o^2\, dV &= c^2 \int \vec{\mathbf{B}}_o^2\, dV \end{aligned}}$$

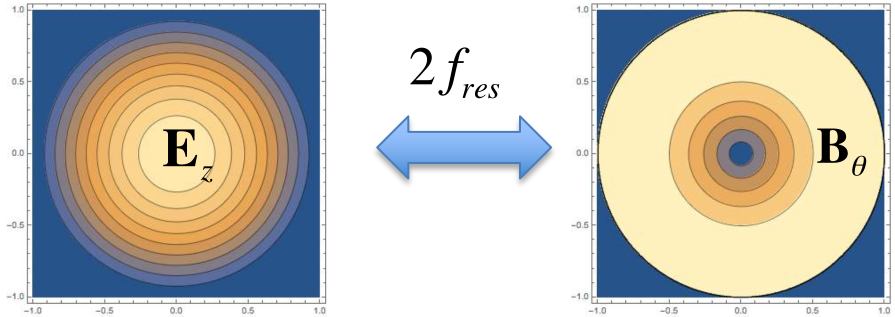


- Each mode has full analogy with a resonant LC circuit or a mechanical oscillator: energy stored in electric field can be compared to potential energy, and energy stored in magnetic filed – to kinetic energy

SI

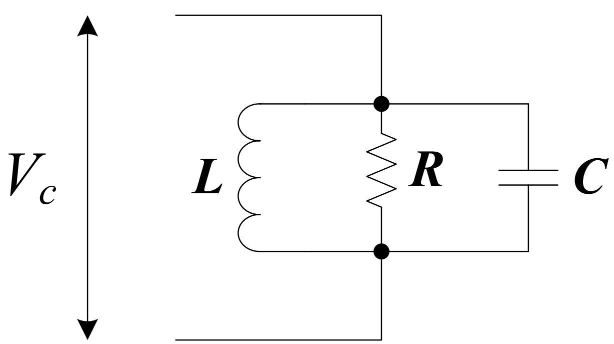
EM Cavity

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_o(\vec{r})\cos(\omega t + \varphi(\vec{r}))$$
$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_o(\vec{r})\sin(\omega t + \psi(\vec{r}))$$
$$\int \vec{\mathbf{E}}_o^{\,2}\, dV = \int \vec{\mathbf{B}}_o^{\,2}\, dV \ \ (\cdot c^2 \ for\ SI)$$

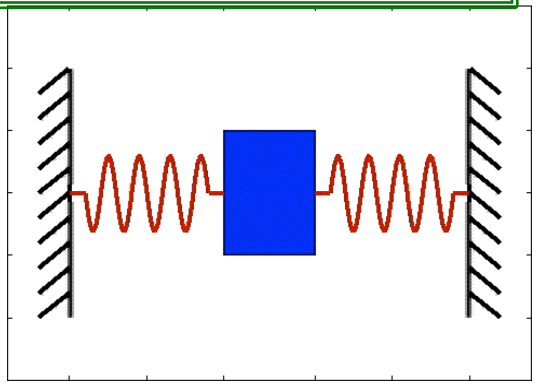


- Each mode has full analogy with a resonant LC circuit or a mechanical oscillator: energy stored in electric field can be compared to potential energy, and energy stored in magnetic filed – to kinetic energy
- Typical energy stored in 5 cell, 700 MHz SRF cavity operating at 20 MV/m is ~ 500 J
- What much more impressive is the intra-cavity power of about 2,000 GW!

$$E_M = \frac{LI^2}{2}; E_E = \frac{Q^2}{2C}; \omega_o = \frac{1}{\sqrt{LC}}$$



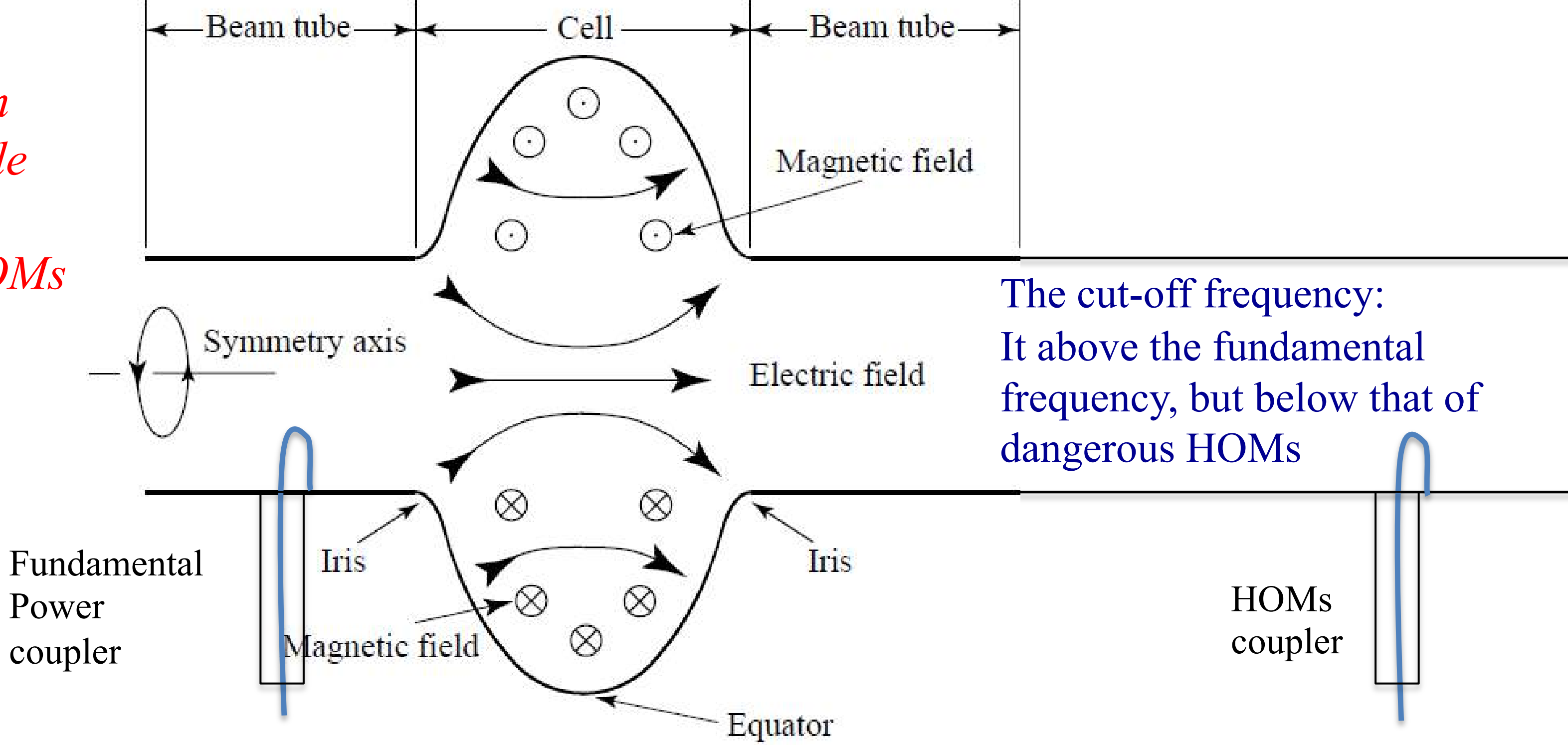
$$E_K = \frac{Mv^2}{2}; E_P = \frac{Kx^2}{2}; \omega_o = \sqrt{\frac{K}{M}}$$



Single cell SRF Cavity

The shape and the size of the cavity determines the resonant frequency for the fundamental mode ω_o

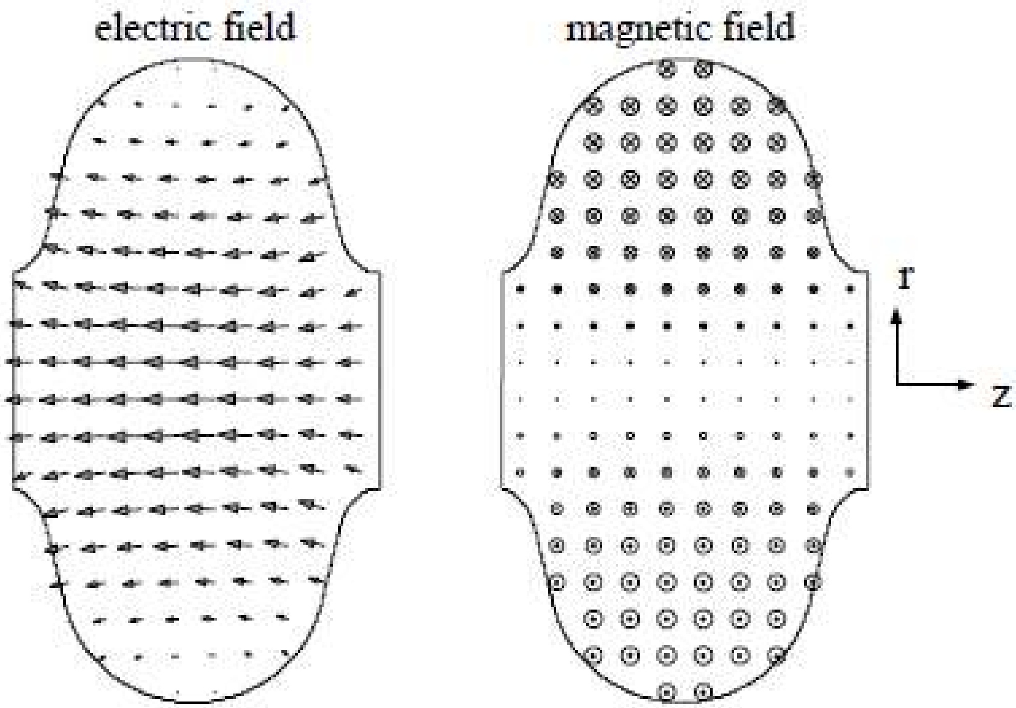
Modes other than fundamental mode are called high-order modes -HOMs



The cut-off frequency:
It above the fundamental frequency, but below that of dangerous HOMs

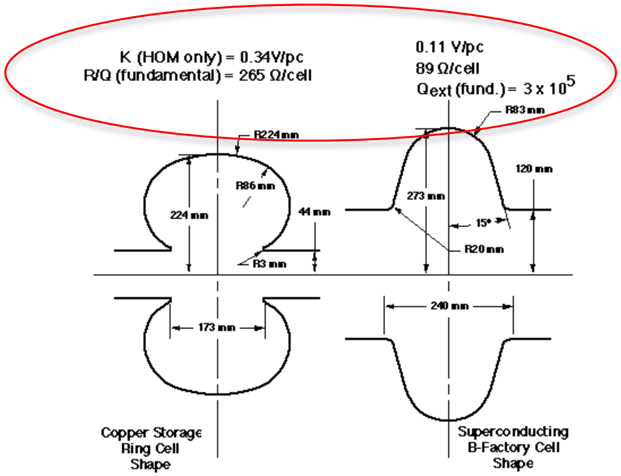
- Fundamental (eigen) mode is trapped inside the cavity and decays exponentially inside the pipe (waveguideıde with $\omega_c > \omega_o$)
- Fundamental power coupler delivers the power from RF transmitter at resonant (eigen) frequency $f_o = \omega_o / 2\pi$ and excites the EM field in the cavity
- HOM couplers are usually used to suppress undesirable fields on the cavity by damping them/ They are usually placed where fundamental frequency fields are very low but HOMs are strong. In this case strong damping of HOM preserves high Q-factor (to be discussed next) at the fundamental frequency

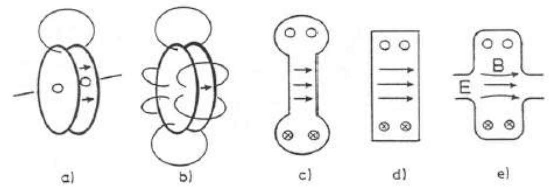
Typical field diagrams

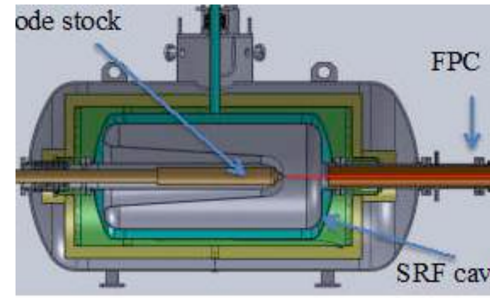


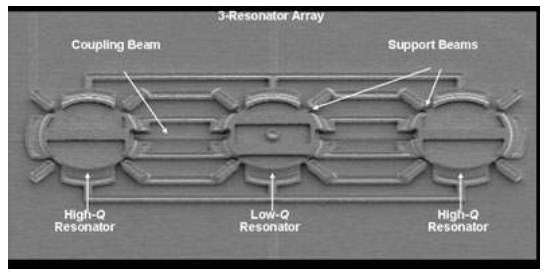
RF cavities come in many shapes, forms and sizes

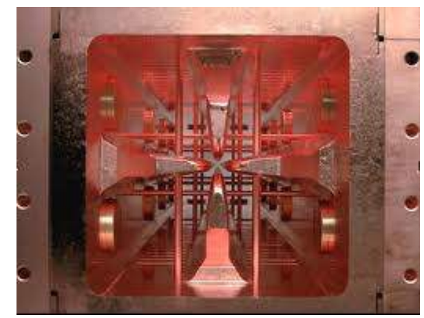
What these mean?

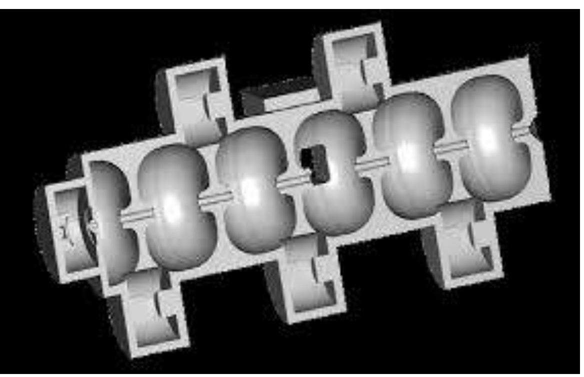


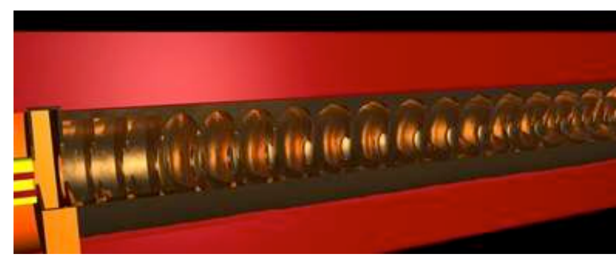






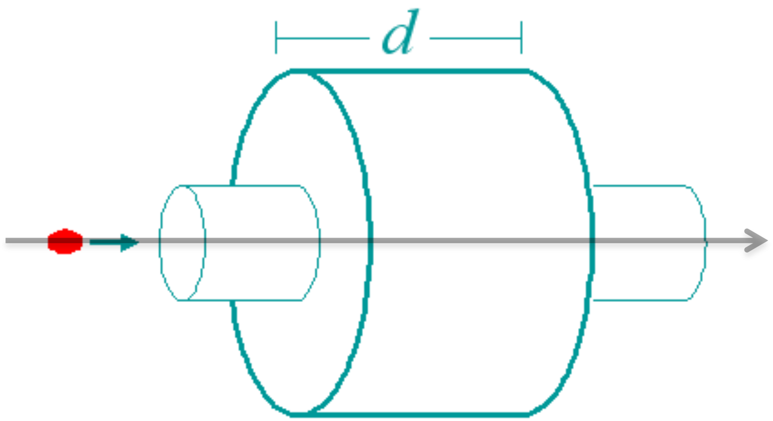






Acceleration inside RF cavity

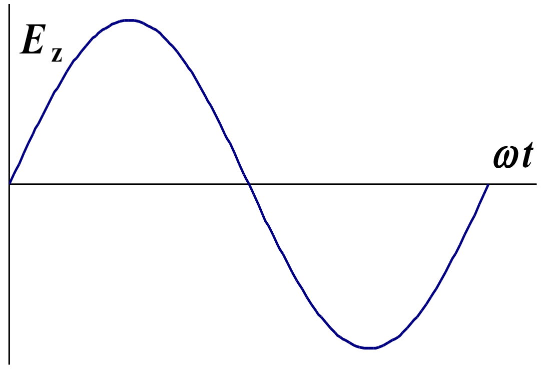
- Let's consider a pillbox cavity terminated by a vacuum pipe for particles to pass
- Let's also consider a charge particle passing on the axis of the cavity the cavity with constant velocity



- Electric field on the axis is

$$\mathbf{E_z}(z,t) = \mathbf{E_o}(z)\cos(\omega_0 t + \varphi)$$

- Specific form of $E_o(z)$ depends on the cavity design
- Energy change of the particle with charge q passing through the cavity is:



$$\Delta E = q\int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos(\omega_0 t + \varphi)\,dz$$

$$t = \frac{z}{\mathrm{v}} \Rightarrow \Delta E = q\int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos\left(\omega_0 \frac{z}{\mathrm{v}} + \varphi\right)dz$$

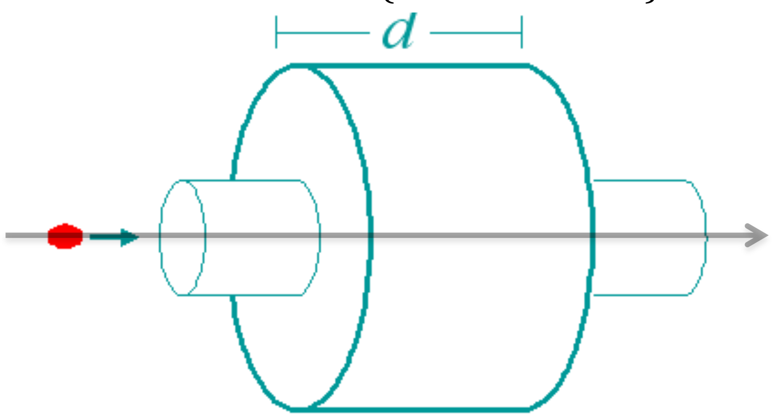
$$\Delta E = qV_{RF}\cos(\varphi + \varphi_o)$$

$$V_{RF} = \sqrt{V_s^2 + V_c^2};\tan(\varphi_o) = \frac{V_c}{V_s};\;\; V_c = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz;\, V_s = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\sin\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz$$

Acceleration inside RF cavity (cont..)

- Now let's consider a pillbox cavity where E_z field is constant and extends from –*d/2* to +*d/2*. Field decays very fact in the pipe

$$\mathbf{E_o}(z) = \begin{Bmatrix} \mathbf{E_o}, |z| \le d/2 \\ 0, |z| > d/2 \end{Bmatrix}$$



$$V_s = \mathbf{E_o} \int_{-d/2}^{d/2} \sin\left(\omega_0 \frac{z}{\mathrm{v}}\right) dz = 0$$

$$V_c = \mathbf{E_o} \int_{-d/2}^{d/2} \cos\left(\omega_0 \frac{z}{\mathrm{v}}\right) dz = \mathbf{E_o} \frac{2\mathrm{v}}{\omega_0} \cdot \sin\frac{\omega_0 d}{2\mathrm{v}} \Rightarrow V_c = \mathbf{E_o} d \cdot \frac{\sin X_t}{X_t}; X_t = \frac{\omega_0 d}{2\mathrm{v}}$$

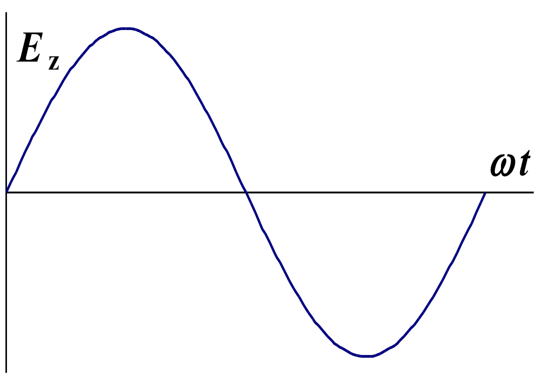
$$V_{RF} = |V_c|; \tan(\varphi_o) = 0;$$

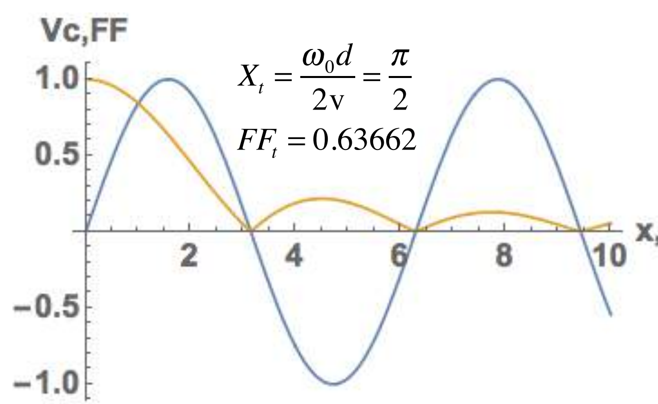
- Thus, the accelerating voltage differs from the ideal $E_o d$ by the transit time factor

$$\frac{V_{RF}}{\mathbf{E_o} d} = FF_t = \left|\frac{\sin X_t}{X_t}\right|; X_t = \frac{\omega_0 d}{2\mathrm{v}}$$

$$\boxed{\Delta E = qV_{RF} \cdot \cos\varphi_o; \varphi_o = \omega t}$$

- Thus making cavity longer than the distance particle passed during ½ of the RF period makes no sense ($X_t = \pi/2$)



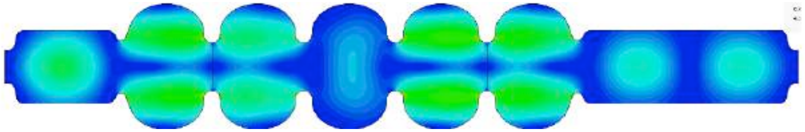


Any cavity, any mode
Approximation – constant velocity

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_o(\vec{r})\cos(\omega t + \varphi(\vec{r})) = E_o\vec{u}_e(\vec{r})\cos(\omega t + \varphi(\vec{r}))$$

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_o(\vec{r})\sin(\omega t + \psi(\vec{r})) = E_o\vec{u}_b(\vec{r})\sin(\omega t + \psi(\vec{r}))$$

$$\int \vec{\mathbf{E}}_o^{\,2}\, dV = c^2 \int \vec{\mathbf{B}}_o^{\,2}\, dV \Leftrightarrow \int \vec{u}_e^{\,2}\, dV = c^2 \int \vec{u}_b^{\,2}\, dV$$



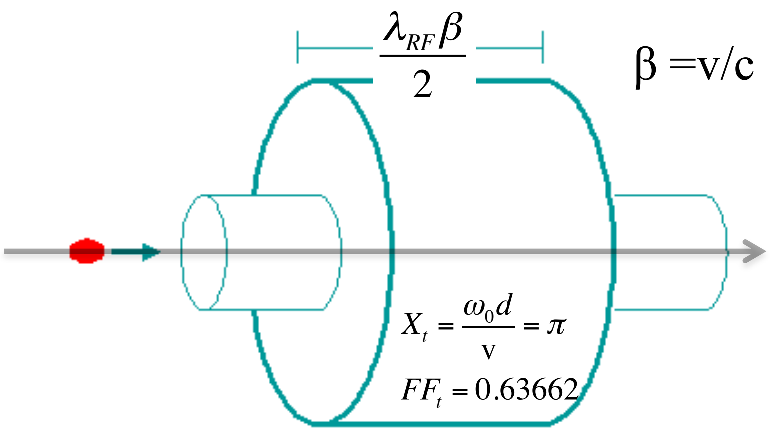
- A charged particle with a constant velocity in any RF system is described as

$$\Delta E = qV_{RF}\cos(\varphi);\ \varphi = \omega t;\ \lambda_{RF} = 2\pi c/\omega$$

$$V_{RF} = \sqrt{V_s^{\,2} + V_c^{\,2}}; V_c = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\cos\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz;\ V_s = \int_{-\infty}^{\infty} \mathbf{E_o}(z)\sin\left(\omega_0 \frac{z}{\mathrm{v}}\right)dz$$

What are β=x cavities

- For heavy particles like protons, it takes a lot of RF cavities to accelerate to velocity comparable to speed of the light
- Hence, there are so called low-β cavities designed for slow particles
- You will see in literature β=0.1, β=0.5… cavities – it means that they are designed. For particle traveling nearly speed of light cavities called β=1.



$$\Delta E = qV_{RF} \cdot \cos\varphi_o ; \varphi_o = \omega t$$

β =1 Pillbox

$$\frac{\omega_0 d}{c} = \pi$$

$$FF_t(\beta = 1) = 0.6366$$

$$FF_t(\beta = 0.8) = 0.4705$$

$$FF_t(\beta = 0.5) = 0$$

Multi-cell cavities

- We learned so far that single cell RF cavity has limited accelerating voltage

$$\boxed{Max\left(V_{RF}\right)=\frac{\mathbf{E_o}\lambda_{RF}}{\pi}}$$

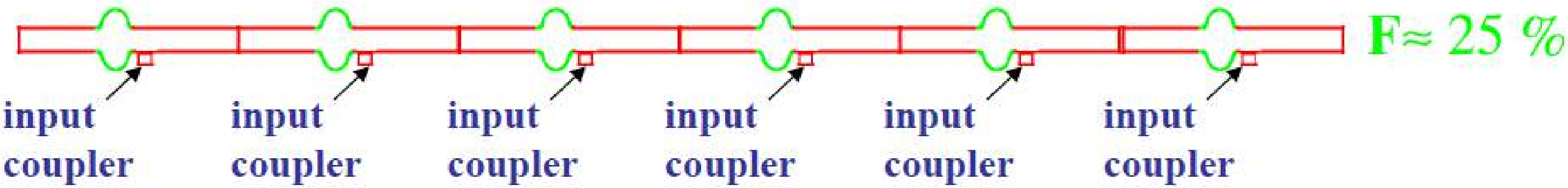
- To gain more energy we can either use more individual cells or use multi-cell cavities
- The first path, while feasible, is expensive (each cavity would need individual transmitter, waveguide, controls, etc.) and less effective – the average accelerating gradient (energy gain per meter of real estate) would be low
- Thus, where the acceleration gradient is important, the accelerator community uses multi-cell cavities

9-cell Tesla design

7-cell

5-cell

Why multi-cell cavities?



higher fill-factor:

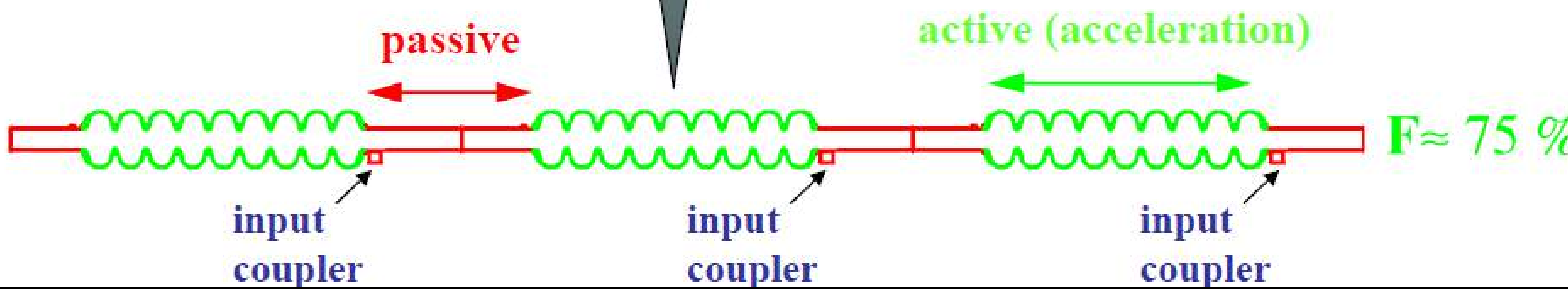
$$F = \frac{\text{active length}}{\text{total length}}$$

⇒ *lower costs*

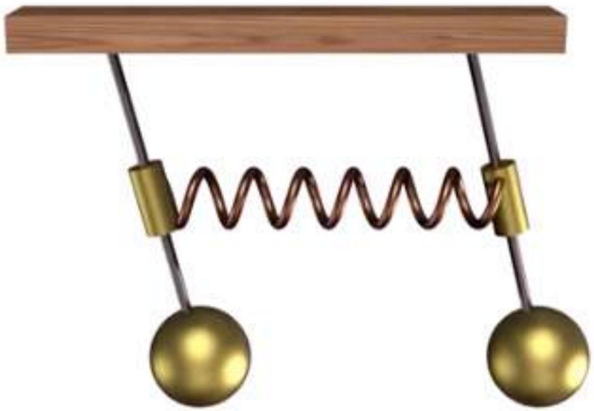
⇒ *better beam*

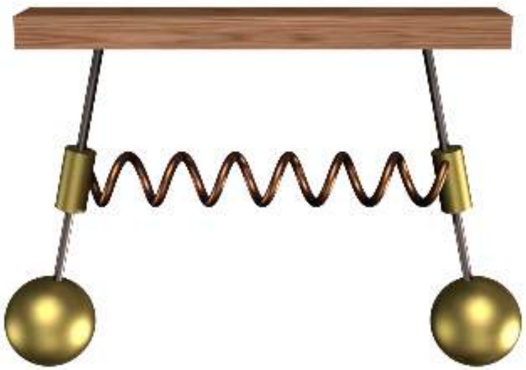
fewer

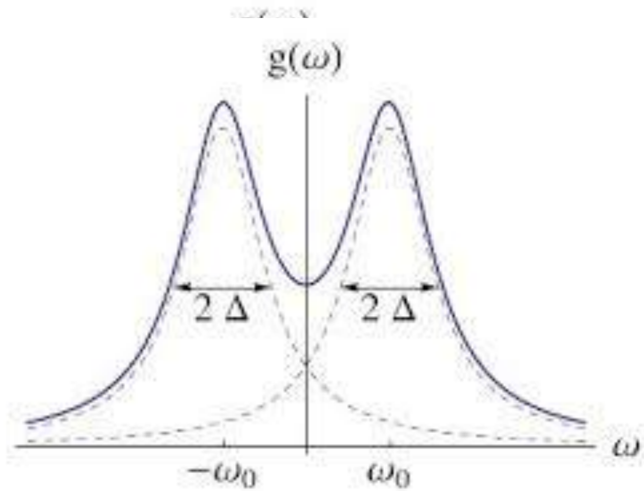
- **input couplers**
- **waveguide elements**
- **RF control systems**
- **...**



Two couple oscillators: 0-mode and π-mode

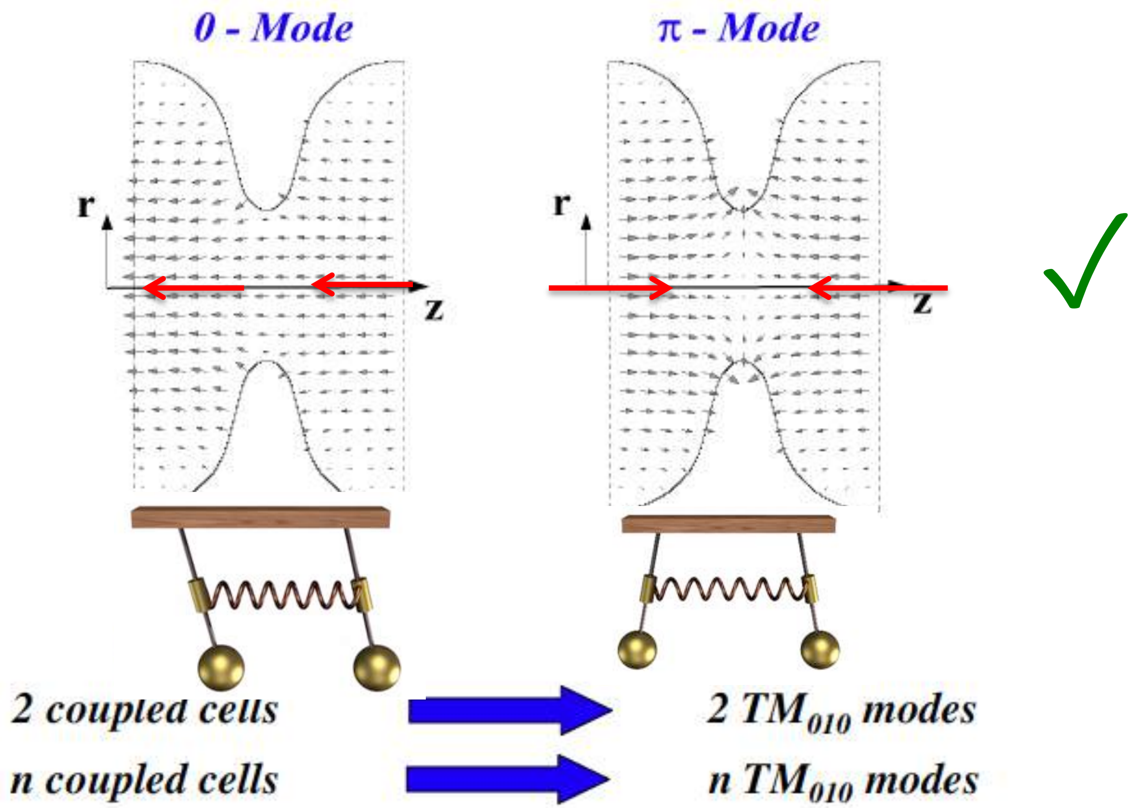






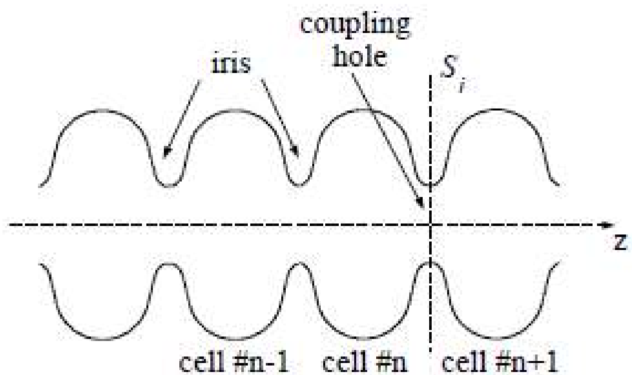
Multi-cell cavities (cont.)

- Several cells can be connected together to form a multi-cell cavity
- Coupling of TM_{010} modes of the individual cells via the iris causes them to split
- 0-mode does not give any advantages – all cavities have the same direction of the field…
- π-mode is of special interest for us:
    - electric field has opposite directions on neighboring cells
    - particle passes through accelerating voltage in a cell in half of RF period
    - when particle crosses to the next cell – it sees again accelerating voltage

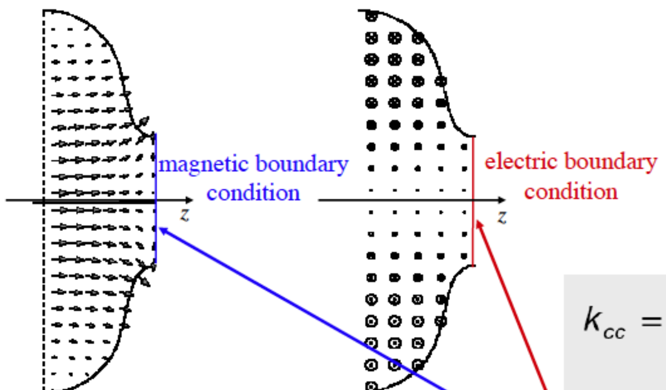


Multi-cell cavities - coupling

- Even though calculating coupling between the cavities is straight forwards, in practice is done using EM cavity codes
- For us is important to know that larger iris provides for stronger coupling and better uniformity of the field
- But increasing the iris reduces the electric field on axis (shunt impedance) and reduces accelerating gradient of such accelerator - hence, there is a compromise



$$\boxed{\begin{aligned} &\frac{d^2x_1}{dt^2} + \omega_o^2 x_1 = -kx_2 \\ &\ldots\ldots \\ &\frac{d^2x_n}{dt^2} + \omega_o^2 x_n = kx_{n-1} - kx_{n+1} \\ &\ldots\ldots \\ &\frac{d^2x_N}{dt^2} + \omega_o^2 x_N = +kx_{N-1} \end{aligned}}$$



$$k_{cc} = \frac{\omega_\pi - \omega_0}{\frac{\omega_\pi + \omega_0}{2}}$$

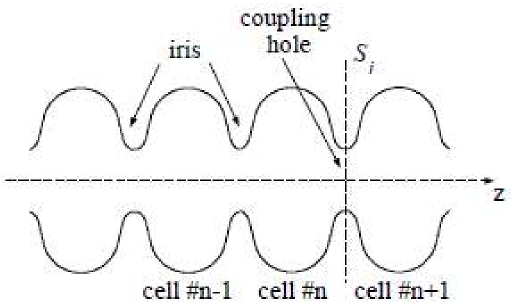
Coupling factor: $k_{cc} = \frac{c}{\omega_0} \int_{S_i} (\vec{e} \times \vec{h}) \cdot \vec{u}_z ds$

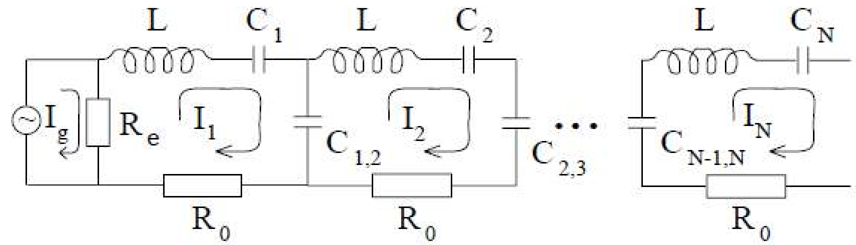
$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

where N is the number of cells,
$n = 1 \ldots N$ is the mode number

Multi-cell cavities (cont.)

- Cavity consisting of n-cell is similar to N-coupled linear oscillators or resonant contours
- They all have nearly identical frequencies, but coupling splits then in n modes

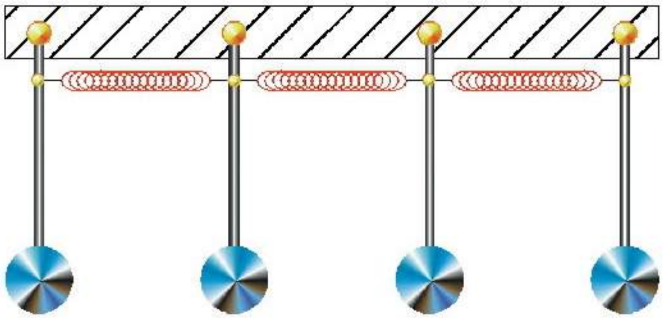




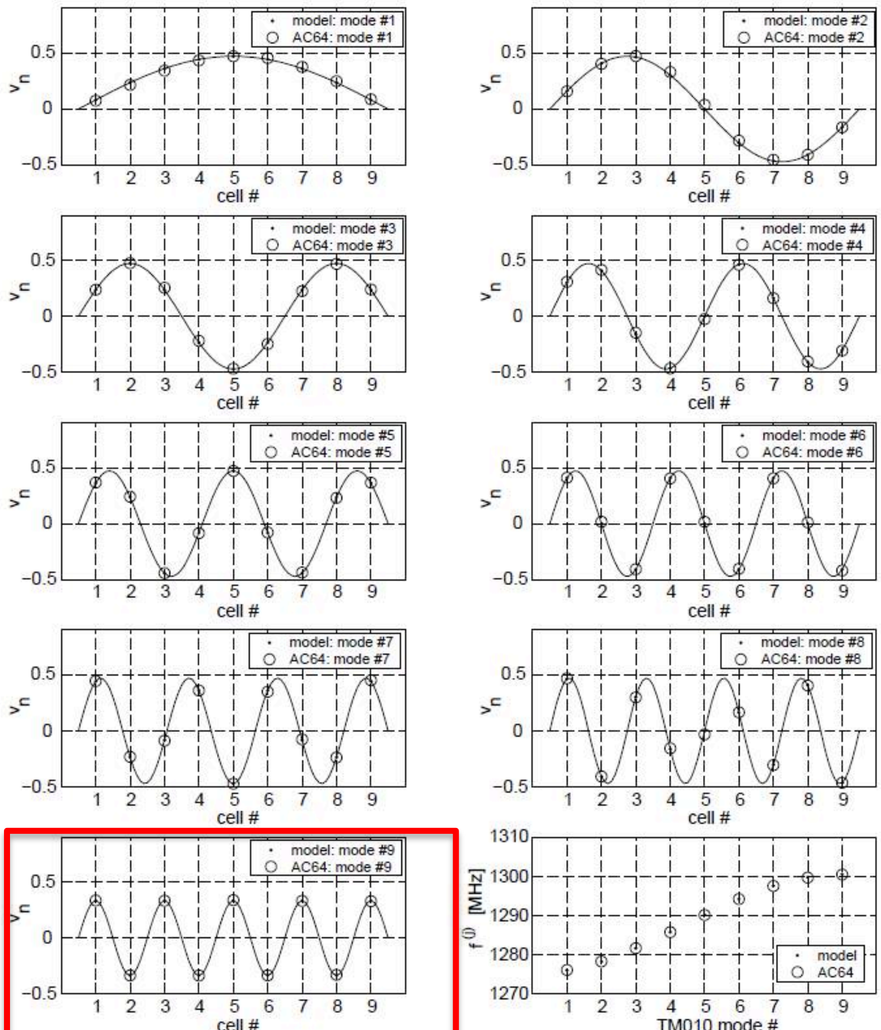
- The width of the pass-band (frequencies of various coupled modes) is determined by the strength of the cell-to-cell coupling *k* and the frequency of the *n*-th mode can be calculated from the dispersion formula

$$\left(\frac{f_n}{f_0}\right)^2 = 1 + 2k\left[1 - \cos\left(\frac{n\pi}{N}\right)\right]$$

where *N* is the number of cells,
n = 1 … N is the mode number.

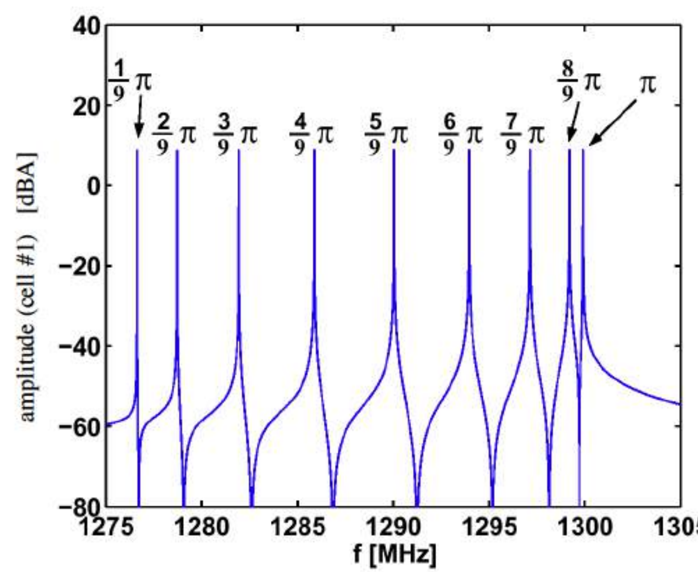


Multi-cell cavity modes



π-mode

- Simulated eigen-modes amplitudes and eigen-frequencies in a 9-cell TESLA cavity, compared to the measured values
- A longer cavity with more cells has more modes in the same frequency range. The number of cells is typically a result of optimization for specific goal.
- The accelerating mode for super-conducting RF cavities is usually the **π**-mode – e.g. at the highest frequency for electrically coupled structures.
- The same considerations are true for HOMs: each HOMs in individual cell would split into N modes



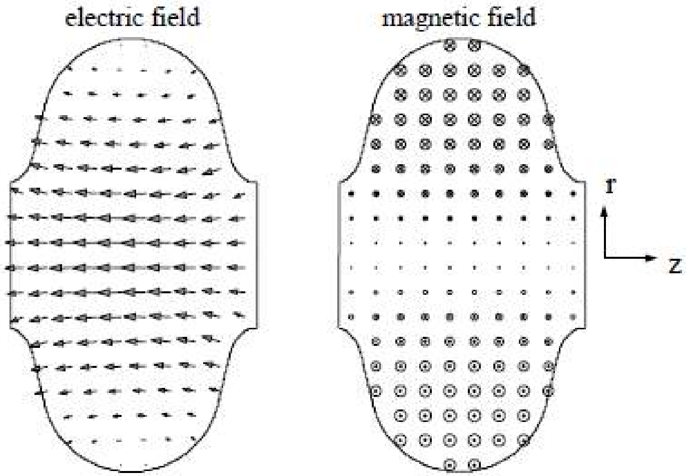
Realistic RF cavity (linac) Figures of merit

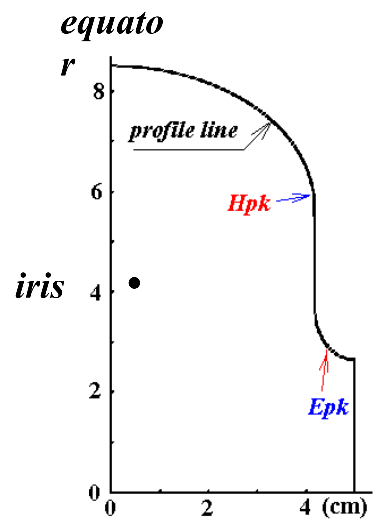
- Final conductivity of the surfaces
    - Approximation of the boundary conditions
    - Surface impedance, losses in the surface
- Main RF cavity characteristics
  - Accelerating voltage, peak electric and magnetic field
    - Q factor: internal, external, total
    - Geometrical factor, G
    - Shunt impedance R_{sh}, R_{sh}/Q
    - Coupling coefficient, *ONE MORE β!*

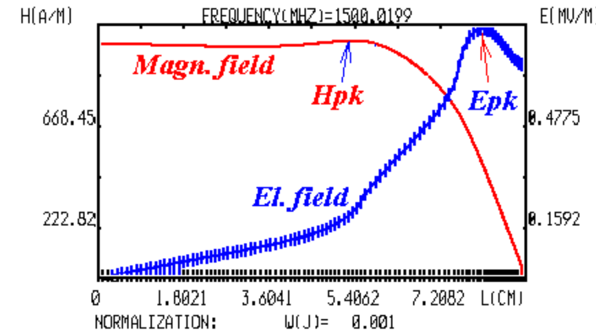
This part is usually related to more "engineering" factors measured in ohms, watts, etc.… – hence, for a change, we are using SI system…

Again, the main idea of this course: you are learning accelerator lingo and basis behind it

Typical SRF Cell fields (simulated using an EM code)







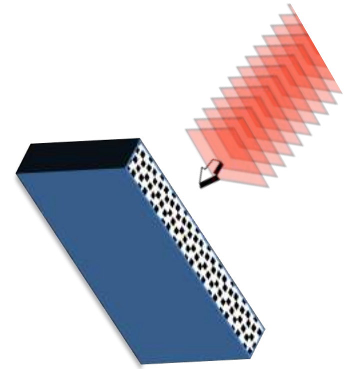
Fields along the profile line

- Important for the cavity performance are the ratios of the peak surface fields to the accelerating field. Peak surface electric field is responsible for field emission; typically for real cavities E_{pk}/E_{acc} **= 2…2.6**, as compared to **1.6** for a pillbox cavity.
- Peak surface magnetic field has fundamental limit (critical field for SRF cavities – will discuss at next lecture); surface magnetic field is also responsible for wall current losses; typical values for real cavities H_{pk}/E_{acc} **= 40…50 Oe/MV/m**, compare this to **30.5** for the pillbox
- In SGS system 1**Oe -> 1 Gs;** /MV/m is 33.3 Gs, hence ratio H_{pk}/E_{acc} is dimensionless and is close to unity: 0.92 for a pillbox cavity, 1.2 – 1.5 for elliptical cavities.
- Tangential magnetic field on the surface induces Ohmic losses and affect Q-factor

EM wave inside a conducting media

- For this course we need to understand what happens when an EM wave interacts with a conducting surface
- Inside the conduction we need to add permittivity and permeability as well its conductivity

$$\vec{\nabla}\cdot\vec{\mathbf{E}}=0; \qquad \vec{\nabla}\cdot\vec{\mathbf{B}}=0; \qquad \vec{\mathbf{J}}=\sigma\vec{\mathbf{E}}$$
$$\vec{\nabla}\times\vec{\mathbf{E}}=-\frac{\partial\vec{\mathbf{B}}}{\partial t} \qquad \vec{\nabla}\times\vec{\mathbf{B}}=\mu\left(\varepsilon\frac{\partial\vec{\mathbf{E}}}{\partial t}+\vec{\mathbf{J}}\right);$$



$$\vec{\mathbf{E}}=\vec{\mathbf{E}}e^{i(\vec{k}\vec{r}-\omega t)};\vec{\mathbf{B}}=\vec{\mathbf{B}}e^{i(\vec{k}\vec{r}-\omega t)}; \quad \vec{k}\cdot\vec{\mathbf{E}}=0;\vec{k}\cdot\vec{\mathbf{B}}=0;$$
$$i\vec{k}\times\vec{\mathbf{E}}=i\omega\vec{\mathbf{B}} \qquad i\vec{k}\times\vec{\mathbf{B}}=\mu\left(-i\omega\varepsilon+\sigma\right)\vec{\mathbf{E}};$$
$$\Rightarrow\vec{k}\times\vec{k}\times\vec{\mathbf{E}}=-i\omega\mu\left(-i\omega\varepsilon+\sigma\right)\vec{\mathbf{E}}$$
$$\left(\vec{k}^2-\mu\varepsilon\omega^2\right)\vec{\mathbf{E}}=\mu i\omega\sigma\vec{\mathbf{E}};\ \vec{k}=k\vec{n}\Rightarrow k=\sqrt{\mu\varepsilon\omega^2+i\mu\omega\sigma}$$

- Equations are just a bit more complicated that in vacuum
- Practical solution is well know for a good conductors when the skin depth is much smaller than the RF wavelength
- And the EM field decays very fast inside the conductor

$$\boxed{\delta=\sqrt{\frac{2}{\mu\sigma\omega}}}$$

$$k=\pm\frac{\omega}{c}\sqrt{\mu\varepsilon+\frac{i\mu\sigma}{\omega}}=\pm(\alpha+i\beta);$$
$$k\cong\begin{cases}\mu\varepsilon\frac{\omega}{c}-\frac{i\sigma}{2\varepsilon\omega};\ \varepsilon\omega>>\sigma\\ (1+i)\sqrt{\frac{\mu\sigma\omega}{2}};\ \sigma>>\omega\varepsilon\end{cases}$$

$Good\ conductor:\sigma>>\varepsilon\omega$

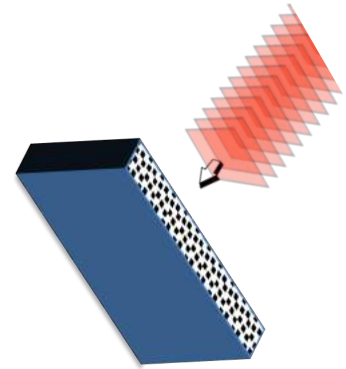
$$\vec{\mathbf{E}}\cong\vec{\mathbf{E}}e^{-\frac{\vec{n}\vec{r}}{\delta}}\cos\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right);\vec{\mathbf{B}}\cong\boxed{\frac{\mu\sigma\delta}{\sqrt{2}}}\vec{\mathbf{E}}e^{-\frac{\vec{n}\vec{r}}{\delta}}\cos\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t-\frac{\pi}{4}\right);\vec{n}\cdot\vec{\mathbf{E}}=0;$$

Note: Inside a good conductor magnetic field is much stronger than eclectic field

SI

Boundary condition for an ideal conductor

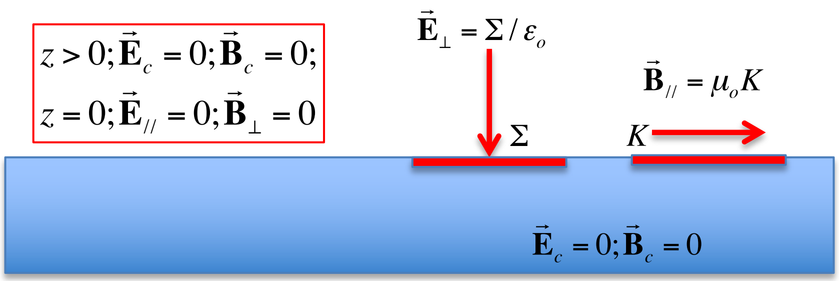
- For an ideal conductor, the condition inside the conductor are simple: both AC electric and magnetic fields are zero



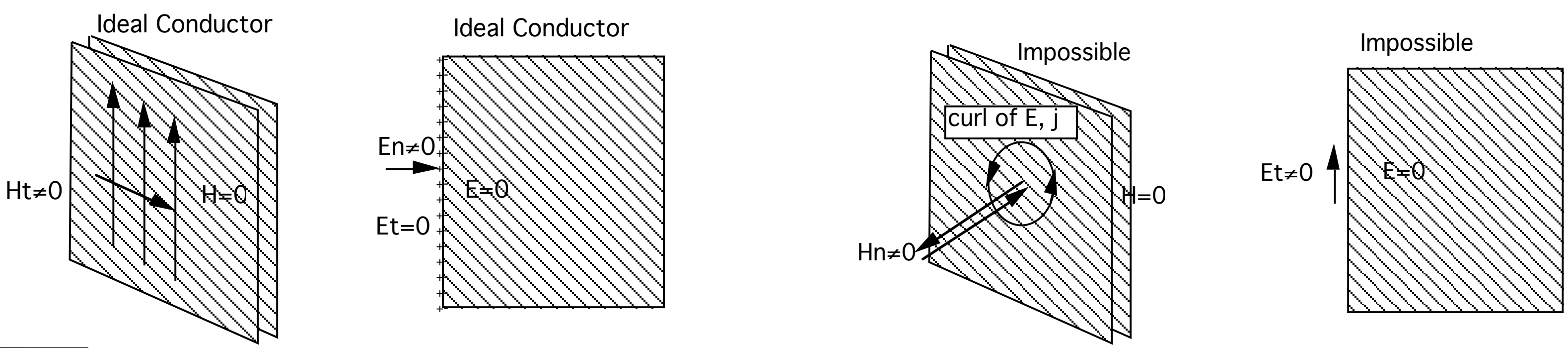
$$\boxed{\begin{aligned} &Ideal\ conductor: \sigma \to \infty \\ &\left|\vec{\mathbf{E}}_{//}\right| = \frac{\sqrt{2}\left|\vec{\mathbf{B}}_{//}\right|}{\mu\sigma\delta} \propto \frac{1}{\sqrt{\sigma}}\left|\vec{\mathbf{B}}_{//}\right| \to 0 \end{aligned}}$$

$$i\omega\vec{\mathbf{B}}_{\perp} \equiv (\vec{k} \times \vec{\mathbf{E}}_{//})_{\perp} : \ \vec{\mathbf{B}}_{\perp} \to 0$$

$$\boxed{\begin{aligned} &z > 0; \vec{\mathbf{E}}_c = 0; \vec{\mathbf{B}}_c = 0; \\ &z = 0; \vec{\mathbf{E}}_{//} = 0; \vec{\mathbf{B}}_{\perp} = 0 \end{aligned}}$$



An ideal conductor compensates magnetic field parallel to it surface by a surface current and the normal eclectic field by a surface charge

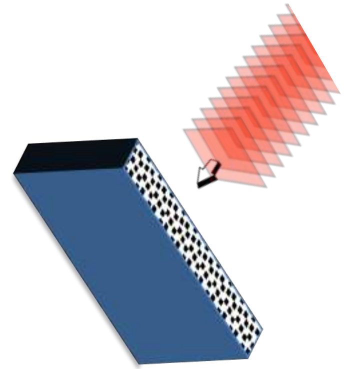


- Simple facts:
  - At the surface of ideal conductor only transverse component of electric field and longitudinal components of the magnetic field are allowed
  - Placing such surface at locations where transverse component of electric field and longitudinal components of the magnetic field are zero would not affect such EM field

SI

Boundary condition for an ideal conductor

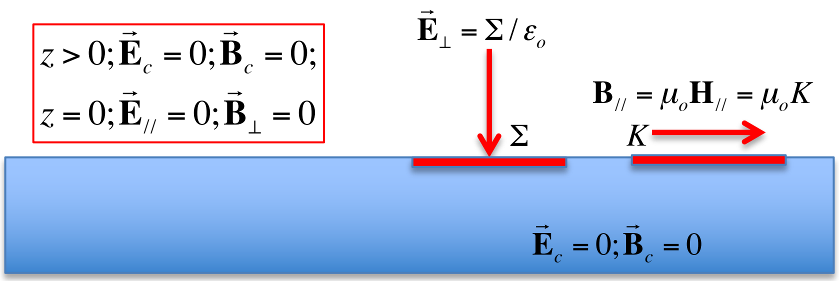
- For an ideal conductor, the condition inside the conductor are simple: both AC electric and magnetic fields are zero



$Ideal\ conductor: \sigma \to \infty$

$$\left|\vec{\mathbf{E}}_{//}\right| = \frac{\sqrt{2}\left|\vec{\mathbf{B}}_{//}\right|}{\mu\sigma\delta} \propto \frac{1}{\sqrt{\sigma}}\left|\vec{\mathbf{B}}_{//}\right| \to 0$$

$$i\omega\vec{\mathbf{B}}_{\perp} \equiv (\vec{k}\times\vec{\mathbf{E}}_{//})_{\perp} : \ \vec{\mathbf{B}}_{\perp} \to 0$$

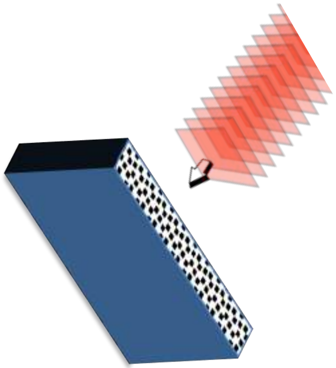


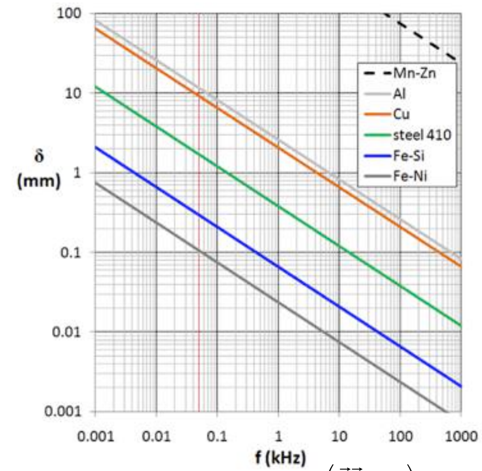
- Good cavities are build using very good conductors (including super-conductors)
- Hence, the electric field component parallel to the surface is very small (nearly zero – "not allowed") while the the magnetic field component parallel to the surface is not limited and in fact is given by the mode structure $\mathbf{B}_{//} = \mu_o \mathbf{H}_{//}$
- This parallel component of the field is compensated by the surface current, which naturally causes dissipation in real conductor

SI

Real: the conducting surface

- As input we have
- Inside the conductor the EM decays with typical length called skin depth





$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

$$\vec{\mathbf{E}}_{//} \cong \mathrm{Re}\,\mathbf{E}_o\, e^{-\frac{\vec{n}\vec{r}}{\delta}} e^{i\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)};\mathbf{H}_{//} \cong \sigma\delta\,\mathrm{Re}\frac{\mathbf{E}_o}{1+i} e^{-\frac{\vec{n}\vec{r}}{\delta}} e^{i\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)};\vec{n}\cdot\vec{\mathbf{E}}=0;$$

$$\vec{\mathbf{H}}_{//} = e^{-x/\delta}\,\mathrm{Re}\,\mathbf{H}_0\, e^{-\frac{\vec{n}\vec{r}}{\delta}} e^{i\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)}$$

$$\left|\vec{\mathbf{E}}_{//}\right| = \frac{\sqrt{2}}{\sigma\delta}\left|\vec{\mathbf{H}}_{//}\right|$$

- The current density is

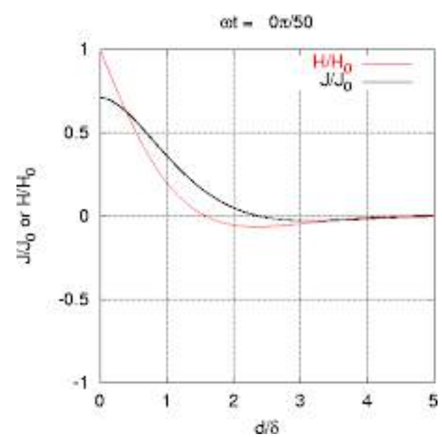
$$J \cong \frac{\sqrt{2}}{\delta}\left|\vec{\mathbf{H}}_{//}\right| e^{-\frac{\vec{n}\vec{r}}{\delta}} \cos\left(\frac{\vec{n}\vec{r}}{\delta}-\omega t\right)$$

$$K = \int J\, d\zeta \cong \vec{\mathbf{H}}_{//}$$

- And Ohmic losses per unit area

$$\frac{P_{loss}}{A} = \int \frac{\left\langle J^2(\zeta)\right\rangle_t}{\sigma} d\zeta \cong \frac{1}{2\delta\sigma}\left|\vec{\mathbf{H}}_{//}\right|^2$$

Surface impedance



At 1 GHz

Conductor	**Skin depth (μm)**
Aluminum	2.52
Copper	2.06
Gold	2,50
Silver	2.02

$$Z_s \equiv \frac{\mathbf{E}_o}{K} = \frac{\mathbf{E}_o}{\mathbf{H}_0} = \frac{1+i}{\sigma\delta} = R_s + iX_s$$

$$\frac{P_{loss}}{A} = \frac{1}{2}R_s\left|\vec{\mathbf{H}}_{//}\right|^2$$

$$R_s = \sqrt{\frac{\omega\mu}{\sigma}}\,[\Omega]$$

Beware of factors 2!

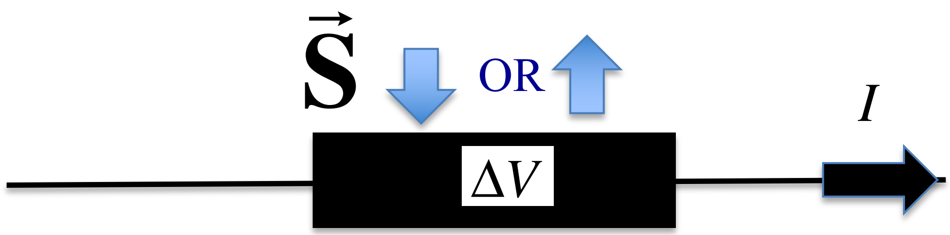
Question

- You should from E&M expression for Pointing vector

$$\vec{\mathbf{S}} = \vec{\mathbf{E}} \times \vec{\mathbf{H}}$$

indicating the flow of EM energy: direction and power density. Depending on you memory to remember "left hand" or "right hand" screw rule, you may get the direction either right or wrong… I have 50% success.

Based on the energy conservation law, please find direction of the EM energy flow in the case of a simple resistor with a current flowing through it. Is it pointed inside the surface of the resistor or outside? Does the result depends on the direction of the current?



$$\Delta V = -\mathbf{E}_z \cdot L = RI; \quad \oint \vec{\mathbf{H}} d\vec{l} = 2\pi r \vec{\mathbf{H}}_\varphi = I$$

Quality factor (SI)

- Let's consider a stand-alone cavity without any external couplers

$$\vec{\mathbf{H}}_0(\vec{r}) = H_o \vec{u}(\vec{r});\ \int |\vec{u}(\vec{r})|^2 dV = 1 \Rightarrow H_o = \sqrt{\frac{\int |\vec{\mathbf{H}}_0(\vec{r})|^2 dV}{\int |\vec{u}(\vec{r})|^2 dV}}$$

- Energy stored in the cavity

$$\boxed{W = \int\left(\varepsilon_o \frac{\vec{\mathbf{E}}^2}{2} + \mu_o \frac{\vec{\mathbf{H}}^2}{2}\right) dV = \frac{\mu_o}{2}\int \vec{\mathbf{H}}_o^2\, dV}$$

- Losses in the walls

$$\boxed{P_{loss} = \oiint \frac{1}{2} R_s \left|\vec{\mathbf{H}}_0\right|^2 dA = \frac{dW}{dt}}$$

- Quality factor (definition)

$$Q_0 \equiv \frac{\omega_0 \cdot (\text{stored energy})}{\text{average power loss}} = \frac{\omega_0 U}{P_c} = 2\pi \frac{1}{T_0}\frac{U}{P_c} = \omega_0 \tau_0 = \frac{\omega_0}{\Delta\omega_0}$$

- It is number of RF oscillation times 2π required for energy inside the cavity to reduce e-fold.

$$Q_0 = \frac{\omega_0 \mu_0 \int_V \left|\vec{\mathbf{H}}_0\right|^2 dV}{\boxed{R_s} \oiint \left|\vec{\mathbf{H}}_0\right|^2 dA} = \frac{\omega_0 \mu_0 \int_V |\vec{u}|^2 dV}{\boxed{R_s} \oiint |\vec{u}|^2 dA} \qquad \int_V |\vec{u}|^2 dV = 1$$

$$\boxed{R_s = \sqrt{\frac{\omega\mu}{\sigma}}\,[\Omega]}$$

Geometry factor: definition

- The ratio of two integrals determining Q-factor depends only on the cavity geometry: geometry defines eigen mode

$$G = \frac{\omega_0 \mu_0 \int_V \left|\vec{\mathbf{H}}_0\right|^2 dV}{\oiint \left|\vec{\mathbf{H}}_o\right|^2 dA} \equiv \frac{\omega_0 \mu_0 \int_V \left|\vec{u}\right|^2 dV}{\oiint \left|\vec{u}\right|^2 dA} \equiv F(geometry); \qquad \vec{r} \to \alpha\vec{r} \to \left\{ \begin{matrix} \omega_0 \to \omega_0 / \alpha \\ dV \to \alpha^3 dV \\ dA \to \alpha^2 dA \end{matrix} \right\}$$

The parameter G is the geometry factor (also known as geometry constant)

Obviously

$$Q_0 = \frac{G}{R_s} \qquad \boxed{R_s = \sqrt{\frac{\omega\mu}{\sigma}}\,[\Omega]}$$

- The geometry factor depends only on the cavity shape and electromagnetic mode, but not its size: Scaling the cavity size x-fold, increases volume as x^3, reduces frequency as 1/x and increasing surface as x^2. Hence, G does not change.
- **It is very useful for comparing different cavity shapes. TEM_{010} mode in a pillbox cavity had *G* = 257 Ohm independent the pillbox cavity length (d):**
  G_{TEM010}= 257 Ohm for any ratio of the length to the radius.
- At f=1.5 GHz for a normal conducting copper (σ= 5.8 × 10^7 S/m) cavity we get δ = 1.7 µm, R_s = 10 mOhm, and $Q_0 = G/R_s$ = 25,700.

Example: a pillbox cavity

- For a 1.5 GHz RF cavity
- normal conducting copper (σ= 5.8 × 10⁷ S/m)

$$\sigma = 5.8 \cdot 10^{7} S/m; \quad \delta = 1.7 \mu m \quad \Rightarrow R_s = 10 m\Omega$$

$$Q_{Cu} = \frac{G}{R_s} = 25{,}700$$

- for superconducting Nb at 1.8 K surface resistance can be as low as few nOhm, but typically is ~20 nOhm.

$$R_s = 20 n\Omega$$

$$Q_{SRF} = \frac{G}{R_s} \propto 1.2 \cdot 10^{10}$$

- Six orders of magnitude in heat losses making SRF cavities very attractive. Even with loss in cooling efficiency 500 to 1,000-fold, there is still three orders of magnitude in cooling.
- Hence, SRF cavity can operate at 30-fold higher accelerating gradient compared with room temperature Cu cavity using the same amount of cooling.

Shunt impedance and R/Q: definitions

- The shunt impedance determines how much acceleration a particle can get for a given power dissipation in a cavity

$$R_{sh} = \frac{V_{RF}^2}{P_{loss}} \sim \frac{E_o^2}{H_o^2} \frac{FF_t^2}{\boxed{R_s} \oiint |\vec{u}|^2 dA}$$

  It characterized the cavity losses.

  Often the shunt impedance is defined as in the circuit theory

$$R_{sh} = \frac{V_{RF}^2}{2P_{loss}}$$

  and, to add to the confusion, a common definition in linacs is

$$r_{sh} = \frac{E_{acc}^2}{P'_{loss}}$$

  where P'_{loss} is the power dissipation per unit length and the shunt impedance is in Ohms per meter.

- A related quantity is the ratio of the shunt impedance to the quality factor, **which is independent of the surface resistivity and the cavity size**:

$$\boxed{\frac{R_{sh}}{Q_0} = \frac{V_{RF}^2}{\omega_0 W}}$$

- This parameter is frequently used as a figure of merit and useful in determining the level of mode excitation by bunches of charged particles passing through the cavity. Sometimes it is called the geometric shunt impedance.
- **Pillbox cavity has *R/Q* = 196 Ohm**.

Dissipated power

- The power loss in the cavity walls is

$$P_{loss} = \frac{V_c^2}{R_{sh}} \equiv \frac{V_c^2}{Q_0 \cdot (R_{sh}/Q_0)} \equiv \frac{V_c^2}{(R_s \cdot Q_0)(R_{sh}/Q_0)/R_s} \equiv \frac{V_c^2 \cdot R_s}{G \cdot (R_{sh}/Q_0)}$$

- To minimize the losses one needs to maximize the denominator.
- The material-independent denominator is *G*R/Q*
- This parameter should be used during cavity shape optimization.

Consider now frequency dependence.

- For normal conductors $R_s \sim \omega^{1/2}$:

$$\boxed{\frac{P_{loss}}{L} \propto \frac{1}{G \cdot (R_{sh}/Q_0)} \cdot \frac{E_{acc}^2 R_s}{\omega} \propto \omega^{-1/2}} \qquad \boxed{\frac{P}{A} \propto \omega^{1/2}}$$

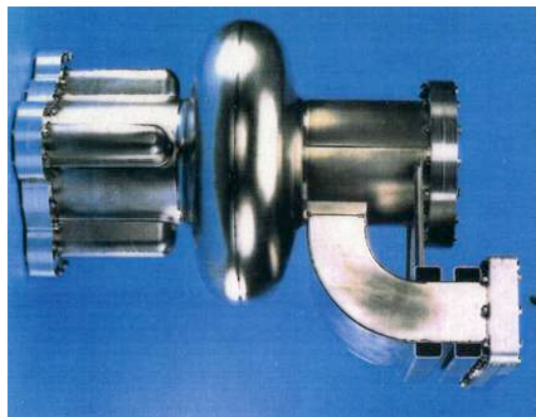
- For superconductors $R_s \sim \omega^2$

$$\boxed{\frac{P_{loss}}{L} \propto \omega} \qquad \boxed{\frac{P}{A} \propto \omega^2}$$

- NC cavities favor high frequencies, SC cavities favor low frequencies.

Pillbox *vs.* "real life" cavity

Quantity	Cornell SC 500 MHz	Pillbox
G	270 Ω	257 Ω
R_a/Q_0	88 Ω /cell	196 Ω/cell
E_{pk}/E_{acc}	2.5	1.6
H_{pk}/E_{acc}	52 Oe/(MV/m)	30.5 Oe/(MV/m)



- **In a high-current storage rings, it is necessary to damp Higher-Order Modes (HOMs) to avoid beam instabilities.**
- The beam pipes are made large to allow HOMs propagation toward microwave absorbers
- This enhances H_{pk} and E_{pk} and reduces R/Q.

Parameters of the 5-cell BNL3 cavity

Parameter	704 MHz BNL3 cavity
V_{acc} [MV]	20
No. of cells	5
Geometry Factor	283
R/Q [Ohm]	506.3
E_{pk}/E_{acc}	2.46
B_{pk}/E_{acc} [mT/MV/m]	4.26
Q_0	$> 2 \times 10^{10}$
Length [cm]	158
Beam pipe radius [mm]	**110**
Operating temperature [K]	1.9

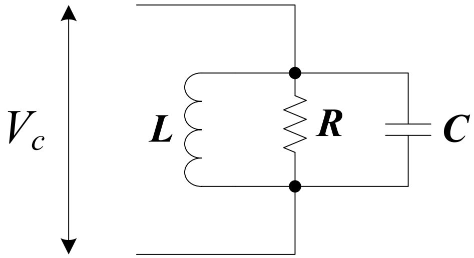
- **It was designed for high current Energy Recovery Linacs. It is necessary to damp dipole Higher-Order Modes (HOMs) to avoid beam instabilities.**
- The beam pipes are made large to allow HOMs propagation toward HOM couplers to damp the modes
- This enhances B_{pk} and E_{pk} and reduces R/Q.

Parallel circuit model

A resonant cavity can be modeled as a series of parallel *RLC* circuits representing the cavity eigen modes. For each mode:

dissipated power $$P_{loss} = \frac{V_c^2}{2R_{sh}}$$

shunt impedance $$R_{sh} = 2R$$

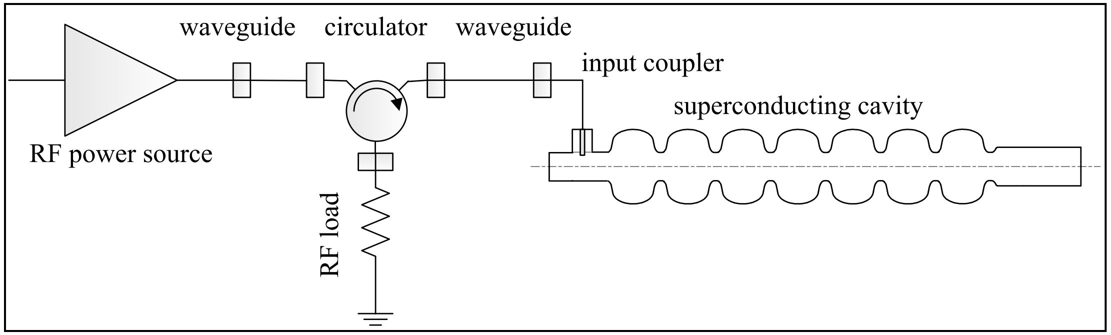


quality factor $$Q_0 = \omega_0 CR = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$$

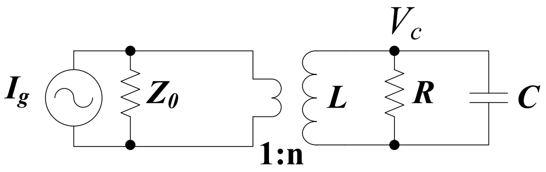
impedance $$Z = \frac{R}{1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \approx \frac{R}{1 + 2iQ\left(\frac{\omega - \omega_0}{\omega_0}\right)}$$

Connecting to a power source

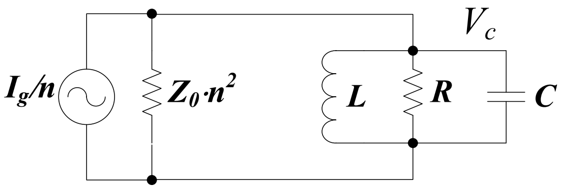
- Consider a cavity connected to an RF power source



- The input coupler can be modeled as an ideal transformer:



or



External & loaded Q factors

- If RF is turned off, stored energy will be dissipated now not only in R, but also in Z_0/n^2, thus

$$P_{tot} = P_o + P_{ext}$$

$$P_o = P_{loss} = \frac{V_c^2}{2R_{sh}} = \frac{V_c^2}{R_{sh}/Q \cdot Q_0} \qquad P_{ext} = \frac{V_c^2}{2Z_0 \cdot n^2} = \frac{V_c^2}{R_{sh}/Q \cdot Q_{ext}}$$

- This is definitions of an external quality factor associated with a coupler.
- Such Q factors can be identified with any external ports on the cavity: input coupler, RF probe, HOM couplers, beam pipes, etc.
- Then the total power loss can be associated with the loaded Q factor of

$$\boxed{\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext1}} + \frac{1}{Q_{ext2}} + \ldots}$$

Coupling parameter β

- Coupling parameter is defined as

$$\beta \equiv \frac{Q_0}{Q_{ext}}$$

e.g.

$$\frac{1}{Q_L} = \frac{1+\beta}{Q_0}$$

- β defines how strongly the couplers interact with the cavity
- Large β implies that the power taken out of the coupler is large compared to the power dissipated in the cavity walls:

$$P_{ext} = \frac{V_c^2}{R/Q \cdot Q_{ext}} = \frac{V_c^2}{R/Q \cdot Q_0} \cdot \beta = \beta P_0$$

- The total power needed from an RF power source is expressed as

$$P_{forward} = (\beta + 1) P_0$$

What we learned about RF accelerators ?

- Several figures of merits are used to characterize accelerating cavities:

$$V_{rf}\,,\ E_{peak}\,,\ H_{peak},\ R_s,\ Q_0,\ Q_{ext},\ R/Q,\ G,\ R_{sh}\ldots$$

- Superconducting RF cavities can have quality factor a million times higher than that of best Cu cavities.
- In a multi-cell cavity every eigen mode splits into a pass-band. The number of modes in each pass-band is equal to the number of cavity cells.
- Coaxial lines and rectangular waveguides are commonly used in RF systems for power delivery to cavities.