Chapter 3
Classical Cyclotron

Abstract This chapter introduces the classical cyclotron, and the theoretical material needed for the simulation exercises. It begins with a brief reminder of the historical context, and continues with beam optics and with the principles and methods which the classical cyclotron leans on, including
- ion orbit in a cyclic accelerator,
- weak focusing and periodic transverse motion,
- revolution period and isochronism,
- voltage gap and resonant acceleration,
- the cyclotron equation.
The simulation of a cyclotron dipole will either resort to an analytical model of the field: the optical element DIPOLE, or will resort to using a field map together with the keyword TOSCA to handle it and raytrace through. An additional accelerator device needed in the exercises, CAVITE, simulates a local oscillating voltage. Running a simulation generates a variety of output files, including the execution listing zgoubi.res, always, and other zgoubi.plt, zgoubi.CAVITE.out, zgoubi.MATRIX.out, etc., aimed at looking up program execution, storing data for post-treatment, producing graphs, etc. Additional keywords are introduced as needed, such as the matching procedure FIT[2]; FAISCEAU and FAISTORE which log local particle data in zgoubi.res or in a user defined ancillary file; MARKER; the 'system call' command SYSTEM; REBELOTE, a 'do loop'; and some more. This chapter introduces in addition to spin motion in accelerator magnets; dedicated simulation exercises include a variety of keywords: SPNTRK, a request for spin tracking, SPNPRT or FAISTORE, to log spin vector components in respectively zgoubi.res or some ancillary file, and the "IL=2" flag to log stepwise particle data, including spin vector, in zgoubi.plt file. Simulations include deriving transport matrices, beam matrix, optical functions and their transport, from rays, using MATRIX and TWISS keywords.

Notations used in the Text

B; B_0	magnetic field; at a reference radius R_0
$\mathbf{B}$; B_R; B_y	field vector; radial component; axial component
$BR = p/q$	magnetic rigidity
C; C_0	orbit length, $C = 2\pi R$; reference, $C_0 = 2\pi R_0$
E	ion energy, $E = \gamma m_0 c^2$
f_{rev}, f_{rf}	revolution and RF voltage frequencies
G	gyromagnetic anomaly, $G = 1.7928$ for proton, -4.184 for helion
h	harmonic number, an integer, $h = f_{\text{rf}}/f_{\text{rev}}$
$k = \frac{R}{B}\frac{dB}{dR}$	radial field index
m; m_0; M	ion mass; rest mass; in units of MeV/c^2
$\mathbf{p}$; p; p_0	ion momentum vector; its modulus; reference
q	ion charge
R; R_0; R_E	equilibrium orbit radius; reference, $R(p_0)$; at energy E
RF	Radio-Frequency
s	path variable
T_{rev}, T_{rf}	revolution and accelerating voltage periods
$\mathbf{v}$; v	ion velocity vector; its modulus
$V(t)$; $\hat{V}$	oscillating voltage; its peak value
W	kinetic energy, $W = \frac{1}{2}mv^2$
x, x', y, y'	radial and axial coordinates $\left[(*)' = \frac{d(*)}{ds}\right]$
α	trajectory deviation, or momentum compaction
$\beta = \frac{v}{c}$; β_0; β_s	normalized ion velocity; reference; synchronous
$\gamma = E/m_0c^2$	Lorentz relativistic factor
Δp, δp	momentum offset
ε_u	Courant-Snyder invariant (u : x, r, y, l, Y, Z, s, etc.)
θ	azimuthal angle
ϕ	RF phase at ion arrival at the voltage gap

3.1 Introduction

Cyclotrons are the most widespread type of accelerator, today, used by thousands, with the production of isotopes as the dominant application. This chapter is devoted to the first cyclic accelerator: the early 1930s *classical* cyclotron which its concept limited to low energy, a few 10s of MeV/nucleon. This limitation overcome a decade later by the azimuthally varying field (AVF) technique, this is the subject of the next chapter.

The classical cyclotron is based on four main principles:

(i) the use of a cylindrical-symmetry magnetic field in the gap of an electromagnet (Fig. 3.1) to maintain ions on a circular trajectory

(ii) transverse vertical confinement of the beam obtained by a slow radial decrease of the magnetic field. A technique known as weak focusing, applied over the years in all cyclic accelerators: microtron, betatron, synchrocyclotron, synchrotron. These weak focusing accelerator species all are still part of the landscape today

(iii) resonant acceleration by synchronization of a fixed-frequency accelerating voltage on the quasi-constant revolution time (Fig. 3.1). and

(iv) use of high voltage, to mitigate the effect of the turn-by-turn RF phase slip.

Resonant acceleration has the advantage that a small gap voltage is enough to accelerate with, in principle, no energy limitation, by contrast with the electrostatic techniques developed at the time, which required the generation of the full voltage, such as the Van de Graaf which was limited by sparking at a few tens of megavolts.

The cyclotron concept goes back to the late 1920s [1], yet it was not until the early 1930s when a cyclotron was first brought to operation [2]. The principles are summarized in Fig. 3.1: an oscillating voltage is applied on a pair of electrodes ("dees") forming an accelerating gap and placed between the two poles of an electromagnet. Ions reaching the gap during the acceleration phase of the voltage wave experience an energy boost; no field is experienced inside the dees. Under the effect of energy increase at the gap every half-revolution, they spiral out in the quasi-constant field of the dipole.
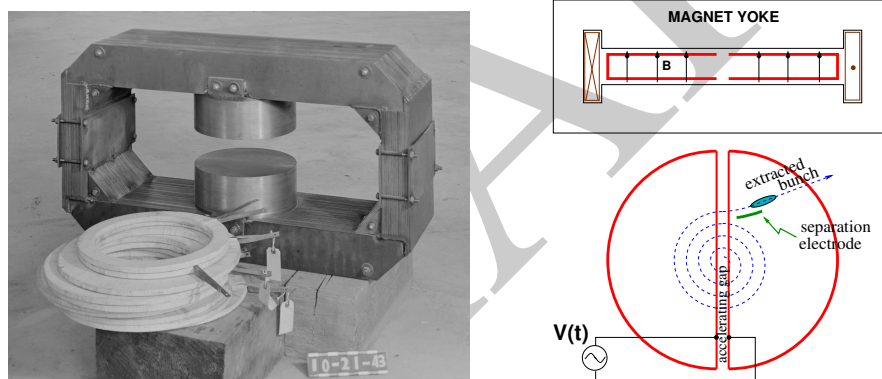
Fig. 3.1 Left: a cyclotron electromagnet, namely here that used for a model of Berkeley's 184-inch cyclotron in the early 1940s [3]. Magnetic field in the gap decreases with radius. Right: a schematic of the resonant acceleration motion; gap after gap, accelerated ions spiral out (bottom) in the quasi-uniform field (top). A double-dee (or, a variant, a single-dee facing a slotted electrode) forms an accelerating gap. The fixed-frequency oscillating voltage $V(t)$ applied is a harmonic of the revolution frequency. Ions experiencing proper voltage phase at the gap, turn by turn, are accelerated. A septum electrode allows beam extraction

The first cyclotron achieved acceleration of H_2^+ hydrogen ions to $80\,keV$ [2], at Berkeley in 1931. The apparatus used a dee-shaped electrode vis-à-vis a slotted electrode forming a voltage gap, the ensemble housed in a $5\,in$ diameter vacuum chamber and placed in the 1.3 Tesla field of an electromagnet. A $\approx 12\,MHz$ vacuum tube oscillator provided $1\,kVolt$ gap voltage.

One goal foreseen in developing this technology was the acceleration of protons to MeV energy range for the study of atom nucleus. And in background, a wealth of potential applications. An 11 in cyclotron followed which delivered a 0.01 μA H_2^+ beam at 1.22 MeV [4], and a 27 in cyclotron later reached 6 MeV (Fig. 3.2) [5]. Targets were mounted at the periphery of the 11-inch cyclotron, disintegrations were observed in 1932. And, in 1933: *'The neutron had been identified by Chadwick in 1932. By 1933 we were producing and observing neutrons from every target bombarded by deuterons.'* [5, M.S. Livingston, p. 22].
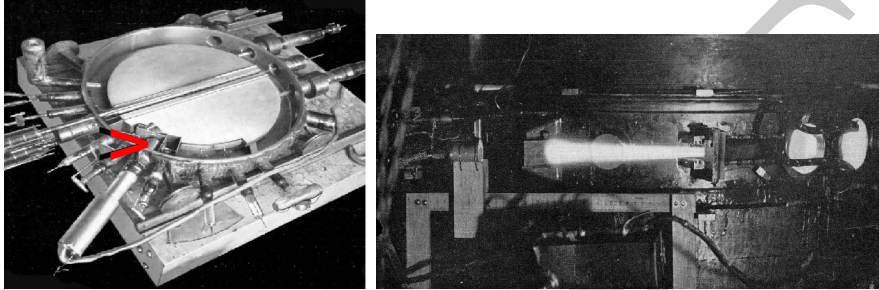
Fig. 3.2 Berkeley 27-inch cyclotron, brought to operation in 1934, accelerated deuterons up to 6 MeV. Left: a double-dee (seen in the vacuum chamber, cover off), 22 in diameter, creates an accelerating gap: 13 kV, 12 MHz radio frequency voltage is applied for deuterons for instance (through two feed lines seen at the top right corner). This apparatus was dipped in the 1.6 Tesla dipole field of a 27 in diameter, 75 ton, electromagnet. A slight decrease of the dipole field with radius, from the center of the dipole, ensures axial beam focusing. With their energy increasing, ions spiral out from the center to eventually strike a target (red arrow). Right: ionization of the air by the extracted beam (1936); the view also shows the vacuum chamber squeezed between the pole pieces of the electromagnet [3]

Fig. 3.3 Berkeley 184 in diameter, 4,000 ton cyclotron during construction [3]. The coil windings around both of the magnetic poles are clearly visible. Following the invention of longitudinal focusing it was actually operated as a synchrocyclotron, in 1946. The man on the right gives the scale
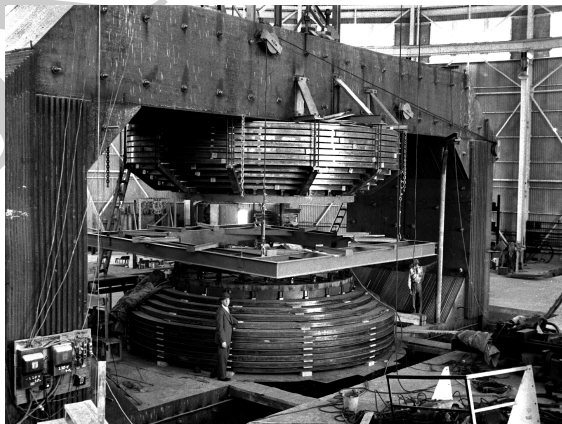
A broad range of applications were foreseen: *"At this time biological experiments were started. [...] Also at about this same time the first radioactive tracer experiments on human beings were tried [...] simple beginnings of therapeutic use, coming a little bit later, in which neutron radiation was used, for instance, in the treatment of cancer. [...] Another highlight from* 1936 *was the first time that anyone tried to make artificially a naturally occurring radio-nuclide.* (*a bismuth isotope*) [5, McMillan, p. 26].

Berkeley's 184 in cyclotron, the largest (Fig. 3.3), commissioned in 1941, was to accelerate Deuterons to 100 MeV for meson production. It's magnet however was diverted to the production of uranium for the atomic bomb during the second world war years [1]. Re-started in 1946, as a consequence of the discovery of phase focusing the accelerator was actually operated as a synchrocyclotron (an accelerator species addressed in Chap. 7).

Limitation in energy

The understanding of the dynamics of ions in the classical cyclotron took some time, and brought two news, a bad one and a good one,

(i) the bad one first: the energy limitation. A consequence of the loss of isochronism resulting from the relativistic increase of the ion mass so that *"[...] it seems useless to build cyclotrons of larger proportions than the existing ones [...] an accelerating chamber of* 37 *in radius will suffice to produce deuterons of* 11 *MeV energy which is the highest possible [...]"* [6], or in a different form: *"If you went to graduate school in the* 1940s, *this inequality* $(-1 < k < 0)$ *was the end of the discussion of accelerator theory"* [7].

(ii) the good news now: the energy limit which results from the mass increase can be removed by splitting the magnetic pole into valley and hill field sectors. This is the azimuthally varying field (AVF) cyclotron technology, due to L.H. Thomas in 1938 [8]. It took some years to see effects of this breakthrough (Fig. 3.4). The AVF is the object of Chap. 4.

With the progress in magnet computation tools, in computer speed and in beam dynamics simulations, the AVF cyclotron ends up being essentially as simple to design and build: it has in a general manner supplanted the classical cyclotron in all energy domains (Fig. 3.4).

3.2 Basic Concepts and Formulæ

The cyclotron was conceived as a means to overcome the technological difficulty of a long series of high electrostatic voltage electrodes in a linear layout, by, instead, repeated recirculation through a single accelerating gap in synchronism with an oscillating voltage (Fig. 3.5). As the accelerated bunch spirals out in the uniform magnetic field, the velocity increase comes with an increase in orbit length; the

Fig. 3.4 Evolution of the number of the various cyclotron species, over the years [9]. From the 1950s on the AVF cyclotron rapidly supplanted the 1930s' classical cyclotron
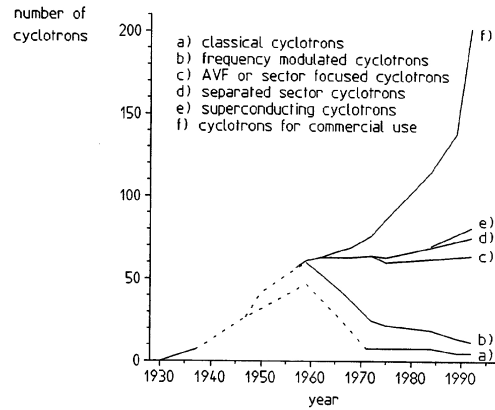
**Fig. 3.5** Resonant acceleration: in an $h = 1$ configuration an ion bunch meets an oscillating field **E** across gap A, at time t, at an accelerating phase; it meets again, half a turn later, at time $t + T_{\mathrm{rev}}/2$, the accelerating phase across gap A', and so on: the magnetic field recirculates the bunch through the gap, repeatedly. Higher harmonic allows more bunches: the next possibility in the present configuration is h=3, and 3 bunches, 120 degrees apart, in synchronism with **E**
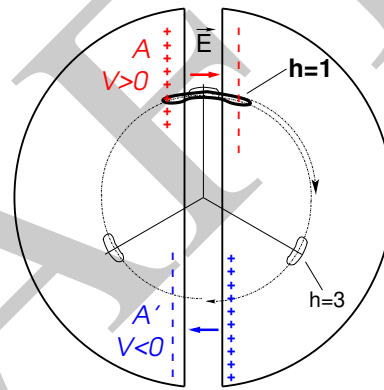
Fig. 3.6 An ion which reaches the double-dee gap at the RF phase $\omega_{\mathrm{rf}} t = \phi_A$ or $\omega_{\mathrm{rf}} t = \phi_B$ is accelerated. If it reaches the gap at $\omega_{\mathrm{rf}} t = \phi_C$ it is decelerated
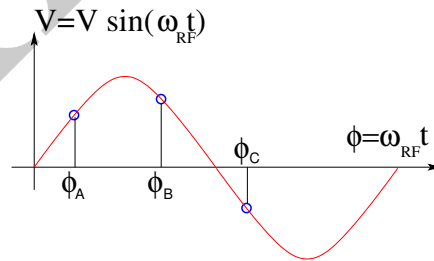
net result is a slow increase of the revolution period T_{rev} with energy, yet, with appropriate fixed $f_{rf} \approx h/T_{rev}$ the revolution motion and the oscillating voltage can be maintained in sufficiently close synchronism, $T_{rev} \approx T_{rf}/h$, that the bunch will transit the voltage gap at an accelerating phase (Fig. 3.6) over a large enough number of turns that it acquires a significant energy boost.

The orbital motion quantities: radius R, ion rigidity BR, revolution frequency f_{rev}, satisfy

$$BR = \frac{p}{q}, \qquad 2\pi f_{rev} = \omega_{rev} = \frac{v}{R} = \frac{qB}{m} = \frac{qB}{\gamma m_0} \qquad (3.1)$$

These relationships hold at all γ, so covering the *classical* cyclotron domain ($v \ll c$, $\gamma \approx 1$) as well as the *isochronous* cyclotron (in which the ion energy increase is commensurate with its mass). To give an idea of the revolution frequency, in the limit $\gamma = 1$, for protons, one has $f_{rev}/B = q/2\pi m = 15.25$ MHz/T.

The cyclotron design sets the constant RF frequency $f_{rf} = \omega_{rf}/2\pi$ at an intermediate value of $h f_{rev}$ along the acceleration cycle. The energy gain, or loss, by the ion when transiting the gap, at time t, is

$$\Delta W(t) = q\hat{V} \sin \phi(t) \quad \text{with} \quad \phi(t) = \omega_{rf}t - \omega_{rev}t + \phi_0 \qquad (3.2)$$

with ϕ its phase with respect to the RF signal at the gap (Fig. 3.6), $\phi_0 = \phi(t = 0)$, and $\omega_{rev}t$ the orbital angle. Assuming constant field B, the increase of the revolution period with ion energy satisfies

$$\frac{\Delta T_{rev}}{T_{rev}} = \gamma - 1 \qquad (3.3)$$

The mis-match so induced between the RF and cyclotron frequencies is a turn-by-turn cumulative effect and sets a limit to the tolerable isochronism defect, $\Delta T_{rev}/T_{rev} \approx 2 - 3\%$, or highest velocity $\beta = v/c \approx 0.22$. This results for instance in a practical limitation to ≈ 25 MeV for protons, and ≈ 50 MeV for D and α particles, a limit however dependent on energy gain per turn.

Over time multiple-gap accelerating structures where developed, whereby a "multiple-Δ" electrode pattern substitutes to a "double-D". An example is GANIL C0 injector with its 4 accelerating gaps and $h = 4$ and $h = 8$ RF harmonic operation [10].

3.2.1 Fixed-Energy Orbits, Revolution Period

In a laboratory frame (O;x,y,z), with (O;x,z) the bend plane (Fig. 3.7), assume $\mathbf{B}|_{y=0} = \mathbf{B}_y$, constant. An ion is launched from the origin with a velocity

$$\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (v \sin \alpha, 0, v \cos \alpha)$$

at an angle α from the z-axis. Solving
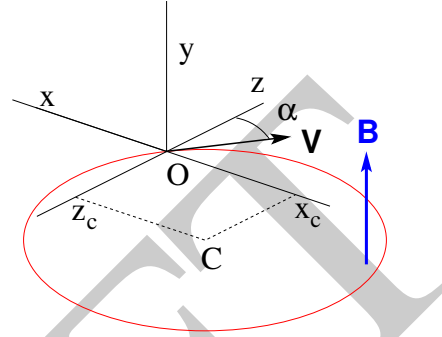
Fig. 3.7 Circular motion of an ion in the plane normal to a uniform magnetic field **B**. The orbit is centered at $x_C = -v\cos\alpha/\omega_{rev}$, $z_C = v\sin\alpha/\omega_{rev}$, its radius is v/ω_{rev}

$$m\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B} \qquad (3.4)$$

with $\mathbf{B} = (0, B_y, 0)$ yields the parametric equations of motion

$$\begin{cases} x(t) = \dfrac{v}{\omega_{rev}}\cos(\omega_{rev}t - \alpha) - \dfrac{v\cos\alpha}{\omega_{rev}} \\ y(t) = \text{constant} \\ z(t) = \dfrac{v}{\omega_{rev}}\sin(\omega_{rev}t - \alpha) + \dfrac{v\sin\alpha}{\omega_{rev}} \end{cases} \qquad (3.5)$$

which result in

$$\left(x + \frac{v\cos\alpha}{\omega_{rev}}\right)^2 + \left(z - \frac{v\sin\alpha}{\omega_{rev}}\right)^2 = \left(\frac{v}{\omega_{rev}}\right)^2 \qquad (3.6)$$

a circular trajectory of radius $R = v/\omega_{rev}$ centered at $(x_C, z_C) = (-\frac{v\cos\alpha}{\omega_{rev}}, \frac{v\sin\alpha}{\omega_{rev}})$.

Stability of the cyclic motion - The initial velocity vector defines a reference closed orbit in the median plane of the cyclotron dipole; a small perturbation in α or v results in a new orbit *in the vicinity* of the reference. An axial velocity component v_y on the other hand, causes the ion to drift away from the reference, vertically, linearly with time, as is no axial restoring force. The next Section will investigate the necessary field property to ensure both horizontal and vertical confinement of the cyclic motion in the vicinity of a reference orbit in the median plane.

3.2.2 Weak Focusing

In the early accelerated turns in a classical cyclotron (central region of the electro-magnet, energy up to tens of keV/u), the accelerating electric field provides vertical focusing for particles with proper RF phase [11, Sect. 8], whereas a flat magnetic field with uniformity $dB/B < 10^{-4}$ is sufficient to maintain isochronism. Beyond this low energy region however, at greater radii, a magnetic field gradient must be introduced to ensure transverse stability: field must decrease with R.

Fig. 3.8 Moving frame $(M_0; s, x, y)$ along the reference circular orbit. The curvature $1/R_0$ is constant along the orbit and $(M_0; s, x, y)$ can be considered equivalent to the cylindrical frame $(C; \theta, R_0, y)$
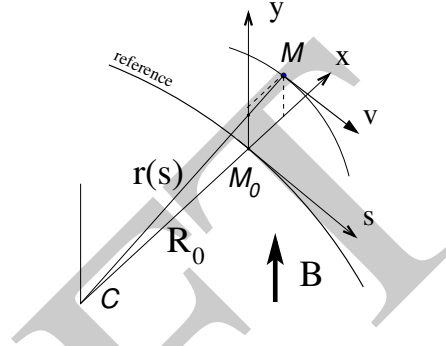
Ion coordinates in the following are defined in the moving frame $(M_0; s, x, y)$ (Fig. 3.8), which moves along the reference orbit (radius R_0), with its origin M_0 the projection of ion location M on the reference orbit; the s axis is tangent to the latter, the x axis is normal to s, the y axis is normal to the bend plane. Median-plane symmetry of the field is assumed, thus the radial field component $B_R|_{y=0} = 0$ at all R (Fig. 3.9).

Consider small motion excursions $x(t) = r(t) - R_0 \ll R_0$; introduce Taylor expansion of the field components,

$$B_y(R_0 + x) = B_y(R_0) + x \left.\frac{\partial B_y}{\partial R}\right|_{R_0} + \frac{x^2}{2!} \left.\frac{\partial^2 B_y}{\partial R^2}\right|_{R_0} + ... \approx B_y(R_0) + x \left.\frac{\partial B_y}{\partial R}\right|_{R_0}$$

$$B_R(0 + y) = y \underbrace{\left.\frac{\partial B_R}{\partial y}\right|_0}_{= \left.\frac{\partial B_y}{\partial R}\right|_{R_0}} + \frac{y^3}{3!} \left.\frac{\partial^3 B_R}{\partial y^3}\right|_0 + ... \approx y \left.\frac{\partial B_y}{\partial R}\right|_{R_0} \qquad (3.7)$$

Using these, and noting $\dot{(*)} = d(*)/dt$, the linear approximation of the differential equations of motion in the moving frame writes

$$F_x = m\ddot{x} = -qvB_y(R) + \frac{mv^2}{R_0 + x} \approx -qv\left(B_y(R_0) + \left.\frac{\partial B_y}{\partial R}\right|_{R_0} x\right) + \frac{mv^2}{R_0}\left(1 - \frac{x}{R_0}\right)$$

$$\rightarrow m\ddot{x} = -\frac{mv^2}{R_0^2}\left(\frac{R_0}{B_0}\left.\frac{\partial B_y}{\partial R}\right|_{R_0} + 1\right)x \qquad (3.8)$$

$$F_y = m\ddot{y} = qvB_R(y) = qv\left.\frac{\partial B_R}{\partial y}\right|_{y=0} y + \text{higher order} \rightarrow m\ddot{y} = qv\frac{\partial B_y}{\partial R}y$$

Fig. 3.9 Axial motion stability requires proper shaping of field lines: B_y has to decrease with radius. The Laplace force pulls a positive charge with velocity pointing out of the page, at I, toward the median plane. Increasing the field gradient (k closer to -1, gap opening up faster) increases the focusing
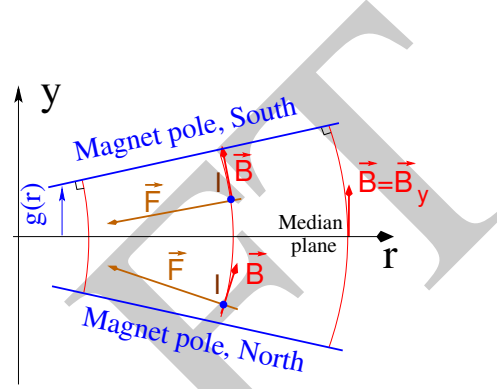
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Fig. 3.10 Geometrical focusing: take k=0; two circular trajectories which start from $r = R_0 \pm \delta R$ (solid lines, going counter-clockwise) undergo exactly one oscillation around the reference orbit $r = R_0$. A negative k (triangles), for axial focusing, decreases the radial convergence; a positive k (square markers) increases the radial convergence - and increases vertical divergence
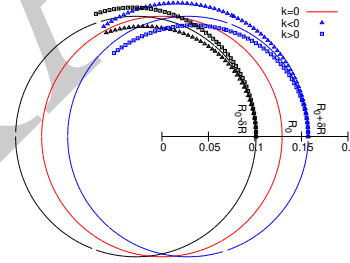
1676   Note $B_y(R_0) = B_0$ and introduce

$$\omega_R^2 = \omega_{\text{rev}}^2\left(1 + \frac{R_0}{B_0}\frac{\partial B_y}{\partial R}\right), \quad \omega_y^2 = -\omega_{\text{rev}}^2\frac{R_0}{B_0}\frac{\partial B_y}{\partial R} \qquad (3.9)$$

1677 substitute in Eqs. 3.8, this yields

Fig. 3.11 Radial motion stability. Trajectory arcs at $p = mv$ are represented: case of $k = 0$ (thin black lines), of $-1 < k < 0$ (thick blue lines), and of $k = -1$ (dashed concentric circles). k decreasing towards -1 reduces the geometrical focusing, increases axial focusing. The resultant of the Laplace and centrifugal forces, $F_t = -qvB + mv^2/r$, is zero at I, motion is stable if F_t is toward I at i, i.e. $qvB_i < mv^2/R_i$, and toward I as well at e, i.e. $qvB_e > mv^2/R_e$

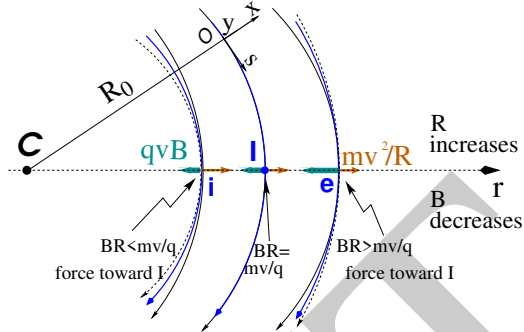$$\ddot{x} + \omega_R^2 x = 0 \qquad \text{and} \qquad \ddot{y} + \omega_y^2 y = 0 \tag{3.10}$$

A restoring force (linear terms in x and y, Eq. 3.10) arises from the radially varying field, characterized by a field index

$$k = \frac{R_0}{B_0} \frac{\partial B_y}{\partial R} \bigg|_{R=R_0, y=0} \tag{3.11}$$

Radial stability: radially this force adds to the geometrical focusing (curvature term "1" in ω_R^2, Eq. 3.9, Fig. 3.10). In the weakly decreasing field $B(R)$ an ion with momentum $p = mv$ moving in the vicinity of the R_0-radius reference orbit experiences in the moving frame a resultant force $F_t = -qvB + m\frac{v^2}{r}$ (Fig. 3.11) of which the (outward) component $f_c = m\frac{v^2}{r}$ decreases with r at a higher rate than the decrease of the Laplace (inward) component $f_B = -qvB(r)$. In other words, radial stability requires BR to increase with R, $\frac{\partial BR}{\partial R} = B + R\frac{\partial B}{\partial R} > 0$, this holds in particular at R_0, thus $1 + k > 0$.

Axial stability requires a restoring force directed toward the median plane. Referring to Fig. 3.9, this means $F_y = -a \times y$ (with a a positive quantity) and thus $B_R < 0$, at all $(r, y \neq 0)$. This is achieved by designing a guiding field which decreases with radius, $\frac{\partial B_R}{\partial y} < 0$. Referring to Eq. 3.11 this means $k < 0$.

From these radial and axial constraints the condition of "weak focusing" for transverse motion stability around the circular equilibrium orbit results, namely,

$$-1 < k < 0 \tag{3.12}$$

Note regarding the geometrical focusing: the focal distance associated with the curvature of a magnet of arc length $\mathcal{L}$ is obtained by integrating $\frac{d^2x}{ds^2} + \frac{1}{R_0^2}x = 0$ and identifying with the focusing property $\Delta x' = -x/f$, namely,

$$\Delta x' = \int \frac{d^2 x}{ds^2} ds \approx \frac{-x}{R^2} \int ds = \frac{-x\mathcal{L}}{R^2}, \text{ thus } f = \frac{R^2}{\mathcal{L}} \tag{3.13}$$

Isochronism: the axial focusing constraint, B decreasing with R, contributes breaking the isochronism (in addition to the effect of the mass increase) by virtue of $\omega_{\text{rev}} \propto B$.

Paraxial Transverse Coordinates

Introduce the path variable s as the independent variable in Eq. 3.10 and neglect the transverse velocity components $(1 + \frac{x}{R_0} \approx 1, \; y \ll 0)$ so that

$$ds = \left[r^2(s)d\theta^2 + dr^2 + dy^2 \right]^{1/2} \approx |\mathbf{v}|dt \tag{3.14}$$
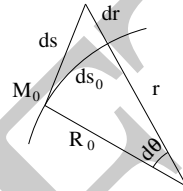
thus the equations of motion in the moving frame (Eq. 3.10) take the form

$$\frac{d^2 x}{ds^2} + \frac{1+k}{R_0^2} x = 0 \quad \text{and} \quad \frac{d^2 y}{ds^2} - \frac{k}{R_0^2} y = 0 \tag{3.15}$$

Given $-1 < k < 0$ the motion is that of a harmonic oscillator, in both planes, with respective restoring constants $(1+k)/R_0^2$ and $-k/R_0^2$, both positive quantities. The solution is a sinusoidal motion,

$$\begin{cases} r(s) - R_0 = x(s) = x_0 \cos \frac{\sqrt{1+k}}{R_0}(s - s_0) + x_0' \frac{R_0}{\sqrt{1+k}} \sin \frac{\sqrt{1+k}}{R_0}(s - s_0) \\ r'(s) = x'(s) = -x_0 \frac{\sqrt{1+k}}{R_0} \sin \frac{\sqrt{1+k}}{R_0}(s - s_0) + x_0' \cos \frac{\sqrt{1+k}}{R_0}(s - s_0) \end{cases} \tag{3.16}$$

$$\begin{cases} y(s) = y_0 \cos \frac{\sqrt{-k}}{R_0}(s - s_0) + y_0' \frac{R_0}{\sqrt{-k}} \sin \frac{\sqrt{-k}}{R_0}(s - s_0) \\ y'(s) = -y_0 \frac{\sqrt{-k}}{R_0} \sin \frac{\sqrt{-k}}{R_0}(s - s_0) + y_0' \cos \frac{\sqrt{-k}}{R_0}(s - s_0) \end{cases} \tag{3.17}$$

Radial and axial wave numbers can be introduced,

$$\nu_R = \frac{\omega_R}{\omega_{\text{rev}}} = \sqrt{1+k} \quad \text{and} \quad \nu_y = \frac{\omega_y}{\omega_{\text{rev}}} = \sqrt{-k} \tag{3.18}$$

i.e., the number of sinusoidal oscillations of the paraxial motion about the reference circular orbit over a turn, respectively radial and axial. Both are less than 1: there is less than one sinusoidal oscillation in a revolution. In addition, as a result of the revolution symmetry of the field,

$$\nu_R^2 + \nu_y^2 = 1 \tag{3.19}$$

₁₇₁₃ *Off-Momentum Orbit*

In a structure with revolution symmetry, the equilibrium trajectory at momentum
$\begin{cases} p_0 \\ p = p_0 + \Delta p \end{cases}$ is at radius $\begin{cases} R_0 \text{ with } B_0 R_0 = \frac{p_0}{q} \\ R \text{ with } BR = \frac{p}{q} \end{cases}$, where $\begin{cases} B = B_0 + \left(\frac{\partial B}{\partial x}\right)_0 \Delta x + ... \\ R = R_0 + \Delta x \end{cases}$
On the other hand

$$BR = \frac{p}{q} \Rightarrow \left[B_0 + \left(\frac{\partial B}{\partial x}\right)_0 \Delta x + ...\right](R_0 + \Delta x) = \frac{p_0 + \Delta p}{q}$$

₁₇₁₄ which, neglecting terms in $(\Delta x)^2$, and given $B_0 R_0 = \frac{p_0}{q}$, leaves $\Delta x \left[\left(\frac{\partial B}{\partial x}\right)_0 R_0 + B_0\right] = \frac{\Delta p}{q}$. With $k = \frac{R_0}{B_0}\left(\frac{\partial B}{\partial x}\right)_0$ this yields
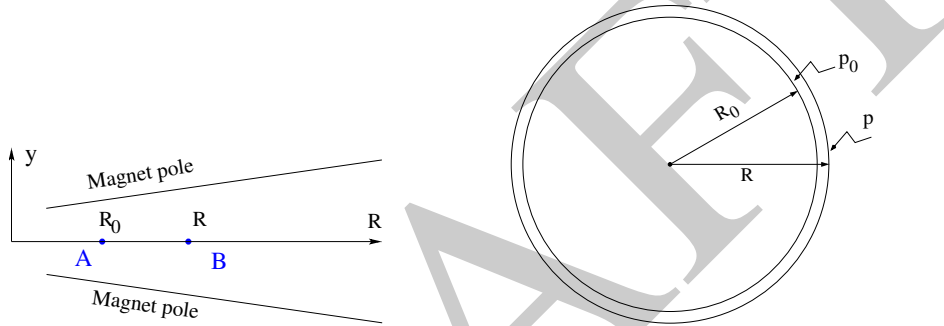
Fig. 3.12 The equilibrium radius at location A is R_0, momentum is p_0, rigidity is $B_0 R_0$. The equilibrium radius at B is R, momentum p, rigidity BR

₁₇₁₅

$$\Delta x = D\,\frac{\Delta p}{p_0} \qquad \text{with} \qquad D = \frac{R_0}{1+k} \qquad \text{the dispersion function} \qquad (3.20)$$

₁₇₁₆ The dispersion D is an s-independent quantity as a result of the revolution symmetry
₁₇₁₇ of the field (k and R=p/qB are s-independent).
₁₇₁₈   To the first order in the coordinates, the vertical coordinates y(s), y'(s) (Eq. 3.17)
₁₇₁₉ are unchanged under the effect of a momentum offset, the horizontal trajectory angle
₁₇₂₀ x'(s) (Eq. 3.16) is unchanged as well (the circular orbits are concentric, Fig. 3.12)
₁₇₂₁ whereas $x(s)$ satisfies

$$x(s, p_0 + \Delta p) = x(s, p_0) + \Delta p\,\left.\frac{\partial x}{\partial p}\right|_{s,p_0} = x(s, p_o) + D\frac{\Delta p}{p_0} \qquad (3.21)$$

Orbit and revolution period lengthening

1723   A δp momentum offset results in (Eq. 3.20)

$$\frac{\delta C}{C} = \frac{\delta R}{R} = \frac{\delta x}{R} = \alpha \frac{\delta p}{p} \quad \text{with} \quad \alpha = \frac{1}{1+k} = \frac{1}{\nu_R^2} \tag{3.22}$$

1724   with α the momentum compaction, a positive quantity: orbit length increases with
1725   momentum. Substituting $\frac{\delta \beta}{\beta} = \frac{1}{\gamma^2} \frac{\delta p}{p}$, the change in revolution period $T_{\text{rev}} = C/\beta c$
1726   with momentum writes

$$\frac{\delta T_{\text{rev}}}{T_{\text{rev}}} = \frac{\delta C}{C} - \frac{\delta \beta}{\beta} = \left(\alpha - \frac{1}{\gamma^2} \right) \frac{\delta p}{p} \tag{3.23}$$

1727   Given that $-1 < k < 0$ and $\gamma \gtrsim 1$, it results that $\alpha - 1/\gamma^2 > 0$: the revolution period
1728   increases with energy, the increase in radius is faster than the velocity increase.

3.2.3 Quasi-Isochronous Resonant Acceleration

1730   The energy W of an accelerated ion (in the non-relativistic energy domain of the
1731   classical cyclotron) satisfies the frequency dependence

$$W = \frac{1}{2} m v^2 = \frac{1}{2} m \left(2\pi R f_{\text{rev}} \right)^2 = \frac{1}{2} m \left(2\pi R \frac{f_{\text{rf}}}{h} \right)^2 \tag{3.24}$$

1732   Observe in passing: given the cyclotron size (radius R), f_{rf} and h set the limit for
1733   the acceleration range. The revolution frequency decreases with energy and the
1734   condition of synchronism with the oscillating voltage, $f_{\text{rf}} = h f_{\text{rev}}$, is only fulfilled
1735   at that particular radius where $\omega_{\text{rf}} = qB/m$ (Fig. 3.13-left). The out-phasing $\Delta\phi$ of
1736   the RF at ion arrival at the gap builds-up turn after turn, decreasing in a first stage
1737   (towards lower voltages in Fig. 3.13-right) and then increasing back to $\phi = \pi/2$ and
1738   beyond towards π. Beyond $\phi = \pi$ the RF voltage is decelerating.
1739   With ω_{rev} constant between two gap passages, differentiating $\phi(t)$ (Eq. 3.2) yields
1740 $\dot{\phi} = \omega_{\text{rf}} - \omega_{\text{rev}}$. Between two gap passages on the other hand, $\Delta\phi = \dot{\phi}\Delta T = \dot{\phi} T_{\text{rev}}/2 =$
1741 $\dot{\phi}\frac{\pi R}{v}$, yielding a phase-shift of

$$\text{half-turn} \quad \Delta\phi = \pi \left(\frac{\omega_{\text{rf}}}{\omega_{\text{rev}}(R)} - 1 \right) = \pi \left(\frac{m\omega_{\text{rf}}}{qB(R)} - 1 \right) \tag{3.25}$$

1742   The out-phasing is thus a gap-after-gap, cumulative effect. Due to this the classical
1743   cyclotron requires quick acceleration (small number of turns), which means high
1744   voltage (tens to hundreds of kVolts). As expected, with ω_{rf} and B constant, ϕ presents
1745   a minimum ($\dot{\phi} = 0$) at $\omega_{\text{rf}} = \omega_{\text{rev}} = qB/m$ where exact isochronism is reached
1746   (Fig. 3.13). The upper limit to ϕ is set by the condition $\Delta W > 0$: acceleration.
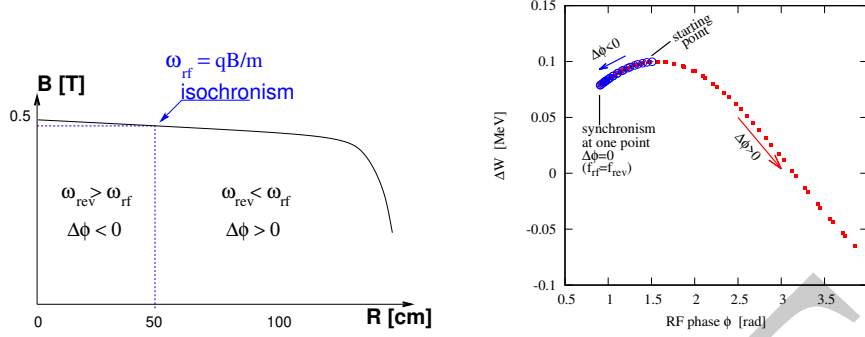
Fig. 3.13 Left: a sketch of the synchronism condition at one point (h=1 assumed). Right: the span in phase of the energy gain $\Delta W = q\hat{V}\sin\phi$ (Eq. 3.2) over the acceleration cycle

Fig. 3.14 A graph of the cyclotron equation (Eq. 3.26), for three different accelerating voltages: 100, 200 and 400 kV/gap (respectively square, circle and triangle markers). The sole settings resulting in $-1 < \cos\phi(E) < 1$, $\forall E$, allow complete acceleration to top energy. $\phi_i = \pi/4$ at injection for instance, does not (upper three curves). $\phi_i = 3\pi/4$ works (lower three curves), with as low as 100 kV/gap
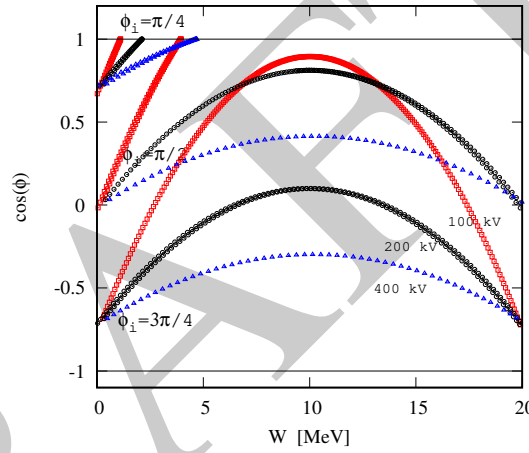
The cyclotron equation determines the achievable energy range, depending on the injection energy E_i, the RF phase at injection ϕ_i, the RF frequency $\omega_{\rm rf}$ and gap voltage $\hat{V}$. It writes [12]

$$\cos\phi = \cos\phi_i + \pi\left[1 - \frac{\omega_{\rm rf}}{\omega_{\rm rev}}\frac{E + E_i}{2M}\right]\frac{E - E_i}{q\hat{V}} \tag{3.26}$$

Equation 3.26 is represented in Fig. 3.14 for various values of the peak voltage and phase at injection ϕ_i. M [eV/c^2] and E [eV] are respectively the rest mass and relativistic energy, $q\hat{V}$ is expressed in electron-volts, the index i denotes injection parameters.

3.2.4 Beam Extraction

From $R = p/qB$ and assuming $B(R) \approx$ constant (this is legitimate as k is normally small), in the non-relativistic approximation ($W \ll M$, $W = p^2/2M$) one gets

$$\frac{dR}{R} = \frac{1}{2}\frac{dW}{W} \tag{3.27}$$

Integrating yields

$$R^2 = R_i^2 \frac{W}{W_i} \tag{3.28}$$

with R_i, W_i initial conditions. From Eqs. 3.27, 3.28, assuming $W_i \ll W$ and constant acceleration rate dW such that $W = n\,dW$ after n turns, one gets the scaling laws

$$R \propto \sqrt{n}, \qquad dR \propto \frac{R}{W} \propto \frac{1}{R} \propto dW, \qquad \frac{dR}{dn} = \frac{R}{2n} \tag{3.29}$$

The turn separation dR is proportional to the energy gain per turn and inversely proportional to the orbit radius.
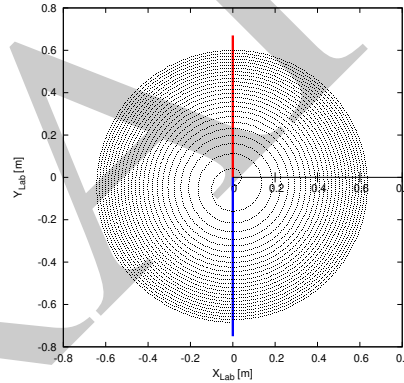
Fig. 3.15 The radial distance between successive turns decreases with energy, in inverse proportion to the orbit radius. The red and blue segments here figure the accelerating gap

    The radial distance between successive turns decreases with energy, toward zero (Fig. 3.15), eventually resulting in insufficient spacing for insertion of an extraction septum.

Orbit modulation

Consider an ion bunch injected in the cyclotron with some (x_0, x_0') conditions in the vicinity of the reference orbit, and assume slow acceleration. While accelerated the bunch undergoes an oscillatory motion around the equilibrium orbit (Eq. 3.16). Observed at the extraction septum this oscillation modulates the distance of the

1770 bunch to the local equilibrium orbit, moving it outwards or inwards depending on
1771 the turn number, which modulates the distance between the accelerated turns. This
1772 effect can be resorted to, so to increase the separation between the final two turns
1773 and so enhance the extraction efficiency [9].

1774 3.2.5 Spin Dance

1775 *"Much of the physics of spin motion can be illustrated using the simplest model of a*
1776 *storage ring consisting of uniform horizontal bending and no straight sections."* [13].

1777 By virtue of this statement, a preliminary introduction to spin motion in magnetic
1778 fields is given in the present chapter. In support to this in addition, comes the fact that
1779 cyclotrons happened to be the first circular machines to acelerate polarized beams
1780 (first acceleration of polarized beams had happened earlier in the 1960s, using
1781 electrostatic columns at voltage generators, when polarized proton and deuteron
1782 sources began operating [14]).

1783 The magnetic field **B** of the cyclotron dipole exerts a torque on the spin angular
1784 momentum **S** of an ion, causing it to precess following the Thomas-BMT differential
1785 equation [15]

$$\frac{d\mathbf{S}}{dt} = \mathbf{S} \times \underbrace{\frac{q}{m} \left[(1+G)\mathbf{B}_\parallel + (1+G\gamma)\mathbf{B}_\perp \right]}_{\boldsymbol{\omega}_{\text{sp}}} \qquad (3.30)$$

1786 where t is the time; $\boldsymbol{\omega}_{\text{sp}}$ the precession vector: a combination of $\mathbf{B}_\parallel$ and $\mathbf{B}_\perp$ compo-
1787 nents of **B** respectively parallel and orthogonal to the ion velocity vector. G is the
1788 gyromagnetic anomaly,

1789 G=1.7928474 (proton), -0.178 (Li), -0.143 (deuteron), -4.184 (^{3}He) ...

1790 **S** in this equation is in the ion rest frame, all other quantities are in the laboratory
1791 frame.

1792 In the case of an ion moving in the median plane of the dipole, $\mathbf{B}_\parallel = 0$, thus the
1793 precession axis is parallel to the magnetic field vector, $\mathbf{B}_y$, so that $\omega_{\text{sp}} = \frac{q}{m}(1 +$
1794 $G\gamma)\mathbf{B}_y$. The spin precession angle over a trajectory arc $\mathcal{L}$ is

$$\theta_{\text{sp, Lab}} = \frac{1}{v} \int_{(\mathcal{L})} \omega_{\text{sp}} \, ds = (1+G\gamma)\frac{\int_{(\mathcal{L})} B \, ds}{BR} = (1+G\gamma)\alpha \qquad (3.31)$$

1795 with α the velocity vector precession (Fig. 3.16). The precession angle in the moving
1796 frame (the latter rotates by an angle α along $\mathcal{L}$) is

$$\theta_{\text{sp}} = G\gamma\alpha \qquad (3.32)$$

1797 thus the number of 2π spin precessions per ion orbit around the cyclotron is $G\gamma$. By
1798 analogy with the wave numbers (Eq. 3.18) this defines the "spin tune"

Fig. 3.16 Spin and velocity vector precession in a constant field, from **S** to **S**′ and **v** to **v**′ respectively. In the moving frame the spin precession along the arc $\mathcal{L} = R\alpha$ is $G\gamma\alpha$, in the laboratory frame the spin precesses by $(1 + G\gamma)\alpha$
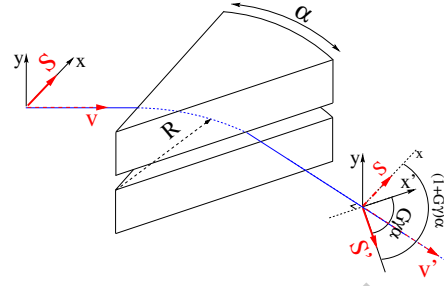
$$\nu_{\mathrm{sp}} = G\gamma \qquad\qquad\qquad (3.33)$$

1799 ## 3.3 Exercises

1800 Note: some of the input data files for these simulations are available in zgoubi
1801 sourceforge repository at

https://sourceforge.net/p/zgoubi/code/HEAD/tree/branches/exemples/book/zgoubiMaterial/cyclotron_classical/

1803 ### 3.1 Modeling a Cyclotron Dipole: Using a Field Map

1804 Solution: page 71

1805 In this exercise, ion trajectories are ray-traced, various optical properties addressed
1806 in the foregoing are recovered, using a field map to simulate the cyclotron dipole.
1807 Fabricating that field map is a preliminary step of the exercise.

1808 The interest of using a field map is that it is an easy way to account for fancy
1809 magnet geometries and fields, including field gradients and possible defects. A
1810 field map can be generated using mathematical field models, or from magnet com-
1811 putation codes, or from magnetic measurements. The first method is used, here.
1812 TOSCA[MOD.MOD1=22.1] keyword [16, *cf.* INDEX] is used to ray-trace through
1813 the map.

1814 *Working hypotheses*: A 2-dimensional $m(R, \theta)$ polar meshing of the median plane
1815 is considered (Fig. 3.17). It is defined in a $(O; X, Y)$ frame and covers an angular
1816 sector of a few tens of degrees. The mid-plane field map is the set of values $B_Z(R, \theta)$ at
1817 the nodes of the mesh. During ray-tracing, TOSCA[MOD.MOD1=22.1] extrapolates
1818 the field along 3D space (R, θ, Z) ion trajectories from the 2D polar map [16].

Fig. 3.17 Principle of a 2D field map in polar coordinates, covering a 180° sector (over the right hand side dee). The mesh nodes $m(R, \theta)$ are distant ΔR radially, $\Delta \theta$ azimuthally. The map is used twice to cover the 360° cyclotron dipole as sketched here, while allowing insertion of an accelerating gap between the two dees
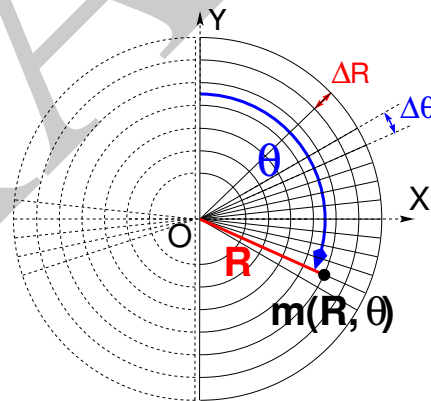
1819 (a) Construct a 180° two-dimensional map of a median plane field $B_Z(R, \theta)$,
1820 proper to simulate the field in a cyclotron as sketched in Fig. 3.1. Use one of
1821 the following two methods: either (i) write an independent program, or (ii) use
1822 zgoubi and its analytical field model DIPOLE, together with the keyword OP-
1823 TIONS[CONSTY=ON] [16, *cf.* INDEX].

1824 Besides: use a uniform mesh (Fig. 3.17) covering from Rmin=1 to Rmax=76 cm,
1825 with radial increment $\Delta R = 0.5$ cm, azimuthal increment $\Delta \theta = 0.5$ [cm]$/R_0$ with R_0

some reference radius (say, 50 cm, in view of subsequent exercises), and constant axial field $B_Z = 5$ kG. The appropriate 6-column formatting of the field map data for TOSCA[MOD.MOD1=22.1] to read is the following:

$$R \cos \theta, \ Z, \ R \sin \theta, \ BY, \ BZ, \ BX$$

with θ varying first, R varying second; Z is the vertical direction (normal to the map mesh), $Z \equiv 0$ in the present case. Note that proper functioning of TOSCA requires the field map to begin with the following line of numerical values:

$$\text{Rmin [cm]} \quad \Delta R \text{ [cm]} \quad \Delta \theta \text{ [deg]} \quad Z \text{ [cm]}$$

Produce a graph of the $B_Z(R, \theta)$ field map content.

(b) Ray-trace a few concentric circular mid-plane trajectories centered on the center of the dipole, ranging in $10 \le R \le 80$ cm. Produce a graph of these concentric trajectories in the $(O; X, Y)$ laboratory frame.

Initial coordinates can be defined using OBJET, particle coordinates along trajectories during the stepwise ray-tracing can be logged in zgoubi.plt by setting IL=2 under TOSCA. In order to find the Larmor radius corresponding to a particular momentum, the matching procedure FIT can be used. In order to repeat the latter for a series of different momenta, REBELOTE[IOPT=1] can be used.

Explain why it is possible to push the ray-tracing beyond the 76 cm radial extent of the field map.

(c) Compute the orbit radius R and the revolution period T_{rev} as a function of kinetic energy W or rigidity BR. Produce a graph, including for comparison the theoretical dependence of T_{rev}.

(d) Check the effect of the density of the mesh (the choice of ΔR and $\Delta \theta$ values, i.e., the number of nodes $N_\theta \times N_R = (1 + \frac{180^o}{\Delta \theta}) \times (1 + \frac{80 \text{ cm}}{\Delta R})$), on the accuracy of the trajectory and time-of-flight computation.

(e) Check the effect of the integration step size on the accuracy of the trajectory and time-of-flight computation, by considering a small $\Delta s = 1$ cm and a large $\Delta s = 10$ cm, at 200 keV and 5 MeV (proton), and comparing with theory.

(f) Consider a periodic orbit, thus its radius R should remain unchanged after stepwise integration of the motion over a turn. However, the size Δs of the numerical integration step has an effect on the final value of the radius:

For two different cases, 200 keV (a small orbit) and 5 MeV (a larger one), provide a graph of the dependence of the relative error $\delta R/R$ after one turn, on the integration step size Δs (consider a series of Δs values in a range $\Delta s : \ 0.1$ mm $\rightarrow 20$ cm). REBELOTE[IOPT=1] do-loop can be used to repeat the one-turn raytracing with different Δs.

3.2 Modeling a Cyclotron Dipole: Using an Analytical Field Model

Solution: page 80

This exercise is similar to exercise 3.1, yet using the analytical modeling DIPOLE, instead of a field map. DIPOLE provides the Z-parallel median plane field $\mathbf{B}(R, \theta, Z = 0) \equiv \mathbf{B}_Z(R, \theta, Z = 0)$ at the projected $m(R, \theta, Z = 0)$ ion location (Fig. 3.18), while $\mathbf{B}(R, \theta, Z)$ at particle location is obtained by extrapolation.
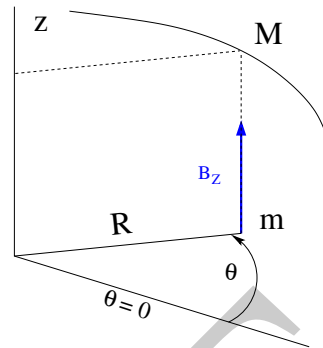
Fig. 3.18 DIPOLE provides the value $B_Z(m)$ of the median plane field at m, projection of particle position $M(R, \theta, Z)$ in the median plane. $\mathbf{B}(R, \theta, Z)$ is obtained by extrapolation

(a) Simulate a 180° sector dipole; DIPOLE requires a reference radius [16, Eqs. 6.3.19-21], noted R_0 here; for the sake of consistency with other exercises, it is suggested to take $R_0 = 50$ cm. Take a constant axial field $B_Z = 5$ kG.

Explain the various data that define the field simulation in DIPOLE: geometry, role of R_0, field and field indices, fringe fields, integration step size, etc.

Produce a graph of $B_Z(R, \theta)$.

(b) Repeat question (b) of exercise 3.1.

(c) Repeat question (c) of exercise 3.1.

(d) As in question (e) of exercise 3.1, check the effect of the integration step size on the accuracy of the trajectory and time-of-flight computation.

Repeat question (f) of exercise 3.1.

(e) From the two series of results (exercise 3.1 and the present one), comment on various pros and cons of the two methods, field map versus analytical field model.

3.3 Resonant Acceleration

Solution: page 84

Based on the earlier exercises, using indifferently a field map (TOSCA) or an analytical model of the field (DIPOLE), introduce a sinusoidal voltage between the two dees, with peak value 100 kV. Assume that ion motion does not depend on RF phase: the boost through the gap is the same at all passes, use CAVITE[IOPT=3] [16, *cf.* INDEX] for that. Note that using CAVITE requires prior PARTICUL in order to specify ion species and data, necessary to compute the energy boost (Eq. 3.2).

(a) Accelerate a proton with initial kinetic energy 20 keV, up to 5 MeV, take harmonic h=1. Produce a graph of the accelerated trajectory in the laboratory frame.

(b) Provide a graph of the proton momentum p and total energy E as a function of its kinetic energy, both from this numerical experiment (ray-tracing data can be stored using FAISTORE) and from theory, all on the same graph.

(c) Provide a graph of the normalized velocity $\beta = v/c$ as a function of kinetic energy, both numerical and theoretical, and in the latter case both classical and relativistic.

1897   (d) Provide a graph of the relative change in velocity $\Delta\beta/\beta$ and orbit length $\Delta C/C$
1898   as a function of kinetic energy, both numerical and theoretical. From their evolution,
1899   conclude that the time of flight increases with energy.

1900   (e) Repeat the previous questions, assuming a harmonic h=3 RF frequency.

3.4 Spin Dance

Solution: page 88

Cyclotron modeling in the present exercise can use Exercise 3.1 or Exercise 3.2
technique (*i.e.*, a field map or an analytical field model), indifferently.

(a) Add spin transport, using SPNTRK [16, *cf.* INDEX]. Produce a listing
(zgoubi.res) of a simulation, including spin outcomes.

Note: PARTICUL is necessary here, for the spin equation of motion (Eq. 3.30) to
be solved [16, Sect. 2]. SPNPRT can be used to have local spin coordinates listed in
zgoubi.res (at the manner that FAISCEAU lists local particle coordinates).

(b) Consider proton case, take initial spin longitudinal, compute the spin preces-
sion over one revolution, as a function of energy over a range 12 keV→5 MeV. Give
a graphical comparison with theory.

FAISTORE can be used to store local particle data, which include spin coor-
dinates, in a zgoubi.fai style output file. IL=2 [16, *cf.* INDEX] (under DIPOLE or
TOSCA, whichever modeling is used) can be used to obtain a print out of particle
and spin motion data to zgoubi.plt during stepwise integration.

(c) Inject a proton with longitudinal initial spin $\mathbf{S}_i$. Give a graphic of the lon-
gitudinal spin component value as a function of azimuthal angle, over a few turns
around the ring. Deduce the spin tune from this computation. Repeat for a couple of
different energies.

Place both FAISCEAU and SPNPRT commands right after the first dipole sector,
and use them to check the spin rotation and its relationship to particle rotation, right
after the first passage through that first sector.

(d) Spin dance: the input data file optical sequence here is assumed to model a
full turn. Inject an initial spin at an angle from the horizontal plane (this is in order
to have a non-zero vertical component), produce a 3-D animation of the spin dance
around the ring, over a few turns.

(e) Repeat questions (b-d) for two additional ions: deuteron (much slower spin
precession), $^3He^{2+}$ (much faster spin precession).

3.5 Synchronized Spin Torque

Solution: page 94

A synchronized spin kick is superimposed on orbital motion. An input data file for
a complete cyclotron is considered as in question 3.4 (d), for instance six 60 degree
DIPOLEs, or two 180 degree DIPOLEs.

Insert a local spin rotation of a few degrees around the longitudinal axis, at the
end of the optical sequence (*i.e.*, after one orbit around the cyclotron). SPINR can be
used for that, rather than a local magnetic field, so to avoid any orbital effect. Track
4 particles on their respective equilibrium orbit, with energies 0.2, 108.412, 118.878
and 160.746 MeV.

Produce a graph of the motion of the vertical spin component S_y along the circular orbit.

Produce a graph of the spin vector motion on a sphere.

3.6 Weak Focusing

Solution: page 97

(a) Consider a 60° sector as in earlier exercises (building a field map and using TOSCA as in exercise 3.1, or using DIPOLE as in exercise 3.2), construct the sector accounting for a non-zero radial index k in order to introduce axial focusing, say $k = -0.03$, assume a reference radius R_0 for a reference energy of 200 keV (R_0 and B_0 are required in order to define the index k, Eq. 3.11). Ray-trace that 200 keV reference orbit, plot it in the lab frame: make sure it comes out as expected, namely, constant radius, final and initial angles zero.

(b) Using FIT[2], find and plot the radius dependence of orbit rigidity, $BR(R)$, from ray-tracing over a BR range covering 20 keV to 5 MeV; superpose the theoretical curve. REBELOTE[IOPT=1] can be used to perform the scan.

(c) Produce a graph of the paraxial axial motion of a 1 MeV proton, over a few turns (use IL=2 under TOSCA, or DIPOLE, to have step by step particle and field data logged in zgoubi.plt). Check the effect of the focusing strength by comparing the trajectories for a few different index values, including close to -1 and close to 0.

(d) Produce a graph of the magnetic field experienced by the ion along these trajectories.

3.7 Loss of Isochronism

Solution: page 106

Compare on a common graphic the revolution period $T_{\text{rev}}(R)$ for a field index value $k \approx -0.95, -0.5, -0.03, 0^-$. The scan method of exercise 3.6, based on REBELOTE[IOPT=1] preceded by FIT[2], can be referred to.

3.8 Ion Trajectories

Solution: page 108

In this exercise individual ion trajectories are computed. DIPOLE or TOSCA magnetic field modeling can be used, indifferently. No acceleration here, ions circle around the cyclotron at constant energy.

(a) Produce a graph of the horizontal $x(s)$ and vertical $y(s)$ trajectory coordinates of an ion with rigidity close to $BR(R_0)$ (R_0 is the reference radius in the definition of the index k), over a few turns around the cyclotron. From the number of turns, give an estimate of the wave numbers. Check the agreement with the expected $\nu_R(k)$, $\nu_y(k)$ values (Eq. 3.18).

(b) Consider now protons at 1 MeV and 5 MeV, far from the reference energy $E(R_0)$; the wave numbers change with energy: consistency with theory can be checked. Find their theoretical values, compare with numerical outcomes.

(c) Consider proton, 200 keV energy, plot as a function of s the difference between $x(s)$ from raytracing and its values from Eq. 3.16. Same for y(s) compared to Eq. 3.17.

IL=2 can be used to store in zgoubi.plt the step-by-step particle coordinates across DIPOLE.

(d) Perform a scan of the wave numbers over $200\,\text{keV}-5\,\text{MeV}$ energy interval, computed using OBJET[KOBJ=5] and MATRIX[IORD=1,IFOC=11], or OBJET[KOBJ=6] and MATRIX[IORD=2,IFOC=11], together with REBELOTE[IOPT=1] to repeat MATRIX for a series of energy values.

3.9 RF Phase at the Accelerating Gap
Solution: page 114

Consider the cyclotron model of exercise 3.6: field index $k = -0.03$ defined at $R_0 = 50\,\text{cm}$, field $B_0 = 5\,kG$ on that radius. two dees, double accelerating gap.

Accelerate a proton from 1 to 5 MeV: get the turn-by-turn phase-shift at the gaps; use CAVITE[IOPT=7] to simulate the acceleration. Compare the half-turn $\Delta\phi$ so obtained with the theoretical expectation (Eq. 3.25). Produce similar graphs $B(R)$ and $\Delta W(\phi)$ to Fig. 3.13.

Accelerate over more turns, observe the particle decelerating.

3.10 The Cyclotron Equation
Solution: page 116

The cyclotron model of exercise 3.3 is considered: two dees, double accelerating gap, uniform field $B = 5\,\text{kG}$, no field gradient needed here (no vertical motion).

(a) Set up an input data file for the simulation of a proton acceleration from 0.2 to 20 MeV. In particular, assume that $\cos(\phi)$ reaches its maximum value at $W_m = 10\,\text{MeV}$; find the RF voltage frequency from $d(\cos\phi)/dW = 0$ at W_m.

(b) Give a graph of the energy-phase relationship (Eq. 3.26), for $\phi_i = \frac{3\pi}{4},\ \frac{\pi}{2},\ \frac{\pi}{4}$, from both simulation and theory.

3.11 Cyclotron Extraction
Solution: page 118

(a) Acceleration of a proton in a uniform field $B = 5\,\text{kG}$ is first considered (field hypotheses as in exercise 3.3). RF phase is ignored: CAVITE[IOPT=3] can be used for acceleration. Take a 100 kV gap voltage.

Compute the distance ΔR between turns, as a function of turn number and of energy, over the range $E : 0.02 \rightarrow 5\,\text{MeV}$. Compare graphically with theoretical expectation.

(b) Assume a beam with Gaussian momentum distribution and *rms* momentum spread $\delta p/p = 10^{-3}$. An extraction septum is placed half-way between two successive turns, provide a graph of the percentage of beam loss at extraction, as a function of extraction turn number. COLLIMA can be used for that simulation and for particle counts, it also allows for possible septum thickness.

(c) Repeat (a) and (b) considering a field with index: take for instance $B_0 = 5\,\text{kG}$ and $k = -0.03$ at $R_0 = R(0.2\,\text{MeV}) = 12.924888\,\text{cm}$.

(d) Investigate the effect of injection conditions (Y_i, T_i) on the modulation of the distance between turns.

Try and confirm numerically that, with slow acceleration, the oscillation is minimized for an initial $|T_i| = |\frac{x_0 \nu_R}{R}|$ (after Ref. [9, p. 133]).

3.12 Acceleration and Extraction of a 6-D Polarized Bunch

Solution: page 123

The cyclotron simulation hypotheses of exercise 3.10-a are considered; account or $k = -0.02$ field index.

Add a short "high energy" extraction line, say 1 meter, following REBELOTE in the optical sequence, ending up with a "Beam_Dump" MARKER for instance.

(a) Create a 1,000 ion bunch with the following initial parameters:

- random Gaussian transverse phase space densities, centered on the equilibrium orbit, truncated at 3 sigma, normalized *rms* emittances $\varepsilon_Y = \varepsilon_Z = 1\,\pi\mu$m, both emittances matched to the 0.2 MeV orbit optics,

- uniform bunch momentum density $0.2 \times (1 - 10^{-3}) \le p \le 0.2 \times (1 + 10^{-3})$ MeV, matched to the dispersion, namely (Eq. 3.21), $\Delta x = D\frac{\Delta p}{p}$,

- random uniform longitudinal distribution $-0.5 \le s \le 0.5$ mm,

Note: two ways to create this object are, (i) using MCOBJET[KOBJ=3] which generates a random distribution, or (ii) using OBJET[KOBJ=3] to read an external particle coordinate file.

Add spin tracking request (SPNTRK), all initial spins normal to the bend plane.

Produce a graph of the three initial 2-D phase spaces: (Y,T), (Z,P), (δl,$\delta p/p$), matched to the 200 keV periodic optics. Provide Y, Z, dp/p, δl and S_Z histograms (HISTO can be used), check the distribution parameters.

(b) Accelerate this polarized bunch to 20 MeV, using the following RF conditions:

- 200 kV peak voltage,

- RF harmonic 1,

- initial RF phase $\phi_i = \pi/4$.

Produce a graph of the three phase spaces as observed downstream of the extraction line. Provide the Y, Z, dp/p, δl and S_Z histograms. Compare the distribution parameters with the initial values.

What causes the spins to spread away from vertical?

3.4 Solutions of Exercises of Chapter 3: Classical Cyclotron

3.1 Modeling a Cyclotron Dipole: Using a Field Map

(a) A field map of a 180° sector of a classical cyclotron magnet.

The first option is retained here: a Fortran program, geneSectorMap.f, given in Tab. 3.1. constructs the required map of a field distribution $B_Z(R, \theta)$, to be subsequently read and raytraced through using the keyword TOSCA [16, *lookup* INDEX].

2059 Regarding the second option: using the analytical model DIPOLE together with
2060 the keyword OPTIONS[CONSTY=ON] to fabricate a field map, examples can be
2061 found for instance in the FFAG chapter exercises (Chap. 10).

Fig. 3.19 Principle 2-D field
map mesh as used by TOSCA,
and the (O;X,Y) coordinate
system. (A), (B): Cartesian
mesh in the (X,Y) plane,
case of respectively 9-point
and a 25-point interpolation
grid; the mesh increments
are ΔX and ΔY. (C) : polar
mesh and increments ΔR
and $\Delta \alpha$ ($\Delta \theta$ in the text), and
(O;X,Y) frame moving along
a reference arc of radius R_M.
The field at particle location is
interpolated from its values at
the closest 3×3 or 5×5 nodes
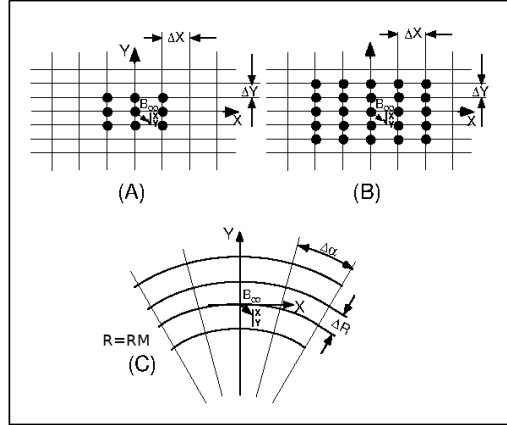
2062 A polar mesh is retained (Fig. 3.19), rather than Cartesian, consistently with
2063 cyclotron magnet symmetry. The program can be compiled (*gfortran -o geneSec-*
2064 *torMap geneSectorMap.f* will provide the executable, geneSectorMap) and run, as
2065 is. The field map is saved under the name geneSectorMap.out, excerpts of the ex-
2066 pected content are given in Tab. 3.2. That name appears under TOSCA in `zgoubi`
2067 input data file for this simulation (Tab. 3.3). Figure 3.20 shows the field over the
2068 180° azimuthal extent (using a gnuplot script, bottom of Tab. 3.2).
2069 Note the following:
2070 (i) the field map azimuthal extent (set at 180° in geneSectorMap) can be changed,
2071 for instance to simulate a 60 deg sector instead;
2072 (ii) the field is vertical being the mid-plane field of dipole magnet. The field
2073 is taken constant in this exercise, $\forall R$, $\forall \theta$ throughout the map mesh, whereas in
2074 upcoming exercises, a *focusing index* will be introduced, which will make $B_Z \equiv$
2075 $B_Z(R)$ an R-dependent quantity (in Chap. 4 which addresses Thomas focusing and
2076 the isochronous cyclotron, exercises will further resort to $B_Z \equiv B_Z(R, \theta)$, an R- and
2077 θ-dependent quantity).
2078 This field map can be readily tested using the example of Tab. 3.3, which raytraces
2079 $E_k = 0.12$, 0.2 and 5.52 MeV protons on circular trajectories centered at the center
2080 of the field map. Trajectory radii, respectively $R = 10.011$, 12.924 and 67.998 cm
2081 (Tab. 3.3), have been prior determined from

$$\text{Rigidity} \quad B\rho = B_0 \times R \quad \text{and} \quad B\rho = p/c = \sqrt{E_k(E_k + 2M)}/c \qquad (3.34)$$

2082 with $B_0 = 0.5$ T (Tab. 3.1) and $M = 938.272$ MeV/c^2 the proton mass.
2083 The optical sequence for this particle raytracing uses the following keywords:

Table 3.1 A Fortran program which generates a 180° mid-plane field map. This angle as well as field amplitude can be changed, a field index can be added. This program can be compiled and run, as is. The field map it produces is logged in geneSectorMap.out

```
C geneSectorMap.f program
      implicit double precision (a-h,o-z)
      parameter (pi=4.d0*atan(1.d0), BY=0.d0, BX=0.d0, Z=0.d0)

      open(unit=2,file='geneSectorMap.out')                              ! Field map storage file.

C----------- Hypotheses :
      AT = 180.d0  /180.d0*pi          ! Angular extent of field map. Can be changed 360, 60 deg, etc.).
      BZ=5.d0                                                            ! Field (kG).
      Rmi=1.d0; Rma=76.d0; RM=50.d0  ! cm. Radial extent of field map; reference radius to define mesh.
      dR = 0.5d0 ; NR = NINT((Rma - Rmi)/dR)+1   ! R-distance between nodes in mesh. Number of R-nodes.
C                                      RdA=RM*dA is the distance between two nodes along R=RM arc,
      RdA = 0.5d0  !     given angle increment dA (dA is the "Delta theta" quantity in the main text).
      NX= NINT(RM*AT /RdA) +1 ; RdA= RM*AT / DBLE(NX -1)  ! exact mesh step at RM, corresponding to NX.
      dA = RdA / RM ;A1 = 0.d0 ; A2 = AT                               ! corresponding delta_angle.
C------------------------------------------------
      write(2,*) Rmi,dR,dA/pi*180.d0,dZ,
     >'    ! Rmi/cm, dR/cm, dA/deg, dZ/cm'
      write(2,*) '# Field map generated using geneSectorMap.f '
      write(2,fmt='(a)') '# AT/rd,  AT/deg, Rmi/cm, Rma/cm, RM/cm,'
     >//' NR, dR/cm, NX, RdA/cm, dA/rd : '
      write(2,fmt='(a,1p,5(e16.8,1x),2(i3,1x,e16.8,1x),e16.8)')
     >'# ',AT, AT/pi*180.d0,Rmi, Rma, RM, NR, dR, NX, RdA, dA
      write(2,*) '# For TOSCA: ',NX,NR,' 1 22.1 1.  !IZ=1 -> 2D ; '
     >//'MOD=22 -> polar map ; .MOD2=.1 -> one map file'
      write(2,*) '#      R*cosA           Z==0,          R*sinA'
     >//'          BY              BZ              BX   ix jr'
      write(2,*) '#      cm              cm              cm '
     >//'          kG              kG              kG '
      write(2,*) '# '
      do jr = 1, NR
        R = Rmi + dble(jr-1)*dR
        do ix = 1, NX
          A = A1 + dble(ix-1)*dA ; X = R * sin(A) ; Y = R * cos(A)
          write(2,fmt='(1p,6(e16.8),2(1x,i0))') Y,Z,X,BY,BZ,BX,ix,jr
        enddo
      enddo
      stop ' Job complete ! Field map stored in geneSectorMap.out.'
      end
```
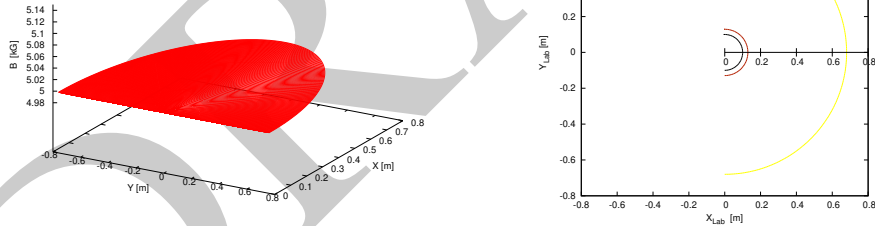

Fig. 3.20 Left: map of a constant magnetic field over a 180 deg sector, 76 cm radial extent. Right: three circular trajectories, at respectively 0.12, 0.2 and 5.52 MeV, computed using that field map

Table 3.3 Simulation input data file FieldMapSector.inc: it is set to allow a preliminary test regarding the field map geneSectorMap.out (as produced by the Fortran program geneSectorMap, Tab. 3.1), by computing three circular trajectories centered on the center of the map. This file also defines the INCLUDE segment between the labels (LABEL1 type [16, Sect. 7.7]) #S_halfDipole and #E_halfDipole

```
FieldMapSector.inc
! Uniform field 180 deg sector. FieldMapSector.inc.
'MARKER'   FieldMapSector_S                                    ! Just for edition purposes.
'OBJET'
64.62444403717985                      ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
3 1
10.011362 0. 0. 0. 0. 0.7745802 'a'     ! p[MeV/c]= 15.007, Brho[kG.cm]= 50.057, kin-E[MeV]=0.12.
12.924888 0. 0. 0. 0. 1.         'b'                          ! kin-E[MeV]=0.2.
67.997983 0. 0. 0. 0. 5.2610112 'c'     ! p[MeV/c]=101.926, Brho[kG.cm]=339.990, kin-E[MeV]=5.52.
1 1 1
'MARKER'   #S_halfDipole
'TOSCA'
0 2   ! IL=2 to log step-by-step coordinates, spin, etc., to zgoubi.plt (avoid, if CPU time matters).
1. 1. 1. 1.      ! Normalization coefficients, for B, X, Y and Z coordinate values read from the map.
HEADER_8                             ! The field map file starts with an 8-line header.
315 151 1 22.1 1.           ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1 if only one map file.
geneSectorMap.out
0 0 0 0     ! Possible vertical boundaries within the field map, to start/stop stepwise integration.
2
1.                                   ! Integration step size. Small enough for orbits to close accurately.
2                                                        ! Magnet positionning option.
0. 0. 0. 0.                                              ! Magnet positionning.
'MARKER'   #E_halfDipole
'FAISCEAU'
'SYSTEM'                  ! This SYSTEM command runs gnuplot, for a graph of the two trajectories.
1
gnuplot <./gnuplot_Zplt.gnu
'MARKER'   FieldMapSector_E                              ! Just for edition purposes.
'END'
```

A gnuplot script to obtain a graph of the orbits, Fig. 3.20:

```
# gnuplot_Zplt.gnu
set key maxcol 1 ; set key t r ; set xtics ;  set ytics ; cm2m = 0.01 ; unset colorbox
set xlabel "X_{Lab}  [m]" ; set ylabel "Y_{Lab} [m]" ; set size ratio 1  ;  set polar
plot for [orbit=1:3] "zgoubi.plt" u ($19==orbit ? $22 :1/0):($10 *cm2m):($19) w l lw 2 lc pal; pause 1
```

The optical sequence for this exercise is given in Tab. 3.4. Compared to the previous sequence (Tab. 3.3), (i) the TOSCA segment has been replaced by an INCLUDE, for the mere interest of making the input data file for this simulation shorter, and (ii) additional keywords are introduced, including

- FIT, which finds the circular orbit for a particular momentum,

- FAISCEAU, a means to check local particle coordinates,

- REBELOTE, which repeats the execution of the sequence (REBELOTE sends the execution pointer back to the top of the data file) for a new momentum value which REBELOTE itself defines, prior.

In order to compute and then plot trajectories (Fig. 3.21), zgoubi proceeds as follows: orbit circles for a series of different radii taken in $[10, 80]$ cm are searched, using FIT to find the appropriate momenta. REBELOTE is used to repeat that fitting on a series of different values of R; prior to repeating, REBELOTE modifies the initial particle coordinate Y_0 in OBJET. Stepwise particle data through the dipole field are logged in zgoubi.plt, due to IL=2 under TOSCA keyword, at the first pass before FIT, and at the last pass following FIT completion. A key point here: a flag,

Table 3.4 Simulation input data file: optical sequence to find cyclotron closed orbits at a series of different momenta. An INCLUDE inserts the #S_halfDipole to #E_halfDipole TOSCA segment of the sequence of Tab. 3.3

```
Uniform field 180 deg. sector. Find orbits.
'MARKER'   FieldMapOrbits_S                                         ! Just for edition purposes.
 'OBJET'
64.62444403717985                         ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1                                                                ! Just one ion.
12.9248888074 0.  0.  0.  0.  1. 'm'            ! This initial radius yields BR=64.6244440372 kG.cm.
1
'INCLUDE'                                                          ! A half of the cyclotron dipole.
1
FieldMapSector.inc[#S_halfDipole:#E_halfDipole]
'FAISCEAU'
'INCLUDE'                                                          ! A half of the cyclotron dipole.
1
FieldMapSector.inc[#S_halfDipole:#E_halfDipole]
'FIT'
1
2  35  0  6.                           ! Vary momentum, to allow fulfilling the following constraint:
1
3.1 1 2 5 0. 1. 0           ! request same radius after a half-turn (i.e., after first 180 deg sector,
!                                            this ensures centering of orbit on center of map).
'FAISCEAU' CHECK   ! Allows quick check of particle coordinates, in zgoubi.res: final should = initial.

'REBELOTE'                                                         ! Repeat what precedes,
15  0.1  0 1                                                       ! 15 times.
1
OBJET 30  10:80     ! Prior to each repeat, first change the value of parameter 30 (i.e., Y) in OBJET.
'SYSTEM'
2
gnuplot <./gnuplot_Zplt.gnu
cp gnuplot_Zplt_XYLab.eps gnuplot_Zplt_XYLab_stage1.eps
'MARKER'   FieldMapOrbits_E                                        ! Just for edition purposes.
'END'
```

―――――――

A gnuplot script to obtain Fig. 3.21:
Note: removing the test '$51==1 ?' on column 51 in zgoubi.plt, would add on the graph the orbit as it is before each FIT.

```
# gnuplot_Zplt.gnu
set key maxcol 1 ; set key t r ; set xtics ; set ytics  ; set size ratio 1 ; set polar ; unset colorbox
set xlabel "X_{Lab}  [m] \n" ; set ylabel "Y_{Lab}  [m] \n" ; cm2m = 0.01 ; sector1=4 ; sector2=8 ; pi = 4.*atan(1.)
lmnt1 = 4; lmnt2=8  ### column numer in zgoubi.plt,  $42: NOEL;  $51: FITLST;  $49: FIT number
plot for [l=lmnt1/4:lmnt2/4] "zgoubi.plt" u ($42==4*l && $51==1 ? $22 +pi*(l-1):1/0):($10 *cm2m):($49) w p ps .3 lc pal
pause 1
```


Fig. 3.21 Circular trajectories in the cyclotron mid-plane, centered on the field map center. The outermost orbit is at R=80 cm by hypothesis, thus $BR = B_0 \times R = 0.4\,\text{T m}$, $E_k = 7.632\,\text{MeV}$. These stepwise (R, θ) data are read from zgoubi.plt, coordinates (Y,X) in zgoubi polar frame nomenclature [16, Sect. 8.3]
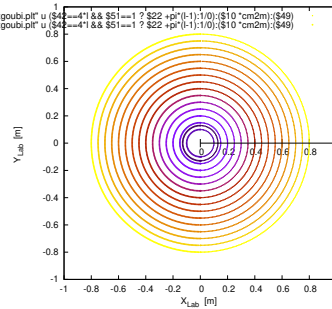
2121 FITLST, recorded in column 51 in zgoubi.plt [16, Sect.8.3], is set to 1 at the last
2122 pass (the last pass follows the completion of the FIT execution and uses updated FIT
2123 variable values).

2124 At the bottom of zgoubi input data file, a SYSTEM command produces a graph
2125 of ion trajectories, by executing a gnuplot script (bottom of Tab. 3.4). Note the test
2126 on FITLST, which allows selecting the last pass following FIT completion. Graphic
2127 outcomes are given in Fig. 3.21.

Fig. 3.22 Numerical (markers) and theoretical (solid lines) values of orbit radius, R, and revolution period, $T_{\rm rev}$, versus kinetic energy (top scale) and rigidity (bottom scale). The mesh density here is $N_\theta \times N_R = 315 \times 151$. The integration step size is $\Delta s = 1$ cm, so ensuring converged results (to $\Delta R/R$ and $\Delta T_{\rm rev}/T_{\rm rev} < 10^{-6}$)
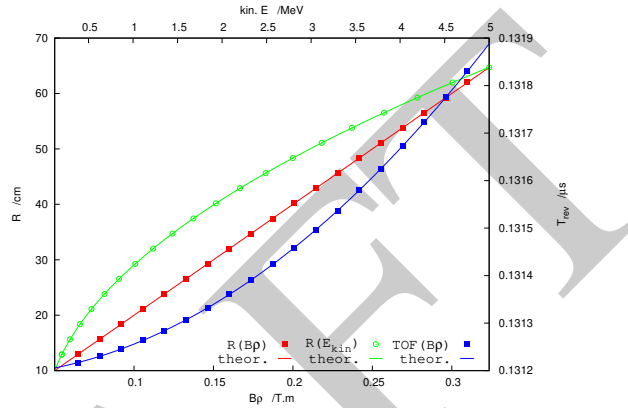
2128 The reason why it is possible to push the raytracing beyond the 76 cm radius field
2129 map extent, without loss of accuracy, is that the field is constant. Thus, referring to
2130 the polynomial interpolation technique used [16, Sect. 1.4], the extrapolation out of
2131 the map will leave the field value unchanged.

2132 (c) Energy and rigidity dependence of orbit radius and time-of-flight.

2133 The orbit radius R and the revolution time $T_{\rm rev}$ as a function of kinetic energy E_k
2134 and rigidity BR are obtained by a similar scan to exercise (b). The results are shown
2135 in Fig. 3.22.

2136 A slow increase of revolution period with energy can be observed, which is due
2137 to the mass increase.

2138 Note that these results are converged for the step size, to high accuracy (see (d)),
2139 due to its value taken small enough, namely $\Delta s = 1$ cm. This corresponds for instance
2140 to 80 steps to complete a revolution for the 120 keV, $R = 12.9$ cm smaller radius
2141 trajectory in Fig 3.21.

2142 (d) Numerical convergence: mesh density.

2143 This question concerns the dependence of the numerical convergence of the
2144 solution of the differential equation of motion [16, Eq. 1.2.1] upon mesh density.

2145 The program used in (b) to generate a field map (Tab. 3.1) is modified to construct
2146 field maps of $B_Z(R, \theta)$ with various radial and azimuthal mesh densities. Changing

Fig. 3.23 Convergence versus
mesh density and step size:
a graph of orbit radius R
(left axis), and revolution
period, T_{rev} (right axis), versus
kinetic energy (top scale) and
rigidity (bottom scale). Solid
markers are for $\Delta s = 1$ cm
and $N_\theta \times N_R = 3 \times 3$ node
mesh, large empty circles
are for $\Delta s = 10$ cm and
$N_\theta \times N_R = 106 \times 151$ node
mesh. Solid lines are from
theory and show convergence
in the case 3×3 nodes and
$\Delta s = 1$ cm
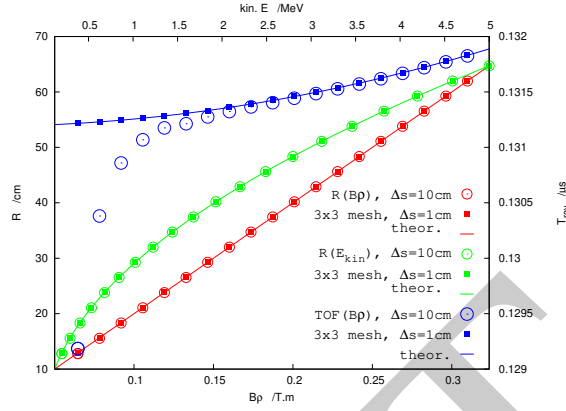
Table 3.5 Field map of a 60° constant field sector as read by TOSCA. The field map is complete, with smallest possible $NX \times NR = 3 \times 3 = 9$ number of nodes. The first line of the header is used by zgoubi (the following 7 are not used), namely, the minimum value of the radius in the map, radius increment, azimuthal increment, and vertical increment (null here, as this is a single, mid-plane map)

```
    1.0     37.50    30.0   0.        ! Rmi/cm, dR/cm, dA/deg, dZ/cm
# Field map generated using geneSectorMap.f
# AT/rd,  AT/deg, Rmi/cm, Rma/cm, RM/cm, NR, dR/cm, NX, RdA/cm, dA/rd :
#  1.04719755E+00   60.   1.   76.   50.   3   37.5   3   26.1799388   0.523598776
# For TOSCA:        3        3  1 22.1 1.  !IZ=1 -> 2D ; MOD=22 -> polar map ; .MOD2=.1 -> one map file
#
#    R*cosA           Z==0,         R*sinA           BY             BZ            BX   ix jr
#     cm               cm             cm             kG             kG            kG
  1.00000000E+00  0.00000000E+00  0.00000000E+00  0.00000000E+00  5.00000000E+00  0.00000000E+00 1 1
  8.66025404E-01  0.00000000E+00  5.00000000E-01  0.00000000E+00  5.00000000E+00  0.00000000E+00 2 1
  5.00000000E-01  0.00000000E+00  8.66025404E-01  0.00000000E+00  5.00000000E+00  0.00000000E+00 3 1
  3.85000000E+01  0.00000000E+00  0.00000000E+00  0.00000000E+00  5.00000000E+00  0.00000000E+00 1 2
  3.33419780E+01  0.00000000E+00  1.92500000E+01  0.00000000E+00  5.00000000E+00  0.00000000E+00 2 2
  1.92500000E+01  0.00000000E+00  3.33419780E+01  0.00000000E+00  5.00000000E+00  0.00000000E+00 3 2
  7.60000000E+01  0.00000000E+00  0.00000000E+00  0.00000000E+00  5.00000000E+00  0.00000000E+00 1 3
  6.58179307E+01  0.00000000E+00  3.80000000E+01  0.00000000E+00  5.00000000E+00  0.00000000E+00 2 3
  3.80000000E+01  0.00000000E+00  6.58179307E+01  0.00000000E+00  5.00000000E+00  0.00000000E+00 3 3
```

Modified TOSCA keyword data, in the case of a 60° sector field map (*compared to Tab.* 3.3, *the sole data line "3 3 1 22.1 1." changes, from "315 151 1 22.1 1." in that earlier* 180° *sector case*):

```
'TOSCA'
0 2        ! IL=2: log step-by-step coordinates, spin, etc., in zgoubi.plt (avoid if CPU time matters).
1. 1. 1. 1.     ! Normalization coefficients, for B, X, Y and Z coordinate values read from the map.
HEADER_8                                        ! The field map file starts with an 8-line header.
3   3   1 22.1 1.          ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1 if only one map file.
geneSectorMap.out
0 0 0 0       ! Possible vertical boundaries within the field map, to start/stop stepwise integration.
2
1.                                  ! Integration step size. Small enough for orbits to close accurately.
2                                                              ! Magnet positionning option.
0. 0. 0. 0.                                                     ! Magnet positionning.
```

these is simply a matter of modifying the quantities dR (radius increment ΔR) and $R\,dA$ (R times the azimuth increment $\Delta\theta$) in the program of Tab. 3.1. The field maps geneSectorMap.out so generated for various (dR, RdA) couples may be saved under different names, and used separately.

Table. 3.5 shows the complete, 9 line, TOSCA field map, in the case of a 60° sector covered in $N_\theta \times N_R = \dfrac{60^o}{\Delta\theta} \times \dfrac{75\,\text{cm}}{\Delta R} = \dfrac{360^o}{120^o} \times \dfrac{75\,\text{cm}}{37.5\text{cm}} = 3 \times 3$ nodes. Six sectors are now required to cover the complete cyclotron dipole: zgoubi input data need be changed accordingly, namely stating TOSCA - possibly via an INCLUDE - six times, instead of just twice in the case of a 180 degree sector.

The result to be expected: with a mesh reduced to as low as $N_\theta \times N_R = 3 \times 3$, compared to $N_\theta \times N_R = 106 \times 151$, radius and time-of-flight should however remain unchanged. This shows in Fig. 3.23 which displays both cases, over a $E_k : 0.12 \rightarrow$ 5 MeV energy span (assuming protons). The reason for the absence of effect of the mesh density is that the field is constant. As a consequence the field derivatives in the Taylor series based numerical integrator are all zero [16, Sect. 1.2]: only B_Z is left in evaluating the Taylor series, however B_Z is constant. Thus R remains unchanged when pushing the ion by a step Δs, and the cumulated path length - the closed orbit length - and revolution time - path length over velocity - end up unchanged. Note: this will no longer be the case when a radial field index is introduced in order to cause vertical focusing, in subsequent exercises.

(e) Numerical convergence: integration step size

Figure 3.23 displays two cases of step sizes, $\Delta s \approx 1$ cm and $\Delta s = 10$ cm.

It has been shown (Fig. 3.22) that $\Delta s \approx 1$ cm is small enough that the numerical integration is converged, agreement with theoretical expectation is quite good.

The difference on the value of R, in the case $\Delta s \approx 10$ cm, appears to be weak, only noticeable at the scale of the graph for R values small enough that the number of steps over one revolution goes as low as $2\pi R/\Delta s \approx 2\pi \times 14.5/10 \approx 9$. The change in time-of-flight due to the larger step size amounts to a relative 10^{-3}.

Step size is critical in the numerical integration, the reason is that the coefficients of the Taylor series that yield the new position vector $\mathbf{R}(M_1)$ and velocity vector $\mathbf{v}(M_1)$, from an initial location M_0 after a Δs push, are the derivatives of the velocity vector [16, Sect. 1.2] and may take substantial values if $\mathbf{v}(s)$ changes quickly. In such case, taking too large a Δs value makes the high order terms significant and the Taylor series truncation [16, Eq. 1.2.4] is fatal to the accuracy (regardless of a possible additional issue of radius of convergence of the series).

(f) Numerical convergence: $\dfrac{\delta R}{R}(\Delta s)$

Issues faced are the following:

- the increase of $\delta R(\Delta s)/R$ at large Δs has been addressed above;

- a small Δs is liable to cause an increase of $\delta R(\Delta s)/R$, due to computer accuracy: truncation of numerical values at a limited number of digits may cause a Δs push to result in no change in $\mathbf{R}(M_1)$ (position) and $\mathbf{u}(M_1)$ (normed velocity) quantities [16, Eq. 1.2.4].

2189   A detailed answer to the question, including graphs, is left to the reader, the
2190 method is the same as in (e).

3.2   Modeling a Cyclotron Dipole: Using an Analytical Field Model

2193   This exercise introduces the analytical modeling of a dipole, using DIPOLE [16,
2194 *lookup* INDEX], and compares outcomes to the field map case of exercise 3.1. The
2195 exercise is not entirely solved, however all the material needed for that is provided,
2196 and indications are given to complete it.

2197   (a) Analytical modeling.

2198   DIPOLE keyword provides an analytical model of the field to simulate a sector
2199 dipole with index, namely [16, *lookup* INDEX]

$$B_Z = \mathcal{F}(\theta)B_0 \left[1 + k \left(\frac{R - R_0}{R_0} \right) + k' \left(\frac{R - R_0}{R_0} \right)^2 + k'' \left(\frac{R - R_0}{R_0} \right)^3 \right] \quad (3.35)$$

2200 R_0 is a reference radius, $B_0 = B_Z(R_0)|_{\mathcal{F} \equiv 1}$ is a reference field value, k is the field
2201 index and k', k" are homogeneous to its first and second derivative with respect to
2202 R (Eq. 3.11). $\mathcal{F}(\theta)$ is an azimuthal form factor, defined by the fringe field model,
2203 presumably taking the value 1 in the body of the dipole. In the present case a
2204 hard-edge field model is considered, so that

$$\mathcal{F} = \begin{cases} 1 & \text{inside} \\ 0 & \text{outside} \end{cases} \text{the dipole magnet} \quad (3.36)$$
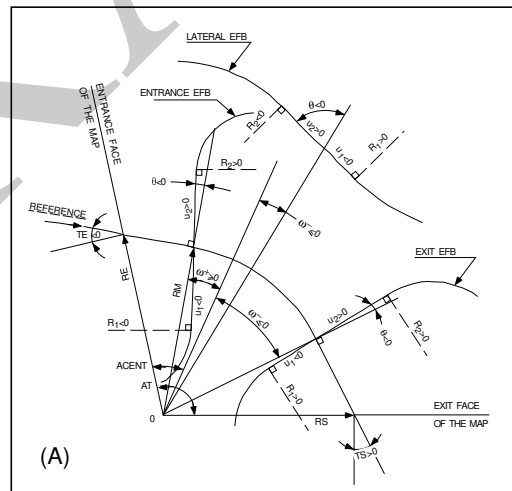
Fig. 3.24 Parameters used to define the geometry of a dipole magnet with index, using DIPOLE. In the text, ACENT is noted ACN [16, Fig. 9]

Setting up the input data list under DIPOLE (Table 3.6) requires close inspection of Fig. 3.24, which details the geometrical parameters such as the full angular opening of the field region that DIPOLE comprises, AT; a reference angle ACN to allow positioning the effective field boundaries at ω^+ and ω^-; field and indices; fringe field regions at $ACN - \omega^+$ (entrance) and $ACN - \omega^-$ (exit); wedge angles, etc.

A 60 deg sector is used here for convenience, it is detailed in Table 3.6 (Table 3.7 provides the definition of a 180 deg sector, for possible comparisons with the present three-sector assembly).

In setting up DIPOLE data the following values have been accounted for:

- $R_0 = 50$ cm, an arbitrary value (consistent with other exercises), more or less half the dipole extent,

- $B_0 = B_Z(R_0) = 5$ kG, as in the previous exercise. Note in passing, $R_0 = 50$ cm thus corresponds to $BR = 0.25$ T m, $E_k = 2.988575$ MeV proton kinetic energy,

- radial field index $k = 0$ for the time being (constant field at all (R, θ)),

- a hard-edge field model for $\mathcal{F}$ (Eq. 3.36). In that manner for instance, two consecutive 60 deg sectors form a continuous 120 deg sector.

A graph of $B_Z(R, \theta)$ can be produced by computing constant radius orbits, for a series of energies ranging in $0.12 - 5.52$ MeV for instance. DIPOLE[IL=2] causes logging of step by step particle data in zgoubi.plt, including particle position and magnetic field vector; these data can be read and plotted, to yield similar results to Fig. 3.20.

(b) Concentric trajectories in the median plane.

The optical sequence of Exercise 3.1-b (Tab. 3.4) can be used, by just changing the INCLUDE to account for a 180° DIPOLE (instead of TOSCA), namely

```
'INCLUDE'
1
3* 60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]
```

wherein 60degSector.inc is the name of the data file of Tab. 3.6 and
                    [#S_60degSectorUnifB:#E_60degSectorUnifB]
is the DIPOLE segment as defined in the latter. Note that the segment represents a 60° DIPOLE, thus it is included 3 times.

The additional keywords in that modified version of the Tab. 3.4 file include

- FIT, which finds the circular orbit for a particular momentum,

- FAISTORE to print out particle data once FIT is completed,

- REBELOTE, which repeats the execution of the sequence (REBELOTE sends the execution pointer back to the top of the data file) for a new momentum value which REBELOTE itself defines.

For the rest, follow the same procedure as for exercise 3.1-b. The results are the same, Fig. 3.21.

(c) Energy and rigidity dependence of orbit radius and time-of-flight.

2246   The orbit radius R and the revolution time T_{rev} as a function of kinetic energy E_k
2247 and rigidity BR are obtained by a similar scan to exercise (b). The procedure is the
2248 same as in exercise 3.1-c. Results are expected to be the same as well (Fig. 3.22).

2249   A comparison of revolution periods can be made using the simulation file of
2250 Table 3.6 which happens to be set for a momentum scan and yields Fig. 3.25, to
2251 be compared to Fig. 3.22: DIPOLE and TOSCA produce the same results as long
2252 as both methods are converged, from the integration step size stand point (small
2253 enough), and regarding TOSCA from field map mesh density stand point in addition
2254 (dense enough).

2255   (d) Numerical convergence: integration step size; $\dfrac{\delta R}{R}(\Delta s)$.

2256   This question concerns the dependence of the numerical convergence of the
2257 solution of the differential equation of motion upon integration step size.

2258   Follow the procedure of exercise 3.1-e: a similar outcome to Fig. 3.23 is expected
2259 - ignoring mesh density with the present analytical modeling using DIPOLE.

2260   The $\dfrac{\delta R}{R}$ dependence upon the integration step size Δs is commented in exer-
2261 cise 3.1-e and holds regardless of the field modeling method (field map or analytical
2262 model).

2263   (e) Pros and cons.

2264   Using a field map is a convenient way to account for complicated one-, two- or
2265 three-dimensional field distributions.

2266   However, using an analytical field model rather, ensures greater accuracy of the
2267 integration method.

2268   CPU-time wise, one or the other method may be faster, depending on the problem.
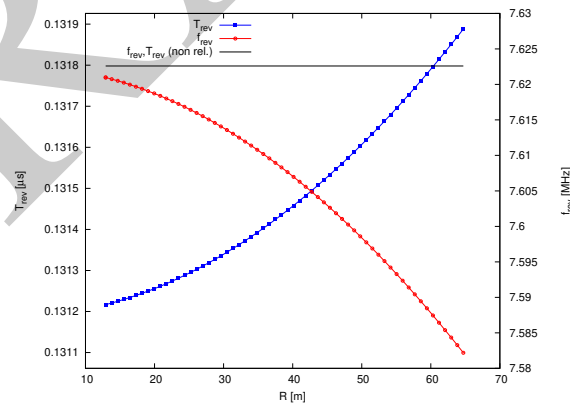
Fig. 3.25 A scan of radius-dependent revolution frequency. An analytical model of a cyclotron dipole is used, featuring uniform field (no radial gradient, at this point)

Table 3.6 Simulation input data file 60degSector.inc: analytical modeling of a dipole magnet, using DIPOLE. That file defines the labels (LABEL1 type [16, Sect. 7.7]) #S_60degSectorUnifB and #E_60degSectorUnifB, for INCLUDEs in subsequent exercises. It also realizes a 60-sample momentum scan of the cyclotron orbits, from 200 keV to 5 MeV, using REBELOTE

Note: this file is available in zgoubi sourceforge repository at

https://sourceforge.net/p/zgoubi/code/HEAD/tree/branches/exemples/book/zgoubiMaterial/cyclotron_classical/ProbMdlAnal/

```
60degSector.inc
! Cyclotron, classical. Analytical model of dipole field. File name: 60degSector.inc
'MARKER'  ProbMdlAnal_S                                           ! Just for edition purposes.
'OBJET'
64.62444403717985                                                 ! 200keV proton.
2
1 1                                                               ! Just one ion.
12.9248888074 0. 0. 0. 0.  1. 'm'                                 ! Closed orbit coordinates for D=p/p_0=1
1                                         ! => 200keV proton. R=Brho/B=64.624444037[kG.cm]/5[kG].
'PARTICUL'              ! Optioanl - using PARTICUL is a way to get the time-of-flight computed,
PROTON                                    ! otherwise, by default \zgoubi\ only requires rigidity.
'FAISCEAU'                                                        ! Local particle coordinates.
'MARKER'   #S_60degSectorUnifB                                    ! Label should not exceed 20 characters.
'DIPOLE'                                                          ! Analytical modeling of a dipole magnet.
2               ! IL=2, only purpose is to logged trajectories in zgoubi.plt, for further plotting.
60. 50.                                                           ! Sector angle AT; reference radius R0.
30.  5. 0. 0. 0.                  ! Reference azimuthal angle ACN; BM field at R0; indices, N, N', N''.
0.  0.                                                            ! EFB 1 is  hard-edge,
4 .1455  2.2670  -.6395 1.1558  0. 0.  0.             ! hard-edge only possible with sector magnet.
30. 0.  1.E6  -1.E6  1.E6  1.E6               ! Entrance face placed at omega+=30 deg from ACN.
0.  0.                                                            ! EFB 2.
4 .1455  2.2670  -.6395 1.1558  0. 0.  0.
-30. 0.  1.E6  -1.E6  1.E6  1.E6              ! Exit face placed at omega-=-30 deg from ACN.
0. 0.                                                             ! EFB 3 (unused).
0 0.     0.     0.     0.    0. 0. 0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
2 10    ! '2' is for 2nd degree interpolation. Could also be '25' (5*5 points grid) or 4 (4th degree).
1.                             ! Integration step size. Small enough for orbits to close accurately.
2  0. 0. 0. 0.                 ! Magnet positionning RE, TE, RS, TS. Could be instead non-zero, e.g.,
!                               2 RE=50. 0. RS=50. 0., as long as Yo is amended accordingly in OBJET.
'MARKER'    #E_60degSectorUnifB                                   ! Label should not exceed 20 characters.
'FAISCEAU'                                                        ! Local particle coordinates.
'FIT'                                      ! Adjust Yo at OBJET so to get final Y = Y0 -> a circular orbit.
1  nofinal
2 30 0 [12.,65.]                                                  ! Variable : Yo.
1 2e-12  199                      ! constraint; default penalty would be 1e-10; maximu 199 calls to function.
3.1 1 2 #End 0. 1. 0                                  ! Constraint: Y_final=Yo.
 'FAISTORE'                                            ! Log particle data here, to zgoubi.fai,
zgoubi.fai                                             ! for further plotting (by gnuplot, below).
1
'REBELOTE'                                             ! Momentum scan, 60 samples.
60 0.2 0      1 60 different rigidities; log to video ; take initial coordinates as found in OBJET.
1                                          ! Change parameter(s) as stated next lines.
OBJET 35 1:5.0063899693          ! Change relative rigity (35) in OBJET; range (0.2 MeV to 5 MeV).
'SYSTEM'
1                                                      ! 2 SYSTEM commands follow.
/usr/bin/gnuplot < ./gnuplot_TOF.gnu &                 ! Launch plot by ./gnuplot_TOF.gnu.
'MARKER'  ProbMdlAnal_E                                 ! Just for edition purposes.
 'END'
```

A gnuplot script, gnuplot_TOF.gnu, to obtain Fig. 3.25:

```
# gnuplot_TOF.gnu
set xlabel "R [m]"; set ylabel "T_{rev} [{/Symbol m}s]"; set y2label "f_{rev} [MHz]"
set xtics mirror; set ytics nomirror; set y2tics nomirror; set key  t l ;  set key spacin 1.2
nSector=6; Hz2MHz=1e-6; M=938.272e6; c=2.99792458e8; B=0.5; freqNonRel(x)= Hz2MHz* c**2*B/M/(2.*pi)
set y2range [7.58:7.63] ; set yrange[1/7.63:1/7.58]
plot \
"zgoubi.fai" u 10:($15 *nSector) axes x1y1 w lp pt 5 ps .6 lw 2 linecolor rgb "blue" tit "T_{rev}" ,\
"zgoubi.fai" u 10:(1/($15*nSector)) axes x1y2 w lp pt 6 ps .6 lw 2 linecol rgb "red" tit "f_{rev}" ,\
freqNonRel(x)  axes x1y2 w l lw 2. linecolor rgb "black"  tit "f_{rev},T_{rev} (non rel.)"  ; pause 1
```

Table 3.7 A 180° version of a DIPOLE sector, where the foregoing quantities $AT = 60^o$, $ACN = \omega^+ = -\omega^- = 30^o$ have been changed to $AT = 180^o$, $ACN = \omega^+ = -\omega^- = 90^o$ - a file used under the name 180degSector.inc in further exercises

Note: this file is available in zgoubi sourceforge repository at
https://sourceforge.net/p/zgoubi/code/HEAD/tree/branches/exemples/book/zgoubiMaterial/cyclotron_classical/ProbMdlAnal/

```
! 180degSector.inc
'MARKER'    #S_180degSectorUnifB                              ! Label should not exceed 20 characters.
 'DIPOLE'                                                     ! Analytical modeling of a dipole magnet.
2
180. 50.                                                     ! Sector angle 180deg; reference radius 50cm.
90.  5. 0. 0. 0.                          ! Reference azimuthal angle; Bo field at R0; indices, N, N', N''.
0.  0.                                                       ! EFB 1 is hard-edge,
4  .1455   2.2670  -.6395  1.1558  0.  0.  0.                ! hard-edge only possible with sector magnet.
90. 0.  1.E6  -1.E6  1.E6  1.E6
0.   0.                                                                          ! EFB 2.
4  .1455   2.2670  -.6395  1.1558  0.  0.  0.
-90. 0.  1.E6  -1.E6  1.E6  1.E6
0.  0.                                                                           ! EFB 3.
0  0.      0.     0.     0.     0. 0. 0.
0.  0.  1.E6  -1.E6  1.E6  1.E6 0.
2   10.
0.5                                       ! Integration step size. Small enough for orbits to close accurately.
2  0. 0. 0. 0.                            ! Magnet positionning RE, TE, RS, TS. Could be isntead non-zero, e.g.,
!                                           2 RE=50. 0. RS=50. 0., as long as Yo is amended accordingly in OBJET.
'MARKER'    #E_180degSectorUnifB                             ! Label should not exceed 20 characters.
```

3.3 Resonant Acceleration

The field map and TOSCA [16, *lookup* INDEX] model of a 180° sector is used here (an arbitrary choice, the analytical field modeling DIPOLE would do as well), the configuration is that of Fig. 3.5 with a pair of sectors.

An accelerating gap between the two dees is simulated using CAVITE[IOPT=3], PARTICUL is added in the sequence in order to specify ion species and data, necessary for CAVITE to operate. Acceleration at the gap does not account for the particle arrival time in the IOPT=3 option: whatever the later, CAVITE boost will be the same as longitudinal motion is an unnecessary consideration, here).

The input data file for this simulation is given in Tab. 3.8. It is resorted to INCLUDE, twice in order to create a double-gap sequence, using the field map model of a 180° sector. The INCLUDE inserts the magnet itself, *i.e.*, the #S_halfDipole to #E_halfDipole TOSCA segment of the sequence of Tab. 3.3. Note: the theoretical field model of Tab. 3.6, segment #S_60degSectorUnifB to #E_60degSectorUnifB (to be INCLUDEd 3 times, twice), could be used instead: exercise 3.2 has shown that both methods, field map and analytical field model, deliver the same results.

Particle data are logged in zgoubi.fai at both occurrences of CAVITE, under the effect of FAISTORE[LABEL=cavity], Tab. 3.8. This is necessary in order to access the evolution of parameters as velocity, time of flight, etc. at each half-turn, given that each half-turn is performed at a different energy

(a) Accelerate a proton.

A proton with initial kinetic energy 20 keV is launched on its closed orbit radius, $R_0 = p/qB = 4.087013$ cm. It accelerates over 25 turns due to the presence to REBELOTE[NPASS=24], placed at the end of the sequence. The energy range, 20 keV to 5 MeV, and the acceleration rate: 0.1 MeV per cavity, 0.2 MeV per turn, determine the number of turns, $NPASS+1 = (5 - 0.02)/0.2 \approx 25$. The accelerated

²²⁹⁵ trajectory spirals out in the fixed magnetic field, it is plotted in Fig. 3.26, reading
²²⁹⁶ data from zgoubi.plt.

Fig. 3.26 Twenty five turn spiral trajectory of a proton accelerated in a uniform 0.5 T field from 20 keV to 5 MeV at a rate of 200 kV per turn (a 100 kV gap voltage). The vertical thick line materializes the gap, the upper half (red) corresponds to the first occurrence of CAVITE in the sequence (Tab. 3.8), the lower half (blue) corresponds to the second occurrence of CAVITE
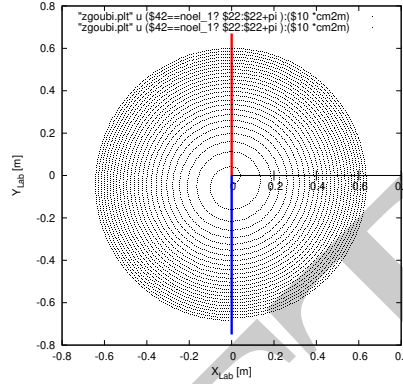
²²⁹⁷ (b) Momentum and energy.

²²⁹⁸ Proton momentum p and total energy E as a function of kinetic energy, from
²²⁹⁹ raytracing (turn-by-turn particle data are read from zgoubi.fai, filled up due to FAI-
²³⁰⁰ STORE) are displayed in Fig. 3.27, together with theoretical expectations, namely,
²³⁰¹ $p(E_k) = \sqrt{E_k(E_k + 2M)}$ and $E = E_k + M$.
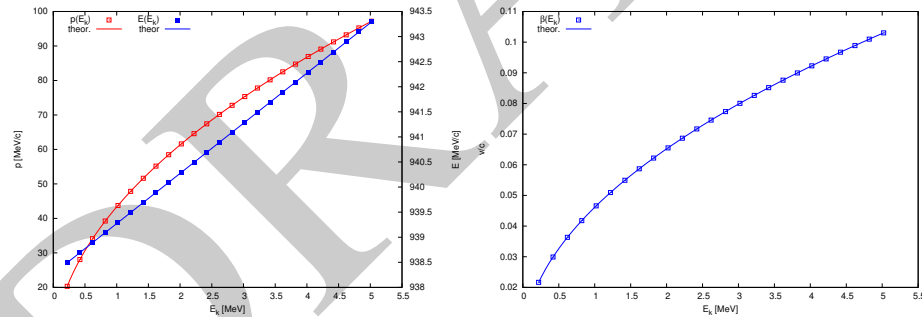
Fig. 3.27 Energy dependence of, left: proton momentum p (left axis) and total energy E (right axis) and of, right: proton normalized velocity $\beta = v/c$. Markers: from raytracing; solid lines: theoretical expectation

²³⁰² (c) Velocity.

²³⁰³ Proton normalized velocity $\beta = v/c$ as a function of kinetic energy from raytracing
²³⁰⁴ is displayed in Fig. 3.27, together with theoretical expectation, namely, $\beta(E_k) =$
²³⁰⁵ $p/(E_k + M)$.

Table 3.8 Simulation input data file: accelerating a proton in a double-dee cyclotron, from 20 keV to 5 MeV, at a rate of 100 kV per gap, independent of RF phase (longitudinal motion is frozen - see question (e) dealing with CAVITE[IOPT=7] for unfrozen motion). Note that particle data are logged in zgoubi.fai (under the effect of FAISTORE) at both occurrences of CAVITE. The INCLUDE file FieldMapSector.inc is taken from Tab. 3.3

```
Cyclotron, classical. Acceleration:  20 keV -> 6 MeV.
'MARKER'  ProbAccelGap_S                                               ! Just for edition purposes.
'OBJET'
64.62444403717985                   ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1                                                                              ! Just one ion.
4.087013 0. 0. 0. 0. 0.3162126 'o'         ! D=0.3162126 => Brho[kG.cm]= 20.435064, kin-E[keV]= 20.
1
'PARTICUL'                                            ! Usage of CAVITE requires partical data,
PROTON                                     ! otherwise, by default \zgoubi\ only requires rigidity.
'FAISTORE'                                                   ! Store particle data, turn-by-turn.
zgoubi.fai   cavity          ! Log coordinates at any occurence of LABEL1=cavity, in zgoubi.fai.
1
'INCLUDE'                                                     ! Insert a 180 deg sector field map.
1
FieldMapSector.inc[#S_halfDipole:#E_halfDipole]
'FAISCEAU'                                                 ! Particle coordinates before RF gap.
'CAVITE'  cavity                                                          ! Accelerating gap.
3                                ! dW = qVsin(phi), independent of time (phi forced to constant).
0. 0.                                                                            ! Unused.
100e3  1.57079632679                                      ! Peak voltage 100 kV; RF phase = pi/2.
'INCLUDE'                                                     ! Insert a 180 deg sector field map.
1
FieldMapSector.inc[#S_halfDipole:#E_halfDipole]
'FAISCEAU'                                                 ! Particle coordinates before RF gap.
'CAVITE'  cavity                                                          ! Accelerating gap.
3                                ! dW = qVsin(phi), independent of time (phi forced to constant).
0. 0.                                                                            ! Unused.
100e3  1.57079632679                                      ! Peak voltage 100 kV; RF phase = pi/2.
'REBELOTE'            ! Repeat NPASS=24 times, for a total of 25 turns; K = 99: coordinates at end of
24   0.1 99                          ! previous pass are used as initial coordinates for the next pass.
'FAISCEAU'                                            ! Local particle coordinates logged in zgoubi.res.

'SYSTEM'
2                                                              ! 2 SYSTEM command follow:
/usr/bin/gnuplot < ./gnuplot_Zplt_XYLab.gnu &                             ! plot trajectories.
/usr/bin/gnuplot < ./gnuplot_awk_Zfai_dTT.gnu &                 ! dC/C, dbta/bta, dT/T graph.

'MARKER'  ProbAccelGap_E                                               ! Just for edition purposes.
'END'
```

Two gnuplot scripts, to obtain respectively Fig. 3.26: _and Fig._ 3.28:
The awk command in gnuplot_awk_Zfai_dTT.gnu takes care of a 1-row shift so to subtract next turn data from currant turn ones.

```
# gnuplot_Zplt_XYLab.gnu
set xtics ;  set ytics ; set xlabel "X_{Lab} [m]" ; set ylabel "Y_{Lab} [m]"
set size ratio 1 ; set polar ; cm2m = 0.01 ; pi = 4.*atan(1.)
set arrow from 0, 0 to 0, 0.67 nohead lc "red" lw 6; set arrow from 0, -0.75 to 0, 0 nohead lc "blue" lw 6
noel_1=6 ; noel_2=11  # 1st CAVITE is element noel_1; 2nd CAVITE is noel_2. Col. $42 in zgoubi.plt is element numb.
plot for [nl=noel_1:noel_2:5] "zgoubi.plt" u ($42==noel_1? $22:$22+pi ):($10 *cm2m) w p pt 5 ps .2 lc rgb "black"

# gnuplot_awk_Zfai_dTT.gnu
set xtics nomirror; set ytics mirror; set xlabel "E_k [MeV]";
set ylabel "{/Symbol Db}/{/Symbol b},  {/Symbol D}C/C,  {/Symbol D}T_{rev}/T_{rev}"; set logscale y; set yrange [:3]
# zgoubi.fai columns: $25: energy; $14: path length; $23: kinetic E; $29: mass; $15: tim
plot "< awk '/#/ {next;}  { if(prev14>0 && prev25>0) print prev24, ($14 -prev14)/prev14 , prev24} \
{ prev14 = $14; prev24 = $24; prev25=$25 }' < zgoubi.fai" u 1:2 w p p pt 5 lc rgb "black" tit "{/Symbol D}C/C" ,\
"< awk '/#/ {next;}  { if(prev14>0 && prev25>0) print prev24, ( -sqrt(prev25**2-$29**2)/prev25 + \
sqrt($25**2-$29**2)/$25 )/(sqrt(prev25**2-$29**2)/prev25) , prev24}  { prev14 = $14; prev24 = $24; prev25=$25 }' \
< zgoubi.fai"  u 1:2 w p pt 6 ps 1.5 lc rgb "red" tit "d{/Symbol b}/{/Symbol b}" ,\
"< awk '/#/ {next;}  { if(prev14>0 && prev25>0) print prev24, ($14 -prev14)/prev14- ( -sqrt(prev25**2-$29**2)/prev25 \
+ sqrt($25**2-$29**2)/$25 )/(sqrt(prev25**2-$29**2)/prev25) , prev24}  { prev14 = $14; prev24 = $24; prev25=$25 }' \
< zgoubi.fai"  u 1:2 w p pt 8 ps 1.5 lc rgb "blue"  tit "{/Symbol D}T/T=dC/C-d{/Symbol b}/{/Symbol b}" ,\
"< awk '/#/ {next;}  { if(prev14>0 && prev15>0) print prev24, ($15-prev15)/prev15 , prev24}  { prev14 = $14; \
prev24 = $24; prev15=$15 }' < zgoubi.fai"  w l lw 2 lc rgb "blue" tit "theor. {/Symbol D}T/T"
```

Fig. 3.28 Relative variation of velocity $\Delta\beta/\beta$ (empty circles), circumference $\Delta C/C$ (solid disks) and revolution time $\Delta T/T$ (triangles), as a function of energy, from raytracing. Theoretical expectation for the latter is also displayed (solid line), for comparison
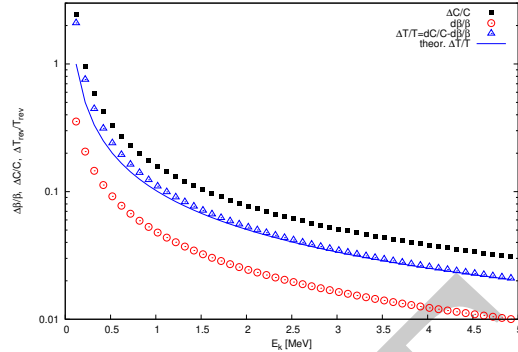
(d) Relative velocity, orbit length and time of flight.

The relative increase in velocity is smaller than the relative increase in orbit length as energy increases (this is what Fig. 3.28 shows). Thus the relative variation of the revolution time, Eq. 3.23, is positive; in other words the revolution time increases with energy, the revolution frequency decreases. Raytracing outcomes are displayed in Fig. 3.28, they are obtained using the gnuplot script given in Tab. 3.8. Note that the path length difference (taken as the difference of homologous quantities in a common line) is always between the two CAVITEs (particle data are logged at the two occurrences of CAVITE), crossed successively, which is half a turn. Same for the difference between homologous velocity data on a common line, it corresponds to two successive crossings of CAVITE, *i.e.*, half a turn. The graph includes the theoretical $\delta T_{\mathrm{rev}}/T_{\mathrm{rev}}$ (Eq. 3.23) for comparison with raytracing; some difference appears in the low velocity regime, this may be due to the large $\Delta\beta$ step imparted by the 100 kV acceleration at the gaps.

(e) Harmonic h=3 RF.

The input data file for this simulation is given in Tab. 3.9. The RF is on harmonic h=3 of the revolution frequency. It has been tuned to ensure acceleration up to 3 MeV. The accelerating gap between the two dees is simulated using CAVITE[IOPT=7]: by contrast with the previous exercise (where CAVITE[IOPT=3] is used), the RF phase at ion arrival at the gap is now accounted for.

Repeating questions (b-d) is straightforward, changing what needs be changed in Tab. 3.9 input data file.

Table 3.9 Simulation input data file: accelerating a proton in a double-dee cyclotron, from 20 keV to 5 MeV, using harmonic 3 RF. The INCLUDE file is taken from Tab. 3.6

```
Cyclotron, classical. Analytical model of dipole field.
'OBJET'
64.62444403717985                                                ! 200keV proton.
2
1 1                                                              ! Just one ion.
12.924888 0. 0. 0. 0.  1.  'm'         ! D=1 => 200keV proton. R=Brho/B=64.624444037[kG.cm]/5[kG].
1
'PARTICUL'                                        ! This is required for spin motion to be computed,
PROTON                                            ! otherwise, by default \zgoubi\ only requires rigidity.
'INCLUDE'
1                                                                ! Include a first 180 deg sector.
./180degSector.inc[#S_180degSectorUnifB:#E_180degSectorUnifB]
'CAVITE'
7
0 22862934.0
285e3 -0.5235987755982988
'INCLUDE'
1                                                                ! Include a second 180 deg sector.
./180degSector.inc[#S_180degSectorUnifB:#E_180degSectorUnifB]
'CAVITE'
7
0 22862934.0                                                     ! RF = 3/T_rev.
285e3 -3.665191429188092                          ! Peak voltage; synchronous phase.
'REBELOTE'
26 0.4 99                                                        ! 26+1 turn tracking.
 'END'
```

3.4 Spin Dance

The DIPOLE analytical field model of exercise 3.2 (Tab. 3.6) is used here, as opposed to using a field map and TOSCA, as it allows more straightforward changes in the field, if desired.

(a) Spin transport.

Spin transport is obtained by adding SPNTRK. PARTICUL is necessary in order to get the Thomas-BMT equation of motion solved [16, Sect. 2]. This results in the input data file given in Tab. 3.10 (excluding FIT and REBELOTE keywords, introduced for the purpose of the following question (b)).

The use of SPNTRK results in the following outcome (an excerpt from zgoubi.res execution listing):

```
     4 Keyword, label(s) :  SPNTRK
          Spin  tracking  requested.
                    Particle  mass       =     938.2721     MeV/c2
                    Gyromagnetic  factor  G =   1.792847
                    Initial spin conditions type  1 :
                    All particles have spin parallel to  X  AXIS
          PARAMETRES  DYNAMIQUES  DE  REFERENCE :
                    BORO     =       64.624 kG*cm
                    beta     =     0.02064411
                    gamma    =     1.00021316
                    gamma*G  =     1.7932295094
          POLARISATION  INITIALE  MOYENNE  DU  FAISCEAU  DE        1  PARTICULES :
                    <SX> =     1.000000
                    <SY> =     0.000000
                    <SZ> =     0.000000
                    <S>  =     1.000000
```

Spin coordinates are logged in zgoubi.res execution listing using SPNPRT. Five sample passes around the cyclotron (four iterations by REBELOTE) result in the following outcomes in zgoubi.res, under SPNPRT:

```
    26 Keyword, label(s) :  SPNPRT
              INITIAL                                    FINAL
        SX        SY        SZ        |S|        SX        SY        SZ        |S|      GAMMA
 m 1  1.000000  0.000000  0.000000  1.000000   0.268269  0.963344  0.000000  1.000000   1.0002
 m 1  1.000000  0.000000  0.000000  1.000000   0.268599  0.963252  0.000000  1.000000   1.0002
 m 1  1.000000  0.000000  0.000000  1.000000   0.268949  0.963154  0.000000  1.000000   1.0003
 m 1  1.000000  0.000000  0.000000  1.000000   0.269319  0.963051  0.000000  1.000000   1.0003
 m 1  1.000000  0.000000  0.000000  1.000000   0.269710  0.962942  0.000000  1.000000   1.0003
```

Table 3.10 Simulation input data file: add spin to the cyclotron simulation of Tab. 3.6. The present input file INCLUDEs six copies of the 60 degree sector DIPOLE defined therein

```
Cyclotron, classical. Analytical model of dipole field. Spin transport.
'MARKER'  ProbAddSpin_S                                          ! Just for edition purposes.
'OBJET'
64.62444403717985                    ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1                                                               ! Just one ion.
12.9248888074 0. 0. 0. 0.  1.  'm'       ! D=1 => 200keV proton. R=Brho/B=64.624444037[kG.cm]/5[kG].
1
'PARTICUL'                                               ! This is required to get the time-of-flight,
PROTON                                     ! otherwise, by default \zgoubi\ only requires rigidity.
'SPNTRK'                                                          ! Request spin tracking.
1                                              ! All spins launched longitudinal (parallel to OX axis).
'INCLUDE'
1
6* ./60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]              ! 6 * 60 degree sector.
'FAISCEAU'                               ! Local particle coordinates.
'FIT'                                 ! Adjust Yo at OBJET so to get final Y = Y0 -> a circular orbit.
1   nofinal
2 30 0 [12.,65.]                                                   ! Variable : Yo.
1  2e-12  199          ! constraint; default penalty would be 1e-10; maximu 199 calls to function.
3.1 1 2 #End 0. 1. 0                                    ! Constraint:  Y_final=Yo.
'FAISCEAU'                                  ! Allows checking that Y = Y0 and T = T0 = 0, here.
'SPNPRT'                                         ! Local spin data, logged in zgoubi.res.
'FAISTORE'                                       ! Log particle data here, to zgoubi.fai,
zgoubi.fai                               ! for further plotting of spin coordinates (by gnuplot, below).
1
'REBELOTE'                                    ! Momentum scan, 60 samples.
60 0.2  0         1 60 different rigidities; log to video ; take initial coordinates as found in OBJET.
1                                                 ! Change parameter(s) as stated next lines.
OBJET 35  1:5.0063899693           ! Change relative rigity (35) in OBJET; range (0.2 MeV to 5 MeV).
'SYSTEM'
1                                                 ! 2 SYSTEM commands follow.
/usr/bin/gnuplot < ./gnuplot_Zfai_spin.gnu &
'MARKER'  ProbAddSpin_E                                          ! Just for edition purposes.
 'END'
```

A gnuplot script to obtain Fig. 3.29:
The file zgoubi.1cm is a copy of zgoubi.fai obtained for a $\Delta s = 1$ cm run; zgoubi.fai is for $\Delta s = 0.5$ cm.

```
# gnuplot_Zfai_spin.gnu
set xlabel "G{/Symbol g}"; set ylabel "Spin precession angle  {/Symbol q}_{sp} / 2{/Symbol p}"
set y2label "relative difference num./theor"; set logscale y2
set xtics; set ytics nomirror; set y2tics; am = 938.27208; G = 1.79284735; pi = 4.*atan(1.); set key t c spacin 1.5
plot \
"zgoubi.fai" u ($31*$25/$29):(((4.*pi -atan($21/$20)))/(2.*pi)) w lp pt 4 ps .7 tit "{/Symbol q}_{sp}/2{/Symbol p}" ,\
"zgoubi.1cm" u ($31*$25/$29):(abs((4*pi-atan($21/$20))/pi*180-$31*$25/$29*360.)) axes x1y2 w lp pt 8 ps .7 tit "1 cm",\
"zgoubi.fai" u ($31*$25/$29):(abs((4.*pi -atan($21/$20))/pi*180-$31*$25/$29*360.)) axes x1y2 w lp pt 8 ps .7 tit "5 mm"
```

(b) Spin precession.

Proton case is considered, simulation is performed using Tab. 3.10 input data file. Initial spin is parallel to the X axis (longitudinal). The particle is raytraced on the circular closed orbit over one revolution, for a particular momentum. Particle data resulting from a FIT (FIT forces orbit closure, by varying the initial Y_0) are logged in zgoubi.fai, by FAISTORE. The computation is repeated using REBELOTE in the very manner that the energy scan was done in exercise 3.2, over an energy range 12 keV→5 MeV.

Figure 3.29 (obtained using the gnuplot script given in Tab. 3.10) displays the resulting energy dependence of the spin precession, $\theta_{sp}(E)$, together with its difference to theoretical expected $\theta_{sp}(E) = G\frac{E}{M} \times 2\pi = G\gamma \times 2\pi$ (proton gyromagnetic anomaly $G = 1.792847$).

Fig. 3.29 $G\gamma$ dependence of the spin precession angle over a revolution around the cyclotron, in the moving frame (left axis), and relative difference to $G\gamma$ for the two integration step sizes $\Delta s = 0.5$ and 1 cm (right axis), Markers are from raytracing, solid lines are to guide the eye
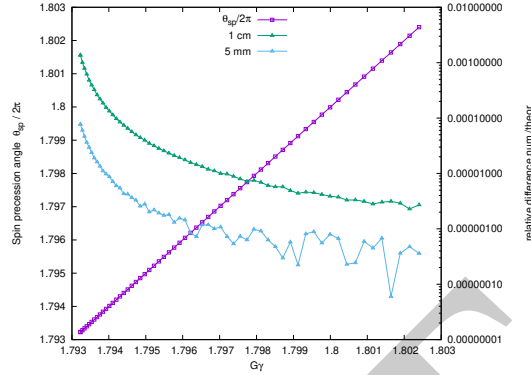
(c) Spin tune.

Two protons are injected with longitudinal initial spin $\mathbf{S}_i \parallel$ OX axis and respective energies 12 keV and 5.52 MeV, thus the following OBJET (a slight modification to Tab. 3.10 data):

```
'OBJET'
64.62444403717985                      ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
2 1
12.9248888074 0. 0. 0. 0. 1.   'm'     ! D=1 => 200keV proton. R=Brho/B=64.624444037[kG.cm]/5[kG].
67.997983 0. 0. 0. 0. 5.2610112 'o'     ! p[MeV/c]=101.926, Brho[kG.cm]=339.990, kin-E[MeV]=5.52.
1 1
```

FAISCEAU following FIT (Tab. 3.10) allows to control that momentum and trajectory radius are matched, which means coordinates at OBJET and current coordinates at FAISCEAU are equal. Inspection of zgoubi.res execution listing shows for instance, after 4 turns:

```
                    OBJET                                              FAISCEAU
      D       Y(cm)    T(mr)    Z(cm)    P(mr)    S(cm)    D-1     Y(cm)    T(mr)    Z(cm)    P(mr)    S(cm)
   1.0000    12.925    0.000    0.000    0.000    0.0000   0.0000  12.925    0.000    0.000    0.000   3.248379E+02
   5.2610    67.998    0.000    0.000    0.000    0.0000   4.2610  67.998   -0.000    0.000    0.000   1.708976E+03
```

A graphic of the projection of the spin motion on the longitudinal axis, over a few turns, from the ray tracing, is given in Fig. 3.30, together with the longitudinal component as of the parametric equations of motion

$$\begin{cases} S_X = \hat{S} \cos(G\gamma\theta) \\ S_Y = \hat{S} \sin(G\gamma\theta) \end{cases} \tag{3.37}$$

The motion amplitude is $\hat{S} = \sin\phi$, with ϕ the angle that the spin vector makes with the vertical precession axis. In this simulation $\mathbf{S}$ is launched parallel to OX, thus $\phi = \pi/2$ and $\hat{S} = 1$.

Now, checking the spin precession:

Placing both FAISCEAU and SPNPRT commands right after the first dipole sector allows checking the spin precession and its relationship to particle rotation, for simplicity right after the first pass through that first sector, as follows. FAISCEAU and SPNPRT (Tab. 3.10) yield, respectively:

```
                    OBJET                                              FAISCEAU
      D       Y(cm)    T(mr)    Z(cm)    P(mr)    S(cm)    D-1     Y(cm)    T(mr)    Z(cm)    P(mr)    S(cm)
   1.0000    12.925    0.000    0.000    0.000    0.0000   0.0000  12.925    0.000    0.000    0.000   3.248379E+02
   5.2610    67.998    0.000    0.000    0.000    0.0000   4.2610  67.998   -0.000    0.000    0.000   1.708976E+03
```
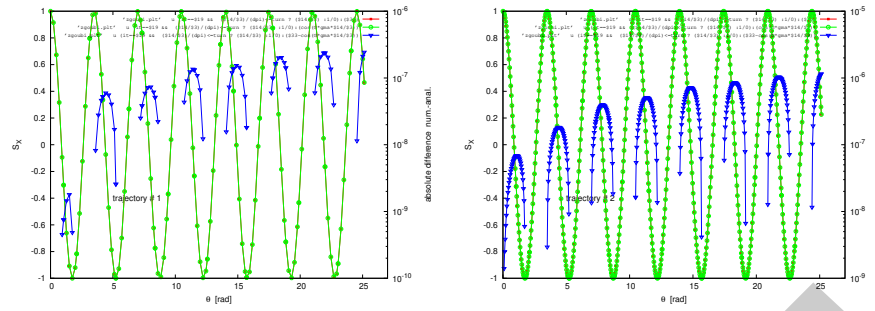
Fig. 3.30 Longitudinal spin component motion (left vertical axis), observed in the moving frame, case of 0.2 MeV energy, R=12.924888 cm (left graph), and of 5.52 MeV energy, R=67.998 cm (right graph). Markers are from ray tracing, the solid line is the theoretical expectation (Eq. 3.37). The right vertical axis (triangle markers; solid line is to guide the eye) shows the absolute difference between both. The oscillation is as expected slightly faster at 5.52 MeV: frequencies are in the ratio $\gamma(5.52\,\text{MeV})/\gamma(0.2\,\text{MeV}) = 1.00566$

		INITIAL				FINAL					--- angles ---	
		SX	SY	SZ	|S|	SX	SY	SZ	|S|	GAMMA	|Si,Sf|	(Z,Sf)
											(deg.)	(deg.)
m	1	1.000000	0.000000	0.000000	1.000000		-0.302266	-0.953224	0.000000	1.000000	1.0002	-107.594	90.000
o	1	1.000000	0.000000	0.000000	1.000000		-0.312396	-0.949952	0.000000	1.000000	1.0059	-108.204	90.000

SPNPRT tells that,

- case of the first particle, tagged 'm' above; its energy is 200 keV, $\gamma = 1.00021315$, its spin tune is $\nu_{sp} = G\gamma = 1.793229$

The computed value of the '(S_i, S_f)' angle between initial and final spin vectors is -107.594 (truncated), negative as spin precession has the sign of proton rotation. Theoretical expectation is $G\gamma\alpha = -107.59377$ deg. The resulting spin components are, as above, $S_X = \cos(-107.59377) = -0.302266$ and $S_Y = \sin(-107.59377) = -0.9532235$.

- case of the second particle, tagged 'o'; its energy is 5.52 MeV, $\gamma = 1.00588315$, its spin tune is $\nu_{sp} = G\gamma = 1.803394$

The computed value of '(S_i, S_f)' is -108.204 (truncated). Theoretical expectation is $G\gamma\alpha = -108.20370$ deg.

Now, accounting for particle rotation in order to get spin coordinates in the laboratory frame:

- the FAISCEAU outcome above shows that, after crossing the 60 deg sector the angles of the two particles have the value $T = 0$, which is expected as they are launched with zero incidence, and as DIPOLE uses a polar coordinate system [16] with particle coordinates computed in the moving (rotating) frame. The latter has also undergone a -60 deg rotation, clockwise, which is therefore the implicit rotation of the particles in the laboratory frame. The spin precession in the laboratory frame results, namely,

- case of the first particle: $(1 + G\gamma)\alpha = -167.59377$ deg.

- case of the second particle: $(1 + G\gamma)\alpha = -168.20370$ deg.

(d) Spin dance.

A 200 keV proton is injected with its initial spin vector at 80 degrees from the vertical axis. The input data file for this simulation is given in Tab. 3.11, together with a gnuplot script for the animation. The latter plots three things, concurrently:

- the circular trajectory of the particle in the (X,Y) plane: this is the curve at Z=0 in Fig. 3.31, a set of points $\{(R\cos(-X), R\sin(-X), 0)\}$ resulting from the step by step integration. Note that X is counted positive clockwise in zgoubi.fai (consistently with the definition of DIPOLE parameters, Fig. 9 in [16]), hence "-X" the rotation angle;

- the spin vector: its foot is attached to the particle (the previous set of points), whereas its tip is at $\{(S_X\cos(-X) - S_Y\sin(-X), S_X\sin(-X) + S_Y\cos(-X), S_Z\}$, with S_X, S_Y, S_Z the spin vector components in the moving frame as read from zgoubi.fai. S_Z is constant as the precession axis is parallel to the Z axis. The $\begin{pmatrix} \cos(-X) & -\sin(-X) \\ \sin(-X) & \cos(-X) \end{pmatrix}$ rotation applied to the (S_X, S_Y) vector accounts for the transformation from the moving frame to the laboratory frame;

- the cycloidal shape trajectory of the tip of the spin vector (the previous set of points).

A frozen view of that spin dance, over about 2.5 proton revolutions around the ring, is given in Fig. 3.31.
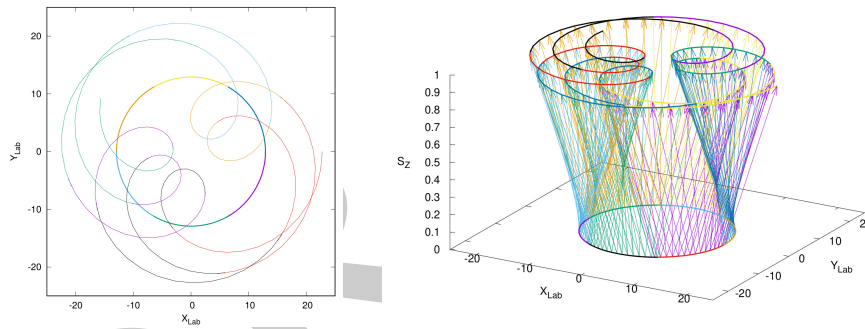
Fig. 3.31 Dance - frozen, here - of the spin of a 200 keV proton over 2.5 turns around the cyclotron. The circle on the left, or bottom closed curve on the right, is the trajectory of the proton. The cycloidal curve represents the motion of the spin vector tip in the moving frame

Table 3.11 Simulation input data file: spin dance, 20 turns around a uniform field cyclotron. The INCLUDE file 60degSector.inc is taken from Tab. 3.6

Note: this animation (input data file & gnuplot script) is available in **zgoubi** sourceforge repository at

https://sourceforge.net/p/zgoubi/code/HEAD/tree/branches/exemples/book/zgoubiMaterial/cyclotron_classical/ProbAddSpin/spinDance/

```
Cyclotron, classical. Spin dance.
'MARKER'  ProbAddSpinDance_S                                  ! Just for edition purposes.
'OBJET'
64.62444403717985                    ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1                                                                    ! Just one ion.
12.9248888074 0. 0. 0. 0.  1.     'm'     ! D=1 => 200keV proton. R=Brho/B=64.624444037[kG.cm]/5[kG].
1
'PARTICUL'                                        ! This is required to get the time-of-flight,
PROTON                                 ! otherwise, by default \zgoubi\ only requires rigidity.
'SPNTRK'                                                     ! Request spin tracking.
4.1                                                          ! All spins are initially
0.984807753012  0. 0.173648177667                            ! at 10 degrees to X axis.
'FAISCEAU'
'INCLUDE'
1
6* ./60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]        ! 6 * 60 degree sector.
'REBELOTE'                                                    ! Multiturn:
19 0.2  99                                         19 additional passes.
'SYSTEM'
1
gnuplot < ./gnuplot_Zplt_SDance.gnu
'MARKER'  ProbAddSpinDance_E                                  ! Just for edition purposes.
'END'
```

A gnuplot script to obtain the spin dance in Fig. 3.31. *Note a "mag" factor, aimed at artificially increasing the amplitude of the vector tip oscillation in this graphic*:

```
set xlabel "X_{Lab}"; set ylabel "Y_{Lab}"; set zlabel "S_Z"; set xtics; set ytics; set ztics  #unset ztics
set zrange  [0:]; set xrange [-25:25]; set yrange [-25:25]; set xyplane 0
dip1=7; dip2=22; dd=3 # positing of 1st and last dipoles in zgoubi.dat sequence, and increment
# magnifies apparent spin tilt    speed up graphic           pi/3                z norm
         mag = 10.         ;         speedUp=1    ;    pi3 = 4.*atan(1.)/3  ;   nz=0.18

# JUST 2D, PROJECTED IN (X,Y) PLANE, FIRST:
set size ratio -1
do for [i=1:239]{ plot \
for [dip=dip1:dip2:dd] "zgoubi.plt" every 1:::speedUp*i u ($19==1 && $42==dip? $10*cos(-$22-pi3*(dip-6.)/3.) :1/0): \
($10*sin(-$22-pi3*(dip-6.)/3.)) w l lw 3 notit ,\
for [dip=dip1:dip2:dd] "zgoubi.plt" every 1:::speedUp*i u ($19==1 && $42==dip? $10*cos(-$22-pi3*(dip-6.)/3.) \
+mag*(cos(-$22-pi3*(dip-6.)/3.)*$33-sin(-$22-pi3*(dip-6.)/3.)*$34) :1/0): \
($10*sin(-$22-pi3*(dip-6.)/3.) +mag*(sin(-$22-pi3*(dip-6.)/3.)*$33+cos(-$22-pi3*(dip-6.)/3.)*$34)) w l notit }
unset size

# 3D, NEXT:
do for [i=1:239]{ splot \
for [dip=dip1:dip2:dd] "zgoubi.plt" every speedUp*i:::speedUp*i u ($19==1&& $42==dip? $10*cos(-$22-pi3*(dip-6)/3):1/0):\
($10*sin(-$22-pi3*(dip-6)/3)):($1*0):(mag*(cos(-$22-pi3*(dip-6)/3)*$33-sin(-$22-pi3*(dip-6)/3)*$34)): \
(mag*(sin(-$22-pi3*(dip-6)/3)*$33+cos(-$22-pi3*(dip-6)/3)*$34)):($35/nz)  w vectors notit ,\
for [dip=dip1:dip2:dd] "zgoubi.plt" every 1:::speedUp*i u ($19==1 && $42==dip? $10*cos(-$22-pi3*(dip-6)/3) :1/0): \
($10*sin(-$22-pi3*(dip-6)/3)):($1*0):($19==1&&$42==dip? $10*cos(-$22-pi3*(dip-6)/3):1/0):($10*sin(-$22-pi3*(dip-6)/3)): \
($1*0) w l lw 3 notit ,\
for [dip=dip1:dip2:dd] "zgoubi.plt" every 1:::speedUp*i u ($19==1 && $42==dip? $10*cos(-$22-pi3*(dip-6)/3)+mag*( \
cos(-$22-pi3*(dip-6)/3)*$33-sin(-$22-pi3*(dip-6)/3)*$34):1/0):($10*sin(-$22-pi3*(dip-6)/3)+mag*(sin(-$22-pi3*(dip-6)/3) \
*$33+cos(-$22-pi3*(dip-6)/3)*$34)):($35/nz):($19==1&&$42==dip? $10*cos(-$22-pi3*(dip-6)/3 +mag*(cos(-$22-pi3*(dip-6)/3) \
*$33 -sin(-$22-pi3*(dip-6)/3)*$34)) :1/0): ($10*sin(-$22-pi3*(dip-6)/3) +mag*(sin(-$22-pi3*(dip-6)/3)*$33+cos(-$22-pi3* \
(dip-6)/3) *$34)):($35/nz) w l lw 3 notit  }
```

2459   (e) Deuteron.

2460   The input data file set up for questions (b-e) can be used *mutatis mutandis*, as
2461   follows.

2462   Raytracing a different particle requires changing the reference rigidity, BORO,
2463   under OBJET, and changing particle data, under PARTICUL. That reference rigidity
2464   is to be determined from the field value in the dipole model (namely, $B_0 = 5\,\text{kG}$).

Particle data for these two particles are (respectively mass (MeV/c^2), charge (C), G factor):

$$deuteron: \quad 1875.612928 \quad 1.602176487 \times 10^{-19} \quad -0.14301$$

$$^3He^{2+}: \quad 2808.391585 \quad 3.204352974 \times 10^{-19} \quad -0.14301$$

3.5 Synchronized Spin Torque

The simulation input data file of exercise 3.4-(d) can be used here, with a few addenda or modifications, as follows:

(i) the initial ion coordinate D (rigidity relative to the reference BORO=64.6244440) under OBJET has to be calculated for the four energies concerned;

(ii) the closed orbit radius at 0.2, 108.412, 118.878 and 160.746 MeV has to be found; calculation is straightforward given that the field considered here is vertical, uniform, namely, B_Z=constant=5 kG, $\forall R$, so that $R = B\rho/B_Z$; otherwise a FIT procedure can be used to find the orbit radius, given the rigidity, as done already in various exercises [16, lookup "closed orbit"], that could help for instance in the presence of a radial index, or field defects;

(iii) initial spins are set vertical for convenience, but this is not mandatory;

(iv) the multiturn tracking is set to a few 10s of turns, in order to allow a few spin precessions;

(v) particle data through DIPOLEs are saved step-by-step all the way in zgoubi.plt by means of IL=2 (the integration step size is 1 cm (Tab. 3.6), thus zgoubi.plt may end up bulky);

(vi) turn-by-turn data are saved in zgoubi.fai by means of FAISTORE;

(vii) SPINR is added at the end of the sequence, to impart on spins the requested X-tilt.

This results in the updated simulation input data file given in Tab. 3.12.

The oscillatory motion of the vertical spin component as the ion orbits around the ring, is displayed in Fig. 3.32. The spin points upward, parallel to the vertical axis at start; SPINR kick is 10 deg in the present case. At $G\gamma = 2$ the spin always finds itself back in the (Y,Z) transverse plane after one proton orbit, this synchronism causes the cumulated spin tilt at SPINR to take the value $N \times 10$ deg (with N the number of orbits). Thus after 18 proton orbits, 36 spin precessions, the spin points downward; it takes 36 orbits, or 226.194 rad, to complete an oscillation. If $G\gamma$ moves away from an integer, the spin tilts with bounded amplitude, within the limits of a cone.

Additional graphs and details are obtained using the simulation file of Tab. 3.13. This file simulates spin motion in three different cases, $G\gamma = 1.79322$, $G\gamma = 2$, integer, yielding an integer number of spin precessions over one proton orbit around the cyclotron, and $G\gamma = 2.5$, half-integer, yielding a half-integer number of spin precessions over one proton orbit. Outcomes are given in Fig. 3.33 which shows the spin motion projected on the (X,Y) plane (horizontal), and on a sphere, step-by-step. The spin kick by SPINR is 20 deg in this case. If $G\gamma = 1.793229$, far from an integer, **S**, initially vertical, remains at a bounded angle to the vertical axis, X-kicked from one circle to another, turn after turn; if $G\gamma = 2$ the spin vector flips by 20 degree in the (Y,Z) plane at SPINR, turn after turn; if $G\gamma = 2.5$, half-integer, the spin vector

Table 3.12 Simulation input data file: superimposition of a turn-by-turn localized 10 deg X-rotation of the spin (using SPINR[$\phi = 0, \mu = 10$]), on top of Thomas-BMT $2\pi G\gamma$ Z-precession. The INCLUDE file 60degSector.inc is taken from Tab. 3.6

```
Cyclotron, classical. Synchronous spin kick.
'MARKER'  ProbAddSpinTorque_S                                    ! Just for edition purposes.
'OBJET'
64.62444403717985                    ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
4 1
12.9248888074 0. 0. 0. 0.  1.     'm'    ! D=1 => 200keV proton. R=Brho/B=64.624444037[kG.cm]/5[kG].
3.0947295453790e2 0. 0. 0. 0. 23.9439548880185 'm'  ! Ggamma=2
3.2492145208941e2 0. 0. 0. 0. 25.1392067607172 'm'  ! Ggamma=2.02
3.8177333586897e2 0. 0. 0. 0. 29.5378429599586 'm'  ! Ggamma=2.1
1 1 1 1
'PARTICUL'                                        ! This is required for spin motion to be computed,
PROTON                                            ! otherwise, by default \zgoubi\ only requires rigidity.
'SPNTRK'                                                         ! Request spin tracking.
4.1
0. 0. 1.                                                ! Initial spin vector is defined here.

'FAISTORE'
zgoubi.fai
1

'INCLUDE'
1
6* ./60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]         ! 6 * 60 degree sector.
'FAISCEAU'
'SPINR'
1                                                   ! Spin rotation,
0. 10.                         1 about the X-axis, by 10 or 20 dgrees as the case may be.

'REBELOTE'                                                  ! Multiturn ray-tracing.
39 0.2  99
'SYSTEM'
1
gnuplot < ./gnuplot_Zplt_spinTilt.gnu
'MARKER'  ProbAddSpinTorque_E                               ! Just for edition purposes.
'END'
```

Fig. 3.32 S_Z motion versus orbital angle, while the ion orbits on a circle. S_Z is constant over a turn and then undergoes a discontinuity upon the 10 deg X-tilt, hence the step function. At $G\gamma = 2$ it takes 36 turns, or 226.194 rad, to complete an oscillation. A graph obtained using zpop: menu 7; 1/1 to open zgoubi.plt; 2/[6,23] for S_Z versus θ; 7 to plot
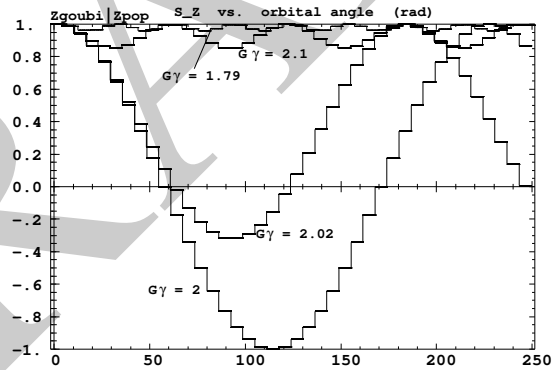
undergoes a half-integer number of precessions over one orbit around the cyclotron, it jumps and alternates between vertical, and the surface of the 20 degree Z-axis cone.

Table 3.13 Simulation input data file: a similar simulation to 3.12, for different $G\gamma$ values, namely 1.79322, 2 and 2.5. The spin kick at SPINR has been changed to 20 deg. Regarding the use of OBJET[IEX] option: IEX=-9 allows inhibiting the tracking for the particle(s) concerned, all the rest left unchanged; it is necessary here to have at least one particle with IEX=1, for proper operation of the gnuplot scripts. The INCLUDE file 60degSector.inc is taken from Tab. 3.6

```
Cyclotron, classical. Synchronized spin kick in a uniform field
'MARKER'  ProbAddSpinSphere_S                                          ! Just for edition purposes.
'OBJET'
64.62444403717985                       ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
3 1
12.924889  0. 0. 0. 0.   1.        'o'                          ! Ggamma=1.793229 ->  0.200MeV;
309.47295  0. 0. 0. 0.  23.943951797 'i'                        ! Ggamma=2   ->   108.411628MeV;
608.30878  0. 0. 0. 0.  47.064911290 'h'                        ! Ggamma=2.5 ->  370.082556MeV.
1 1 1                        ! For any particle: set to 1 to enable ray-tracing, or to -9 to ignore.
'PARTICUL'                                      ! This is required for spin motion to be computed,
PROTON                                          ! otherwise, by default \zgoubi\ only requires rigidity.
'SPNTRK'                                                          ! Request spin tracking.
4.1                                                              ! All initial spins taken parallel to Z axis.
0. 0. 1.

'SPNPRT'  PRINT

'INCLUDE'
1
6* ./60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]             ! 6 * 60 degree sector.
'FAISCEAU'
'SPINR'
1                                                                ! Spin rotation,
0. 20.                                                 1 about the X-axis, by 20 degree here.

'REBELOTE'                              ! REBELOTE[K=99] for multiturn ray-tracing,
39 0.2  99                                             ! 39+1 turns total.
'SYSTEM'
3
gnuplot <./gnuplot_Zspnprt_spinOscillation.gnu
gnuplot < ./gnuplot_Zplt_spinTilt.gnu
gnuplot <./gnuplot_Zplt_spinTilt_3D.gnu
'END'
'MARKER'  ProbAddSpinSphere_E                                          ! Just for edition purposes.
'END'
```

————

A gnuplot script to produce spin components versus turn, reading from zgoubi.SPNPRT.Out, Fig. 3.33:

```
# gnuplot_Zspnprt_spinOscillation.gnu
set xlabel "turn"; set ylabel "S_X,    S_Y,    S_Z"; set key b l ; nbtrj=3 # number of trajectories tracked
do for [it=1:nbtrj] {  unset label; set label sprintf("particle %3.5g",it) at 10, 0.8
plot [] [-1:1] \
   'zgoubi.SPNPRT.Out' every nbtrj::(it+2) u ($22):($13) w lp lw .3 pt 4 ps .8 lc rgb "red" ,\
   'zgoubi.SPNPRT.Out' every nbtrj::(it+2) u ($22):($14) w lp lw .3 pt 6 ps .8 lc rgb "blue" ,\
   'zgoubi.SPNPRT.Out' every nbtrj::(it+2) u ($22):($15) w lp lw .3 pt 8 ps .8 lc rgb "black" ; pause .5
set terminal postscript eps blacktext color  enh
set output sprintf('gnuplot_Zspnprt_spinOsc_trj%i.eps',it); replot; set terminal X11; unset output }
```

A gnuplot script to produce 2D *spin motion projection of Fig.* 3.33:

```
# gnuplot_Zplt_spinTilt.gnu
set xlabel "S_X"; set ylabel "S_Y"; set size ratio -1; set xrange [-1:1]; set yrange [-1:1]; set key t l
nbtrj=3 # number of trajectories tracked
do for [it=1:nbtrj] {  unset label; set label sprintf("particle %i",it) at -.9, .8
  plot 'zgoubi.plt' u ($19==it? $33 :1/0):($34) w lp lw .3 ps .2 lc rgb "blue"; pause .5
  set terminal postscript eps blacktext color  enh
  set output sprintf('gnuplot_Zplt_SX-SY_trj%i.eps',it); replot; set terminal X11; unset output }
```

A gnuplot script to produce the projection on a sphere of Figs. 3.33:

```
# gnuplot_Zplt_spinTilt_3D.gnu
set xlabel "X"; set ylabel "Y"; set zlabel "Z"; set xrange [-1:1]; set yrange [-1:1]; set zrange [-1:1]
set xyplane 0; set view equal xyz; set view 49, 339; unset colorbox
set urange [-pi/2:pi/2]; set vrange [0:2*pi]; set parametric; R = 1.   # radius of sphere
nbtrj=3 # number of trajectories tracked
do for [it=1:nbtrj] {  unset label; set label sprintf("  particle %i",it) at -1, .9, 1.
  splot  R*cos(u)*cos(v),R*cos(u)*sin(v),R*sin(u) w l lw .2 lc rgb "cyan" notit ,\
  'zgoubi.plt'   u ($19==it? $33 :1/0):($34):($35) w lp lw .2 ps .4 lc palette ;  pause .5
  set terminal postscript eps blacktext color  enh
  set output sprintf('gnuplot_Zplt_S3D_trj%i.eps',it); replot; set terminal X11; unset output }
```
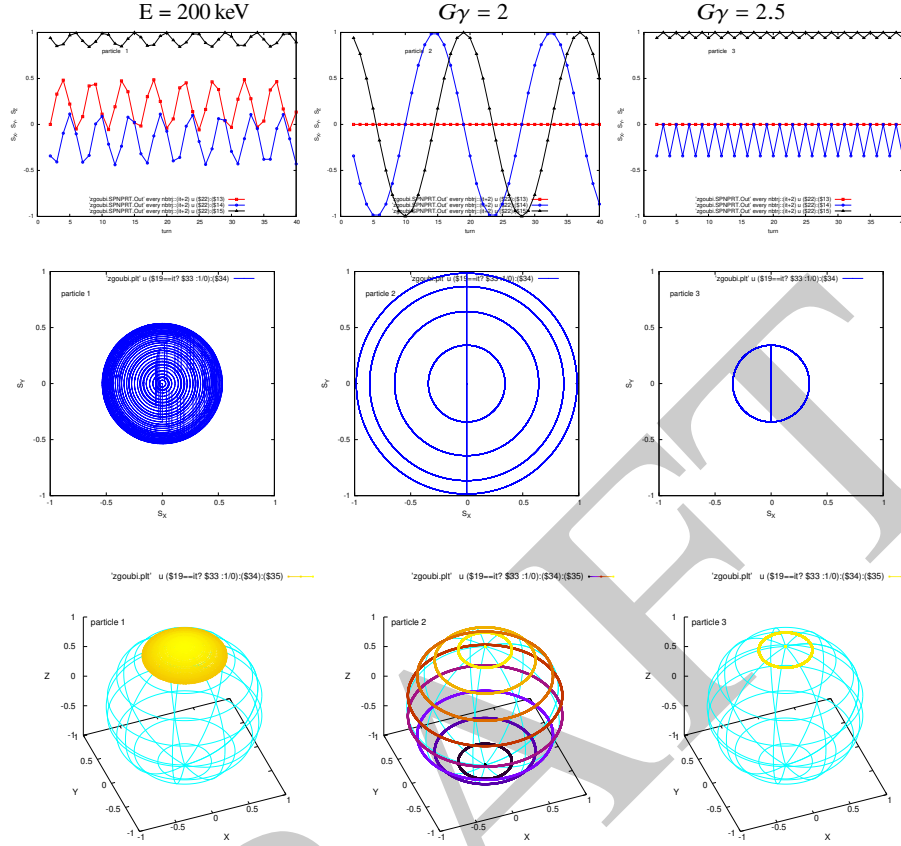
Fig. 3.33 Top row: spin coordinates versus turn; middle row: projection in the median plane (the segment between two consecutive circles materializes the location of the X-kick by SPINR); bottom row: projection on a sphere. $G\gamma = 1.793229$: far from an integer, **S** remains within a cone of reduced aperture. $G\gamma = 2$: the spin vector oscillates between up and down orientations, by 20 deg steps; it takes 180/20=9 orbits for the X-precession at SPINR to flip the spin; $G\gamma = 2.5$: the spin vector finds itself back in the (Y,Z) plane at the location of SPINR, after one orbit and a half-integer number of precessions; it alternates between vertical and 20 deg from vertical, after each orbit around the cyclotron

3.6 Weak Focusing

(a) Add a field index.

To the first order in R, in the median plane (Z=0) and noting $R = R_0 + dR$, $B_Z(R_0) = B_0$, $B_Z(R) = B$, the field writes (Sect. 3.2.2) $B(R) = B_0 + dR \left.\frac{\partial B}{\partial R}\right|_{R_0}$. With $k = \frac{R_0}{B_0} \frac{\partial B}{\partial R}$ (Eq. 3.11) this yields

$$B(R) = B_0 + \frac{B_0}{R_0} \, k \, dR \tag{3.38}$$

Assume the earlier 200 keV conditions as a reference, thus take

$R_0 = 12.9248888$ cm as the 200 keV radius, whereas $B_0 = B(R_0) = 5$ kG.

Take $k = -0.03$, a slow decrease of the field with R - proper to ensure appropriate vertical focusing with marginal impact on the radial extent of the cyclotron. For instance, with that index value the 5 MeV orbit is at a radius of 75.75467 cm (see OBJET in Tab. 3.3) (giving $B = 0.3235$ T along the orbit), whereas if k=0 then $R = 75.75467$ cm is the 6.8463 MeV orbit radius ($B = 0.3788$ T).

The field map is generated using a similar Fortran program to that of exercise 3.1 (see Tab. 3.1), *mutatis mutandis*, namely, introducing a reference radius R_0 and field index k. The resulting program is given in Tab. 3.14, it can be compiled and executed, as is, excerpts of the field data file so obtained are given in Tab. 3.15, a graph $B_Z(R, \theta)$ is given in Fig. 3.34. The orbit radius is assessed for three different energies, and appears to be in accord with theoretical expectation (Fig. 3.34-right). Comparison with Fig. 3.20-right shows the effect of the negative index on the radial distribution of the orbits, including a radius about 20% greater in the 5 MeV range. The input data file to find these trajectories is given in Tab. 3.16:

- the file defines an INCLUDE segment, #S_60degSectorIndx to #E_60degSectorIndx, used in subsequent exercises;

- the file is set to allow a preliminary test regarding the field map geneSectorMapIndex.out (as produced by the program given in Tab. 3.14), by computing three circular trajectories centered on the center of the map, at respectively 20 keV, 200 keV (the reference energy for the definition of the gradient index k) and 5 MeV (a large radius);

- note that once the FIT procedure is completed, zgoubi continues in sequence, so raytracing the 3 ions through the field map with, this time, IL set to 2 under TOSCA for stepwise particle data to be logged in zgoubi.plt.
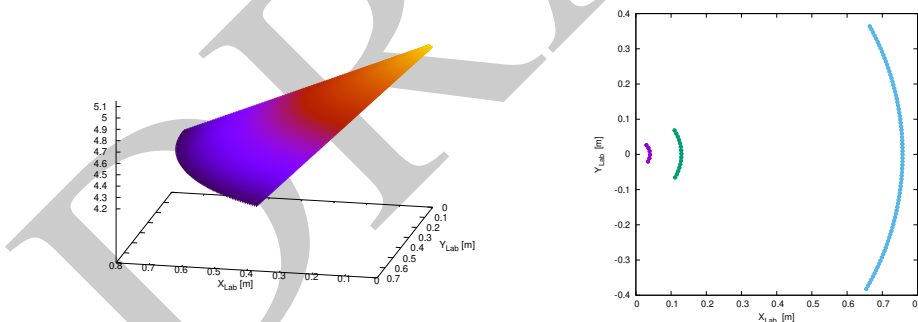
Fig. 3.34 Left: field map of a 60 deg magnetic sector with radial index, 76 cm radial extent. The field decreases from the center of the ring (at $(X_{Lab}, Y_{Lab}) = (0, 0)$). Right: three circular arc of trajectories over a sextant, at respectively from left to right: 0.02 MeV, 0.2 MeV (energy on the reference radius) and 5 MeV

Table 3.14 A Fortran program which generates a 60° mid-plane field map with non-zero transverse field k. The field map it produces is logged in geneSectorMapIndex.out

```fortran
C geneSectorMapIndex.f program
      implicit double precision (a-h,o-z)
      parameter (pi=4.d0*atan(1.d0), BY=0.d0, BX=0.d0, Z=0.d0)

      open(unit=2,file='geneSectorMapIndex.out')                    ! Field map storage file.

C----------- Hypotheses :
      AT = 60.d0  /180.d0*pi        ! Angular extent of field map. Can be changed 360, 60 deg, etc.).
      B0 = 5.d0 ;R0 = 12.9248888074d0           ! field at R0 (kG); 200keV radius (cm), B(R0)=B0=5kG.
      ak = -0.03d0                                      ! Field index, defined at R0.
      Rmi=1.d0; Rma=76.d0; RM=50.d0  ! cm. Radial extent of field map; reference radius to define mesh.
      dR = 0.5d0 ; NR = NINT((Rma - Rmi)/dR)+1   ! R-distance between nodes in mesh. Number of R-nodes.
C                              RdA=RM*dA is the distance between two nodes along R=RM arc,
      RdA = 0.5d0  !       given angle increment dA (dA is the "Delta theta" quantity in the main text).
      NX= NINT(RM*AT /RdA) +1 ; RdA= RM*AT / DBLE(NX -1)  ! exact mesh step at RM, corresponding to NX.
      dA = RdA / RM ; A1 = 0.d0 ; A2 = AT                         ! corresponding delta_angle.
C--------------------------------------------
      write(2,*) Rmi,dR,dA/pi*180.d0,dZ,
     >'    ! Rmi/cm, dR/cm, dA/deg, dZ/cm'
      write(2,*) '# Field map generated using geneSectorMapIndex.f '
      write(2,fmt='(a)') '# AT/rd,  AT/deg, Rmi/cm, Rma/cm, RM/cm,'
     >//' NR, dR/cm, NX, RdA/cm, dA/rd : '
      write(2,fmt='(a,1p,5(e16.8,1x),2(i3,1x,e16.8,1x),e16.8)')
     >'# ',AT, AT/pi*180.d0,Rmi, Rma, RM, NR, dR, NX, RdA, dA
      write(2,*) '# For TOSCA: ',NX,NR,' 1 22.1 1.  !IZ=1 -> 2D ; '
     >//'MOD=22 -> polar map ; .MOD2=.1 -> one map file'
      write(2,*) '# '
      write(2,*) '#    R*cosA          Z==0,        R*sinA'
     >//'         BY             BZ           BX  ix jr'
      write(2,*) '#    cm             cm            cm '
     >//'         kG             kG           kG '
      do jr = 1, NR
        R = Rmi + dble(jr-1)*dR
        BZ = B0 + B0/R0 * ak * (R - R0)
        do ix = 1, NX
          A = A1 + dble(ix-1)*dA ; X = R * sin(A) ; Y = R * cos(A)
          write(2,fmt='(1p,6(e16.8),2(1x,i0))') Y,Z,X,BY,BZ,BX,ix,jr
        enddo
      enddo
      stop ' Job complete ! Field map stored in geneSectorMapIndex.out.'
      end
```

Table 3.15 First and last few lines of the field map file geneSectorMapIndex.out. The file starts
with an 8-line header, the first one of which is effectively used by zgoubi, the following 7 are just
comments

```
     1.      0.5     0.57142857142857140        0.          ! Rmi/cm, dR/cm, dA/deg, dZ/cm
# Field map generated using geneSectorMapIndex.f
# AT/rd,  AT/deg, Rmi/cm, Rma/cm, R0/cm, NR, dR/cm, NX, RdA/cm, dA/rd :
#   1.04719755E+00   6.0E+01   1.0E+00   7.60E+01   1.29248888E+01 151   5.0E-01 106  4.98665501E-01  9.97331001E-03
# For TOSCA:          106          151  1 22.1 1.  !IZ=1 -> 2D ; MOD=22 -> polar map ; .MOD2=.1 -> one map file
#
#       R*cosA            Z==0,        R*sinA          BY            BZ            BX   ix jr
#        cm                cm            cm            kG            kG            kG
   1.00000000E+00  0.00000000E+00  0.00000000E+00  0.00000000E+00  5.13839448E+00  0.00000000E+00 1 1
   9.99950267E-01  0.00000000E+00  9.97314468E-03  0.00000000E+00  5.13839448E+00  0.00000000E+00 2 1
   9.99801073E-01  0.00000000E+00  1.99452974E-02  0.00000000E+00  5.13839448E+00  0.00000000E+00 3 1
   9.99552432E-01  0.00000000E+00  2.99154662E-02  0.00000000E+00  5.13839448E+00  0.00000000E+00 4 1
   9.99204370E-01  0.00000000E+00  3.98826594E-02  0.00000000E+00  5.13839448E+00  0.00000000E+00 5 1
   ...............................
   4.05947602E+01  0.00000000E+00  6.42500229E+01  0.00000000E+00  4.26798081E+00  0.00000000E+00 102 151
   3.99519665E+01  0.00000000E+00  6.46516850E+01  0.00000000E+00  4.26798081E+00  0.00000000E+00 103 151
   3.93051990E+01  0.00000000E+00  6.50469164E+01  0.00000000E+00  4.26798081E+00  0.00000000E+00 104 151
   3.86545219E+01  0.00000000E+00  6.54356779E+01  0.00000000E+00  4.26798081E+00  0.00000000E+00 105 151
   3.80000000E+01  0.00000000E+00  6.58179307E+01  0.00000000E+00  4.26798081E+00  0.00000000E+00 106 151
```

A gnuplot script to obtain Fig. 3.34:

```
# PLOT THE FIELD MAP:
set xtics mirror ; set ytics mirror ; set xlabel "X_{Lab} [m]" ; set ylabel "Y_{Lab} [m]" ; cm2m = 0.01
set zrange [:5.15] ; set view 66, 192 ; unset colorbox
splot "geneSectorMapIndex.out" u ($1 *cm2m):($3 *cm2m):($5) w p lc palette notit ; pause 1

# PLOT THREE TRAJECTORIES
set xtics ;  set ytics ; set xlabel "X_{Lab}  [m]" ; set ylabel "Y_{Lab}  [m]" ; cm2m = 0.01 ; set size ratio 1
plot for [trj=1:3] \
  "zgoubi.plt" u ($19==trj ? $10*cm2m*cos($22) :1/0):($10*cm2m*sin($22)) w p pt 7 ps .6 notit ; pause 1
```

Table 3.16 Simulation input data file FieldMapSectorIndex.inc: a file to test trajectories for a field map with radial index. This file also defines the INCLUDE segment between the LABEL_1s #S_60degSectorIndx and #E_60degSectorIndx

```
FieldMapSectorIndex.inc
! Uniform field sector with index. INCLUDE file FieldMapSectorIndex.inc
'MARKER'   FieldMapSectorIdx_S                                      ! Just for edition purposes.
'OBJET'
64.62444403717985                    ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
3 1
4.003593 0. 0. 0. 0. 0.3162126 'o'         ! p[MeV/c]= 6.126277, Brho[kG.cm]=20.435, kin-E[MeV]=0.02.
12.92488 0. 0. 0. 0. 1.       'o'    ! Reference ; p[MeV/c]=193.739, Brho[kG.cm]=BORO, kin-E[MeV]=0.2.
75.75467 0. 0. 0. 0. 5.0063900 'o'              ! p[MeV/c]=969.934, Brho[kG.cm]=323.535, kin-E[MeV]=5.
1 1 1
'MARKER'   #S_60degSectorIndx
'TOSCA'
0 2     ! IL=2 to log step-by-step coordinates, spin, etc., in zgoubi.plt (avoid, if CPU time matters).
1. 1. 1. 1.    ! Normalization coefficients, for B, X, Y and Z coordinate values read from the map.
HEADER_8                                       ! The field map file starts with an 8-line header.
106 151 1 22.1 1.           ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1 if only one map file.
geneSectorMapIndex.out
0 0 0 0      ! Possible vertical boundaries within the field map, to start/stop stepwise integration.
2
1.0                                 ! Integration step size. Small enough for orbits to close accurately.
2                                                             ! Magnet positionning option.
0. 0. 0. 0.                                                   ! Magnet positionning.
'MARKER'   #E_60degSectorIndx
 'FIT2'                                    ! This matching procedure finds the closed orbit radius.
3  nofinal
2 30 0 [2.,10.]      ! Variable : Y_0, trajectory 1
2 40 0 [10.,15.]     ! Variable : Y_0, trajectory 2
2 50 0 [50.,80.]     ! Variable : Y_0, trajectory 3
3  1e-20  9999       ! Penalty; max numb of calls to function
3.1 1 2 #End 0. 1. 0  ! Constraint : Y_final=Y_0, trajectory 1
3.1 2 2 #End 0. 1. 0  ! Constraint : Y_final=Y_0, trajectory 2
3.1 3 2 #End 0. 1. 0  ! Constraint : Y_final=Y_0, trajectory 3

!  Carry on with coordinates as found, yet with IL=2 under TOSCA so to log trajectories in zgoubi.plt.
'TOSCA'         ! IL=2: log step-by-step coordinates, spin, etc., in zgoubi.plt (avoid if CPU time matters).
1. 1. 1. 1.        ! Normalization coefficients, for B, X, Y and Z coordinate values read from the map.
HEADER_8                                       ! The field map file starts with an 8-line header.
106 151 1 22.1 1.           ! IZ=1 for 2D map; MOD=22 for polar frame; .MOD2=.1 if only one map file.
geneSectorMapIndex.out
0 0 0 0      ! Possible vertical boundaries within the field map, to start/stop stepwise integration.
2
1.0                                 ! Integration step size. Small enough for orbits to close accurately.
2                                                             ! Magnet positionning option.
0. 0. 0. 0.                                                   ! Magnet positionning.
'FAISCEAU'                                   ! Local particle coordinates logged in zgoubi.res.
'SYSTEM'              ! This SYSTEM command runs gnuplot, for a graph of the two trajectories.
1
gnuplot <./gnuplot_Zplt.gnu
'MARKER'   FieldMapSectorIdx_E                                      ! Just for edition purposes.
'END'
```

2538   (b) R-dependence of orbit rigidity.

2539   The method is similar to exercise 3.1-(b) (see Tab. 3.4.): FIT finds the closed orbit
2540   radius R for a given ion rigidity, and REBELOTE is used to repeat for a series of
2541   different momenta, 20 here. The input data file for this exercise is given in Tab. 3.17,
2542   it includes a 21 ion 1-turn raytracing, in sequence with the previous 21-orbit finding.
2543   Raytracing outcomes for $k = -0.03$, $R_0 = R(E = 200\,\text{keV}) = 12.924888\,\text{cm}$,
2544 $B_0 = B(R_0) = 0.5\,\text{T}$ are given in Fig. 3.35, together with theoretical expectation
2545   (with B(R) from Eq. 3.7)

$$\text{Rigidity} \quad BR(R) = B_0 \left(1 + \frac{R - R_0}{R_0} k \right) R \tag{3.39}$$
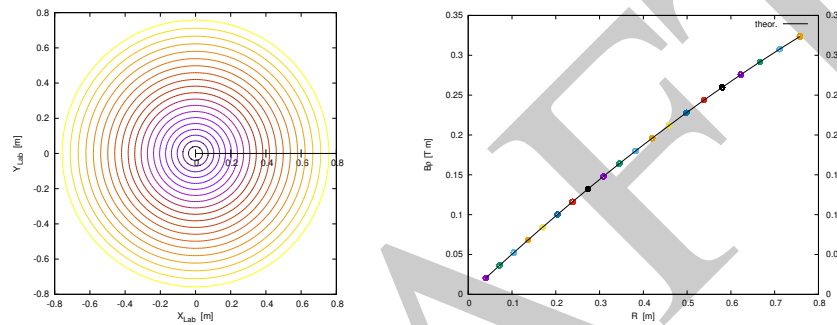
Fig. 3.35 Case of field index k=-0.03. Left: closed orbits at a series of different rigidities. Right:
comparison of $B\rho(R)$ from raytracing outcomes (markers) and from theory (solid line, Eq. 3.39)

2546   (c) Paraxial motion.

2547   A proton with energy 1 MeV is considered, here. DIPOLE [16, *lookup* INDEX]
2548   is used rather than a field map, so allow to freely change the k index value (using
2549   TOSCA instead would require computing a new field map when changing k).

2550   The input data for a 60 deg sector are given in Tab. 3.18, essentially a copy of
2551   the uniform dipole field case of Tab. 3.6 in which the index value $k = -0.03$ has
2552   been added (line 3 under DIPOLE). The input data sequence for multiturn trajectory
2553   computation around the cyclotron is given in Tab. 3.19: in a first stage, orbit finding
2554   is performed by FIT, for 1 MeV energy; in a subsequent second stage, 4 protons with
2555   their initial horizontal coordinates taken on the closed orbit, and differing by their
2556   initial vertical take-off angle, are tracked over 120 sectors, *i.e.*, 20 turns around the
2557   ring.

2558   Fig. 3.36 displays the vertical sine motion. Stronger index (k closer to -1) results
2559   in stronger vertical focusing, hence more oscillations as expected from Eq. 3.18 and
2560   smaller motion amplitude as expected from Eq. 3.17. The latter can be written

Table 3.17 Simulation input data file: scan orbits for momentum dependence. Two problems are stacked, executed in sequence: in a first stage FIT finds a closed orbit, whose coordinates are logged in initialRs.fai file when FIT is completed, following what REBELOTE repeats for an additional 20 momenta; in a second stage OBJET grabs the 21-set of ion coordinates from initialRs.fai and these ions are raytraced over 6 sectors, *i.e.*, one full turn. The INCLUDE file FieldMapSectorIndex.inc is taken from Tab. 3.16

```
Uniform field sector with index. Scan orbits.
'MARKER'    scanSectorIdx_S                                       ! Just for edition purposes.
'OBJET'
64.62444403717985                      ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1                                                               ! Just one ion.
4.0039   0. 0. 0. 0. 0.3162126 'o'     ! p[MeV/c]= 6.126277, Brho[kG.cm]=20.435, kin-E[MeV]=0.02.
1
'FAISCEAU'                                        ! Local particle coordinates logged in zgoubi.res.
'INCLUDE'
1
./FieldMapSectorIndex.inc[#S_60degSectorIndx:#E_60degSectorIndx]
 'FIT'                                  ! This matching procedure finds the closed orbit radius.
1  nofinal
2 30 0 [3.,80.]      ! Variable : Y_0
1  1e-15  99         ! Penalty; max numb of calls to function
3.1 1 2 #End 0. 1. 0  ! Constraint :  Y_final=Y_0
'FAISTORE'
initialRs.fai                                                     ! Log coordinates in initialRs.fai.
1
'REBELOTE'          ! A do-loop. Repeat the above, after changing particle rigidity to a new value.
20 0.2  0 1          ! 20 diffrnt rigidities; I/O options; coordinates as from OBJET; changes follow:
1                            ! Parameter 35 to be changed, in OBJET: relative momentum, namely,
OBJET 35  0.3162126:5.0063900                 ! for energy scan from 0.02 MeV to 5 MeV.

'OBJET'
64.62444403717985                      ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
3
1 999 1
1 999 1
1.  1.  1.  1.  1.  1.  1. '*'
0.  0.  0.  0.  0.  0.  0.
0
initialRs.fai
'FAISCEAU'                                        ! Local particle coordinates logged in zgoubi.res.
'INCLUDE'
1
6* ./FieldMapSectorIndex.inc[#S_60degSectorIndx:#E_60degSectorIndx]      ! INCLUDE 6 times.
'SYSTEM'
2
gnuplot <./gnuplot_Zplt_orbits.gnu                    ! Plot orbits around the cyclotron.
gnuplot <./gnuplot_Zplt_scanBrho.gnu                          ! Plot R(Brho).
'MARKER'    scanSectorIdx_E                              ! Just for edition purposes.
'END'
```

A gnuplot script to obtain orbits, Fig. 3.35:

```
set xtics ;  set ytics ; set xlabel "X_{Lab}  [m]" ; set ylabel "Y_{Lab}  [m]" ; cm2m = 0.01; set polar; set size ratio 1
unset colorbox; pi = 4.*atan(1.); TOSCA1=12; dT=3 # number of 2nd TOSCA & increment in zgoubi.plt listing
plot for [trj=2:21] \
    "zgoubi.plt" u ($19==trj ? $22+($42-TOSCA1)/dT*pi/3 :1/0):($10*cm2m ):($19) w l lw 2 lc palette notit ; pause 1
```

A gnuplot script to obtain Bρ(R), Fig. 3.35:

```
set xtics ;  set ytics nomirror ; set y2tics; set xlabel "R  [m]" ; set ylabel "B{/Symbol r} [T m]"
B0=0.5; R0=12.924888e-2; k=-0.03; Brho(x)= B0* (1.+  (x-R0)/R0* k )*x ; kGcm2Tm=1e-3;  cm2m = 0.01
plot for [trj=2:21] \
    "zgoubi.plt" u ($19==trj? $10*cm2m :1/0):($40*(1.+$2)*kGcm2Tm) w p pt 6 ps 1.2 notit ,\
  Brho(x) axes x1y2 w l lw 2 lc rgb "black" tit "theor." ; pause 1
```

Table 3.18 Simulation input data file sectorWithIndex.inc: definition of a dipole with index, case of analytical field modeling, namely here k=-0.03 and reference radius $R_0 = 50$ cm. Definition of the [#S_60degSectorWIdx:#E_60degSectorWIdx] segment

```
# sectorWithIndex.inc
'MARKER'    #S_60degSectorWIdx                          ! Label should not exceed 20 characters.
'DIPOLE'                                                ! Analytical modeling of a dipole magnet.
2               ! IL=2, only purpose is to logged trajectories in zgoubi.plt, for further plotting.
60. 50.                                                 ! Sector angle AT; reference radius.
30. 5. -0.03 0. 0.  ! Reference azimuthal angle ACN; BM field at R0; indices, N (=k=-0.03) at R0=50cm.
0.  0.                                                  ! EFB 1 is  hard-edge,
4 .1455   2.2670  -.6395  1.1558  0. 0.  0.            ! hard-edge only possible with sector magnet.
30. 0.  1.E6  -1.E6  1.E6  1.E6
0.  0.                                                  ! EFB 2.
4 .1455   2.2670  -.6395  1.1558  0. 0.  0.
-30. 0.  1.E6  -1.E6  1.E6  1.E6
0.  0.                                                  ! EFB 3 (unused).
0  0.      0.      0.      0.     0. 0.  0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
4   10.
0.5                           ! Integration step size. Small enough for orbits to close accurately.
2  0. 0. 0. 0.                                 ! Magnet positionning RE, TE, RS, TS.
'MARKER'    #E_60degSectorWIdx                          ! Label should not exceed 20 characters.
'END'
```

$$Z(s) = P_0 \frac{R_0}{\sqrt{-k}} \sin \frac{\sqrt{-k}}{R_0}(s - s_0) \quad \text{and} \quad \hat{Z} = P_0 \frac{R_0}{\sqrt{-k}} \tag{3.40}$$

Note that this vertical oscillation results in a modulation of the field along the trajectory (see question (d) of this exercise) which results in a radial oscillation, a second order Y-Z coupling effect (extremely weak), displayed in Fig. 3.37.
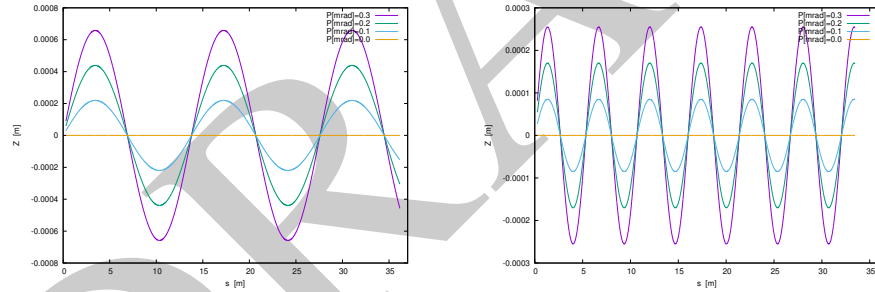
Fig. 3.36 Vertical sine motion over a few turns around the cyclotron, at 1 MeV. Vertical take-off angles are $P_0 = 0$, 0.1, 0.2, 0.3 mrad. Left: k=-0.03, $\nu_Z = \sqrt{0.03} \approx 0.173$ oscillations per turn; right: for k=-0.2, $\nu_Z = \sqrt{0.2} \approx 0.447$ oscillations per turn

(d) Magnetic field.

The magnetic field experienced by 1 MeV protons with four different take-off angles P_0 (Fig. 3.36), along their respective trajectories, case of an index value $k = -0.03$, is displayed in Fig. 3.38. It is essentially constant as expected.

Table 3.19 Simulation input data file: scan orbits for momentum dependence; the file actually stacks two simulations, executed in sequence; the second simulation uses data produced by the first one, as follows. The first part of the file finds the closed orbits, they depend on the vertical excursion and are not exactly zero, due to the field index; closed orbit coordinates so found are logged in initialRs.fai when FIT is completed. The second part of the file starts at the second occurrence of OBJET which reads initial particle coordinates from initialRs.fai and tracks these particles through a sequence of 120 sector dipoles, *i.e.*, 20 turns. The [#S_60degSectorWIdx:#E_60degSectorWIdx] segment of Tab. 3.18 is INCLUDEd, here

```
Uniform field sector with index. Scan orbits.
'MARKER'   1MeVVMotion_S                                        ! Just for edition purposes.
!                                        First stage: find closed orbit at 1 MeV, for some k value.
'OBJET'
64.62444403717985                       ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
1.1
1  1  1  4   1  1
0.  1.  0.  0.1  0.  1.
30.107900 0.  0.  0.   0. 2.2365445724 'm'          ! 1 MeV proton -> Brho/Brho_ref = 2.2365445724.
'INCLUDE'
1
./sectorWithIndex.inc[#S_60degSectorWIdx:#E_60degSectorWIdx]          ! DIPOLE case R0=50cm k=-0.03.
 'FIT'                                   ! This matching procedure finds the closed orbit radius.
1   nofinal
2  40  0  .9                                        ! Variable : Y_0. Variation can be up to 90%.
1  1e-15  99                                        ! Penalty; max numb of calls to function.
3.1 1 2 #End 0. 1. 0                                ! Constraint :  Y_final=Y_0.
'FAISTORE'
initialRs.fai                                              ! Log coordinates in initialRs.fai.
1
!                                        Second stage: raytrace the four particles over 20 turns.
'OBJET'
64.62444403717985                       ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
3
1 999 1
1 999 1
1. 1. 1. 1. 1. 1. 1. '*'
0.  0.  0.  0.  0.  0.  0.
0
initialRs.fai
'FAISCEAU'                                          ! Local particle coordinates logged in zgoubi.res.
'INCLUDE'
1
120 * sectorWithIndex.inc[#S_60degSectorWIdx:#E_60degSectorWIdx]    ! INCLUDE 120 sectors (20 turns).
'FAISCEAU'                                          ! Local particle coordinates logged in zgoubi.res.
'SYSTEM'
2
gnuplot <./gnuplot_Zplt_1MeVVMotion.gnu
gnuplot <./gnuplot_Zplt_1MeVBField.gnu
'MARKER'   1MeVVMotion_E                                        ! Just for edition purposes.
'END'
```

_____

A gnuplot script to obtain Figs. 3.36, 3.37:

```
# gnuplot_Zplt_1MeVVMotion.gnu
set xtics ;  set ytics ; set xlabel "s  [m]" ; set ylabel "Z  [m]" ; cm2m = 0.01; unset colorbox ; set xrange [:36]
plot for [trj=4:1:-1] \
   "zgoubi.plt" u ($19==trj && $42>10? $14*cm2m :1/0):($12*cm2m ):($19) w l lw 2 tit "P[mrad]=0.".(trj-1) ; pause 1
```

_____

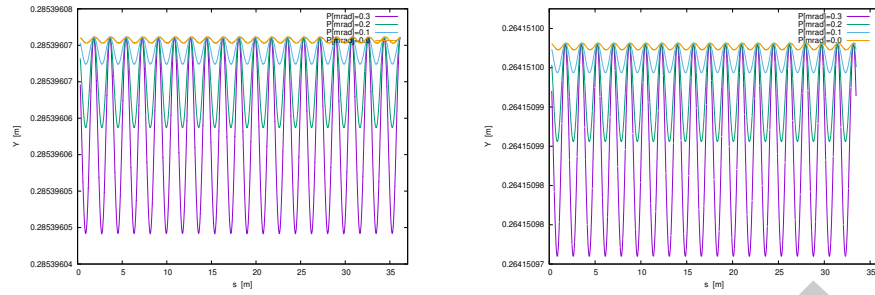*A gnuplot script to obtain Fig.* 3.38:

```
# gnuplot_Zplt_1MeVBField.gnu
set xtics;  set ytics; set xlabel "s  [m]"; set ylabel "Y  [m]"; cm2m = 0.01; unset colorbox
plot for [trj=4:1:-1] \
   "zgoubi.plt" u ($19==trj && $42>10? $14*cm2m :1/0):($10*cm2m ):($19) w l lw 2 tit "P[mrad]=0.".(trj-1) ; pause 1
```

Fig. 3.37 Horizontal motion at 1 MeV, 20 turns around the cyclotron, for vertical take-off angles $P_0 = 0$, 0.1, 0.2, 0.3 mrad. Left: k=-0.03, $\nu_R = \sqrt{1 + 0.03} \approx 1.015$ oscillations per turn; right: for k=-0.2, $\nu_R = \sqrt{1 + 0.2} \approx 1.095$ oscillations per turn

Fig. 3.38 Magnetic field experienced by 1 MeV protons with four different take-off angles P_0 (Fig. 3.36), along their respective trajectories. Case $k = -0.03$. The stepwise structure of these $B_Z(s)$ curves is due to the fact that field variations are at the limit of computer truncation related accuracy
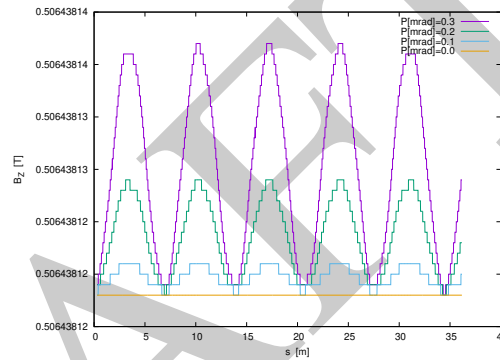
3.7 Loss of Isochronism

In order to scan $T_{\text{rev}}(R)$ for different k values, DIPOLE [16, *lookup* INDEX] is used here, as it allows to easily vary k and subsequently find the closed orbit using FIT. The method of exercise 3.6 is employed to obtain a scan. The input data file of Tab. 3.17 is a good starting point to do this exercise, changing the INCLUDE to account for DIPOLE instead of a field map modeling using TOSCA: the proper INCLUDE formatting can be reproduced from Tab. 3.19. IL under DIPOLE may be set at IL=0 as zgoubi.plt is not used here. Introduce FAISTORE to store local particle data after FIT (that includes time of flight, the quantity of interest here, which requires PARTICUL[PROTON] following OBJET).

The new input data file so built for this simulation, is given in Tab. 3.20.

This input data file is run for four different k values, namely, under DIPOLE (*cf.* Tab. 3.18), the line "30. 5. -0.03 0. 0." is successively changed to $\begin{cases} 30.\ 5.\ 0\ 0.\ 0. \\ 30.\ 5.\ -0.5\ 0.\ 0. \\ 30.\ 5.\ -0.95\ 0.\ 0. \end{cases}$.

The corresponding zgoubi.fai files are saved under dedicated copies for plotting, see "gnuplot script gnuplot_Zfai_scanTrev.gnu" at the bottom of Tab. 3.20.

The results of these T_{rev} scans are displayed in Fig. 3.39. In the case $k = 0$ the loss of isochronism is only due to the relativistic change of the mass, a non-zero k

Table   3.20 Simulation   input   data   file: scan   revolution   time.   The [#S_60degSectorWIdx:#E_60degSectorWIdx] segment of Tab. 3.18 is INCLUDEd, here

```
Uniform field sector with index. Scan orbits.
'MARKER'    isoChroLoss_S                                          ! Just for edition purposes.
'OBJET'
64.62444403717985                       ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1                                                              ! Just one ion.
4.0039   0. 0. 0. 0. 0.3162126 'o'        ! p[MeV/c]= 6.126277, Brho[kG.cm]=20.435, kin-E[MeV]=0.02.
1
'PARTICUL'                                     ! Necessary as time of flight computation is needed,
PROTON                                         ! otherwise, by default \zgoubi\ only requires rigidity.
'INCLUDE'
1
./sectorWithIndex.inc[#S_60degSectorWIdx:#E_60degSectorWIdx]        ! DIPOLE case R0=50cm k=-0.03.
'FIT2'                                         ! This matching procedure finds the closed orbit radius.
1   nofinal
2 30 0 [0.5,80.]        ! Variable : Y_0
1 1e-15  99            ! Penalty; max numb of calls to function
3.1 1 1 2 #End 0. 1. 0    ! Constraint :  Y_final=Y_0
'FAISCEAU'                                     ! Local particle coordinates logged in zgoubi.res.
'FAISTORE'
zgoubi.fai
1
'REBELOTE'             ! A do-loop. Repeat the above, after changing particle rigidity to a new value.
20 0.2  0 1            ! 20 diffrnt rigidities; I/O options; coordinates as from OBJET; changes follow:
1                      ! Parameter 35 to be changed, in OBJET: relative momentum, namely,
OBJET 35  0.3162126:5.00639    ! Acceleration to 5MeV. Commented here, for use in subsequent exercises.
! OBJET 35  0.3162126:2.2365445724    ! Substitute to previous, for energy scan from 0.02 MeV to 1 MeV.
'SYSTEM'
1
gnuplot <./gnuplot_Zfai_scanTrev.gnu                              ! Plot revolution time.
'MARKER'    isoChroLoss_E                                         ! Just for edition purposes.
'END'
```

A gnuplot script to obtain Fig. 3.39:

```
# gnuplot_Zfai_scanTrev.gnu
set xtics ;  set ytics nomirror ; set y2tics; set xlabel "R  [m]" ; set ylabel "T_{rev} [{/Symbol m}s]"
cm2m = 0.01; nSec=6; set y2label "T_{rev}  at  k=0[{/Symbol m}s]" ; set key c r
plot "zgoubi_k0.fai"    u ($10 *cm2m):($15 * nSec) w lp pt 4 ps 1.2 lc rgb "black" tit "k=0" ,\
     "zgoubi_k0.fai"    u ($10 *cm2m):($15 * nSec) axes x1y2 w lp pt 4 ps 1.2  lc rgb "black" notit ,\
     "zgoubi_k0.03.fai" u ($10 *cm2m):($15 * nSec) w lp  pt 7 ps 1.2 tit "k=-03" ,\
     "zgoubi_k0.5.fai"  u ($10 *cm2m):($15 * nSec) w lp  pt 8 ps 1.2 tit "k=-5" ,\
     "zgoubi_k0.95.fai" u ($10 *cm2m):($15 * nSec) w lp  pt 9 ps 1.2 tit "k=-95" ; pause 1
```

augments the effect. The loss of isochronism is the cause of the $\approx 20\,$MeV proton energy limit of the classical cyclotron.
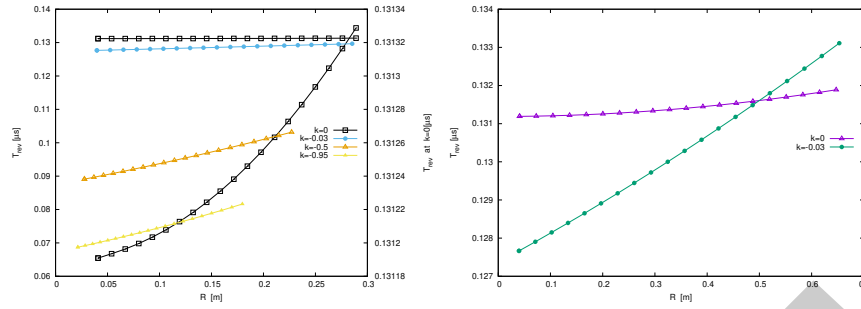
Fig. 3.39 A scan of the revolution time, from 0.02 to 1 MeV, and its dependence on the field index k. The right vertical axis only concerns the case $k = 0$ where the change in revolution time is weak and only due to the mass increase (in $T_{\text{rev}} = 2\pi\gamma m_0/qB$). The right graph shows, up to 5 MeV, the relatively important contribution of the focusing index, even a weak k=-0.03, compared to the effect of the mass increase (k=0 curve). Markers are from raytracing, solid lines are from theory

3.8 Ion Trajectories

A `zgoubi` data file is set up for computation of particle trajectories, taking a field value on reference radius of $B_0(R_0) = 0.5$ T, and reference energy 200 keV (proton). These hypotheses determine the reference radius value. DIPOLE [16, *lookup* INDEX] is used (Tab. 3.21), for its greater flexibility in changing magnet parameters, field and radial field index amongst other, compared to using TOSCA and a field map.

(a) Transverse motion.

It first has to be checked that there is consistency between initial orbital radius Y_0 in OBJET at 200 keV proton energy and the value of the reference radius R_0 in DIPOLE (Eq. 3.35). Its theoretical value is $R_0 = BORO/5[kG] = 12.924889$ cm, a closed orbit finding using FIT can be performed, or it can be referred to the solutions of earlier exercises, to check agreement with raytracing outcomes.

(b) Wave numbers at 1 and 5 MeV.

These considerations result in the input data file given in Tab. 3.22, to compute multiturn trajectories. ; note that $R_0 = 12.924889$ cm therein, whereas a value of $R_0 = 50$ cm may be taken instead in other exercises. Field index derivatives k', k'', ... are taken null in the present exercise.

Three particles with paraxial radial and axial motions are raytraced over a few turns. Their starting radius is the closed orbit radius for the respective energies, while a 0.1 mrad take-off angle is imparted to each particle both vertically and horizontally.

The value of the focusing index k_E at an energy E can be expressed in terms of DIPOLE data which are, the index value k at R_0 (Eq. 3.11), reference radius R_0, and field $B_0 = B_Z(R_0)$, namely,

$$k_E = \frac{R_E}{B_E}\frac{\partial B}{\partial R} = \frac{R_0 + \Delta R}{B_0 + \Delta B}\frac{\partial B}{\partial R} \approx k\,\frac{1 + \Delta R/R_0}{1 + k\Delta R/R_0} \approx k\left[1 + (1-k)\frac{\Delta R}{R_0}\right]$$

Table 3.21 Input data file 60DegSectorR200.inc: it defines DIPOLE as a sequence segment comprised between the "LABEL_1" type labels [16, Sect. 7.7] #S_60DegSectorR200 and #E_60DegSectorR200. DIPOLE here, has an index $k = -0.03$, reference radius $R_0 \equiv R_0(E_k = 200\,\mathrm{keV}) = 12.924888\,\mathrm{cm}$ and $B_0 = B(R_0) = 0.5\,\mathrm{T}$. Note that (i) this file can be run on its own: it has been designed to provide the transport MATRIX of that DIPOLE; (ii) for the purpose of some of the exercises, IL=2 under DIPOLE, optional, causes the printout of particle data in zgoubi.plt, at each integration step (this is at the expense of CPU time, and memory volume)

```
60DegSectorR200.inc
'OBJET'
64.62444403717985                                              ! 200keV proton.
5
0.01 0.001 0.01 0.001 0. 0.0001
12.9248888074 0.  0.  0. 0. 1.                                  ! 200keV. R=Brho/B=*/.5.
'DIPOLE'   #S_60DegSectorR200               ! Analytical modeling of a dipole magnet.
2     ! IL=2, purpose: log stepwise particle data in zgoubi.plt. Avoid if unused: I/Os take CPU time.
60. 12.924888                                  ! Sector angle AT; reference radius R0.
30. 5. -0.03  0. 0.           ! Reference azimuthal angle ACN; BM field at R0; indices, N, N', N''.
0.  0.                                                        ! EFB 1 is  hard-edge,
4 .1455   2.2670   -.6395  1.1558  0. 0. 0.           ! hard-edge only possible with sector magnet.
30. 0.  1.E6  -1.E6  1.E6  1.E6
0.  0.                                                        ! EFB 2.
4 .1455   2.2670   -.6395  1.1558  0. 0. 0.
-30. 0.  1.E6  -1.E6  1.E6  1.E6
0. 0.                                                        ! EFB 3 (unused).
0  0.       0.      0.      0.      0. 0. 0.
0. 0.  1.E6  -1.E6  1.E6  1.E6 0.
4   10.
0.5                              ! Integration step size. Small enough for orbits to close accurately.
2  0. 0. 0. 0.                                     ! Magnet positionning RE, TE, RS, TS.
'FAISCEAU'  #E_60DegSectorR200
'MATRIX'
1 0
'END'
```

Table 3.22 Simulation input data file: raytrace a few turns around the cyclotron, three particles with different momenta, and 0.1 mrad horizontal and vertical take-off angles. The INCLUDE segment is taken from Tab. 3.21
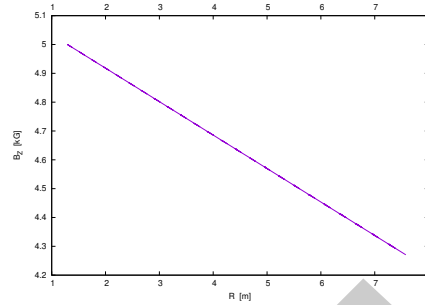
```
'MARKER'   ProbProjTraj_S
 'OBJET'
64.62444403717985                         ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
3 1
12.924888 0.1 0. 0.1 0. 1. 'o'       ! A particle with kin-E=0.2 MeV and 0.1 mrad take-off angles.
30.107898 0.1 0. 0.1 0. 2.2365445 'm'       ! p[MeV/c]=433.306, Brho[kG.cm]=144.535, kin-E[MeV]=1.
75.754671 0.1 0. 0.1 0. 5.0063900 'o'       ! p[MeV/c]=969.934, Brho[kG.cm]=323.535, kin-E[MeV]=5.
1 1 1
'INCLUDE'
1
6* 60DegSectorR200.inc[#S_60DegSectorR200:#E_60DegSectorR200]     ! 6 sectors for an overall 360 deg.
'REBELOTE'
 9 0.1 99                                        ! There will be a total of 9+1=10 tunrs.
'SYSTEM'
1
gnuplot < ./gnuplot_Zplt_traj.gnu                    ! Plot the projected Y(s) and Z(s) motions.
'MARKER'   ProbProjTraj_E
'END'
```

A gnuplot script to obtain Fig. 3.41:

```
# gnuplot_Zplt_traj.gnu
set xtics nomirror; set x2tics; set ytics; set xlabel 's  /C_E '; set ylabel 'Y  [cm]'
set palette defined ( 1 "red", 2 "blue", 3 "black" ) ; unset colorbox
array R[3]; R[1]=0.12924888; R[2]=0.301078986; R[3]=0.75754671; pi = 4.*atan(1.); cm2m = 0.01
sector1=3    # number (NOEL) of 1st DIPOLE in \zgoubi.res (col. 42 in zgoubi.plt)
# in zgoubi.plt, col. 19: particle number; col. 42: keyword number; col. 14: distance; col. 10: Y ; col. 12: YZ
plot for [sector=1:6] for [trj=1:3] 'zgoubi.plt' u ($19==trj && $42==sector1+2*(sector-1)? $14*cm2m/(2.*pi*R[$19]) :1/0) \
:($10*cm2m-R[trj]):($19) w p ps .2 lc palette notit ; pause 1

set ylabel 'Z  [cm]'; plot for [sector=1:6] for [trj=1:3] 'zgoubi.plt' u ($19==trj && $42==sector1+2*(sector-1)? $14*cm2m \
/(2.*pi*R[$19]) :1/0):($12):($19) w p ps .2 lc palette notit ; pause 1
```

Fig. 3.40 In DIPOLE field model (Eq. 3.35), $\frac{\partial B}{\partial R}$ is constant: this graph shows the linear decrease of the field $B_Z(R)$ (Eq. 3.38), obtained from the raytracing of particles circulating in the median plane on orbits spanning a 0.2 to 5 MeV energy range

with ΔR assumed small, $\partial B/\partial R = kB/R_E$ an energy independent quantity, and the index E denoting a quantity taken at the reference energy. The latter property is illustrated in Fig. 3.40, produced using the input data file of Tab. 3.23.

Table 3.23 Simulation input data file for a magnetic field scan. The INCLUDE segment is taken from Tab. 3.21

```
Field and derivative dB/dR, as a finction of R
'MARKER'    ProbProjTrajB_S
'OBJET'
64.62444403717985                   ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
1 1                                                                    ! Just one ion.
12.924888 0.1 0. 0.1 0. 1. 'o'        ! A particle with kin-E=0.2 MeV and 0.1 mrad take-off angles.
1
'INCLUDE'
1              ! IL=2 is necessary under DIPOLE, for step-by-step log of particle data in zgoubi.plt.
60DegSectorR200.inc[#S_60DegSectorR200:#E_60DegSectorR200]                 ! One sector is enough.
'FIT'
1
2 30 0 [12,80]              ! Vary particle's Y0 at OBJET, to have it match its D (=Brho/BORO).
1  1e-20
3.1 1 2 #End 0. 1. 0                                          ! Consrain Y_final=Y0.
'REBELOTE'
25 0.1 0 1                                      ! Scan parameter 35 (relative rigidity, D) in OBJET.
1
OBJET 35 1:5.00639             ! Scan relative rigidity D from 1 (200 keV) to 5.0063900 (5 MeV).
'SYSTEM'
1
gnuplot < ./gnuplot_Zplt_field.gnu                          ! Plot B(R), as read fron zgoubi.plt.
'MARKER'    ProbProjTrajB_E
'END'
```

_____

A gnuplot script to obtain Fig. 3.40:

```
# gnuplot_Zplt_field.gnu
set xtics nomirror; set x2tics; set ytics; set xlabel 's  /C_E '; set ylabel 'Y  [cm]'
set palette defined ( 1 "red", 2 "blue", 3 "black" ) ; unset colorbox
array R[3]; R[1]=0.12924888; R[2]=0.301078986; R[3]=0.75754671; pi = 4.*atan(1.); cm2m = 0.01
sector1=3    # number (NOEL) of 1st DIPOLE in \zgoubi.res (col. 42 in zgoubi.plt)
# in zgoubi.plt, col. 19: particle number; col. 42: keyword number; col. 14: distance; col. 10: Y ; col. 12: YZ
plot  for [i=1:6] for [trj=1:3]
  'zgoubi.plt' u ($19==trj && $42==sector1 +2*(i-1) ? $14*cm2m /(2.*pi*R[$19]) :1/0) \
  :($10*cm2m-R[trj]):($19) w p ps .2 lc palette notit ; pause 1

set ylabel 'Z  [cm]' ;
plot  for [i=1:6] for [trj=1:3]
  'zgoubi.plt' u ($19==trj && $42==sector1 +2*(i-1) ? $14*cm2m \
  /(2.*pi*R[$19]) :1/0):($12):($19) w p ps .2 lc palette notit ; pause 1
```
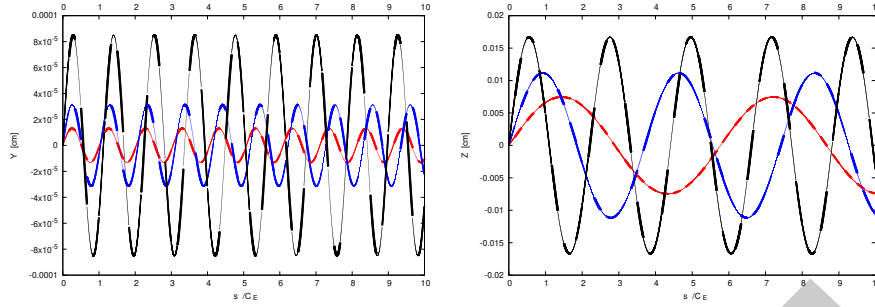
Fig. 3.41 Radial (left) and axial (right) paraxial motion around respectively the 200 keV (smallest amplitude), 1 MeV (intermediate) and 5 MeV (greatest amplitude) closed orbit (the latter is circular, in the median plane, with radius respectively $R_{200\,keV} = 12.924888$ cm, $R_{1\,MeV} = 30.107898$ cm and $R_{5\,MeV} = 75.754671$ cm). The horizontal axis in this graph is s/C_E: path length over closed orbit circumference at energy E, the vertical axis is the motion excursion

The resulting radial and axial motions over 10 turns are displayed in Fig. 3.41, which also illustrates, for paraxial motion at some reference energy, the energy dependence of the focusing strength (or wave number) and of the motion amplitude.

Table 3.24 Wave numbers, from numerical raytracing (columns denoted "ray-tr."), from theory, and from discrete Fourier transform ('DFT' cols.) from a multi-turn tracking

| E (MeV) | k_E | $\nu_R =$ | | | $\nu_Z =$ | | |
		ray-tr.	$\sqrt{1+k}$	DFT	ray-tr.	$\sqrt{-k}$	DFT
0.2	-0.03	0.98520	0.9849	0.98513	0.17320	0.1732	0.17321
1	-0.07279	0.96187	0.96292	0.96291	0.26980	0.26979	0.26981
5	-0.20586	0.89083	0.89115	0.89115	0.45326	0.45371	0.45371

An estimate of the wave numbers can be obtained as the inverse of the number of turns per oscillation, namely,

$$\nu_R = \left.\frac{C_E}{\Delta s_M}\right|_E \quad \text{and} \quad \nu_Z = \left.\frac{C_E}{\Delta s_M}\right|_E$$

with Δs_M the measured distance between two consecutive maxima in the sinusoid of concern in Fig. 3.41, C_E the closed orbit length for the energy of concern. Both quantities are obtained from motion records in zgoubi.plt. This yields the values of Tab. 3.24, where they are compared with the theoretical expectations, namely (Eq. 3.18), $\nu_R = \sqrt{1+k}$ and $\nu_Z = \sqrt{-k}$.

The maximum amplitude of the oscillation is obtained from zgoubi.plt records as well, this yields the results of Tab. 3.25. For comparison, the theoretical values are (Eqs. 3.16, 3.17 with respectively $x_0 = 0$, $x_0' = T_0$ and $y_0 = 0$, $y_0' = P_0$) $\hat{Y} = T_0 \frac{R_E}{\sqrt{1+k}}$ and $\hat{Z} = P_0 \frac{R_E}{\sqrt{-k}}$. wherein R_E denotes the closed orbit radius at energy E (for the record: $R_E \equiv R_0$ at energy $E = 200$ keV, in the foregoing).

Table 3.25 Maximum amplitude of the oscillation, from raytracing (columns denoted "ray-tr.") and from theory. R_E is the closed orbit radius for the energy of concern, $T_0 = P_0 = 0.1$ mrad is the trajectory angle at the origin, positions at the origin are zero

E (MeV)	k	$\hat{Y}$ ray-tr.	$T_0 \frac{R_E}{\sqrt{1+k}}$	$\hat{Z}$ ray-tr.	$P_0 \frac{R_E}{\sqrt{-k}}$
(MeV)		$(\times 10^{-5})$		$(\times 10^{-5})$	
0.2	-0.03	1.3123	1.3125	7.4622	7.4624
1	-0.072787	3.1270	3.1267	1.1160	1.1160
5	-0.20586	8.5010	8.5008	1.6697	1.6697

2624 (c) Comparison with theory.

2625 Figure 3.42 shows the difference between numerical and theoretical vertical mo-
2626 tion excursion, using an *ad hoc* gnuplot script. An integration step size $\Delta s = 2$ cm is
2627 used in the numerical integration.

Fig. 3.42 Vertical excursion of a 1 MeV trajectory over 20 turns (left vertical axis), and difference with theoretical expectation as per Eq. 3.17 (right vertical axis). The plot shows two sinusoidal curves: a segmented one, thicker, from numerical integration, and a thinner one, superimposed, from Eq. 3.17
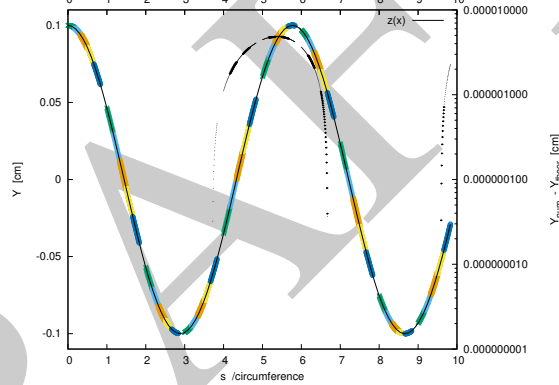
2628 (d) A scan of energy dependence of wave numbers.

2629 A scan of the wave numbers over 200 keV − 5 MeV energy range, computing tunes
2630 with MATRIX, is performed using the input data file given in Tab. 3.26 (essentially
2631 a copy of the input data file of Tab. 3.23, with an INCLUDE accounting for 6
2632 DIPOLEs [16, *lookup* INDEX]).

2633 OBJET[KOBJ=5] generates 13 particles with paraxial horizontal, vertical and
2634 longitudinal sampling, proper to allow the computation of the first order transport
2635 coefficients and wave numbers by MATRIX. REBELOTE repeats MATRIX com-
2636 putation for a series of different particle rigidities. It is preceded by FIT which finds
2637 the closed orbit. MATRIX includes a PRINT command, which causes the transport
2638 coefficients (and various other outcomes of MATRIX computation) to be logged
2639 in zgoubi.MATRIX.out. This allows producing the graphic in Fig. 3.43 - using the
2640 gnuplot script given at the bottom of Tab. 3.26.

Table 3.26 Simulation input data file: for this wave number scan, the INCLUDE segment is taken from Tab. 3.21

```
Field and derivative dB/dR, as a finction of R
'MARKER'    ProbMATRIX_S
'OBJET'
64.62444403717985                          ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
5                                                          ! Define 13 particles for MATRIX computation.
.001 .01 .001 .01 .001 .00001                              ! Sampling of the initial coordinates.
12.924888 0. 0. 0. 0. 1.              ! Reference: p[MeV/c]=193.739, Brho[kG.cm]=BORO, kin-E[MeV]=0.2.
'INCLUDE'
1                ! IL=2 is necessary under DIPOLE, for step-by-step log of particle data in zgoubi.plt.
6* 60DegSectorR200.inc[#S_60DegSectorR200:#E_60DegSectorR200]              ! Six 60 degree sectors.
'FIT'
1
2 30 0 [12,80]                        ! Vary particle's Y0 at OBJET, to have it match its D (=Brho/BORO).
1  1e-10
3.1 1 2 #End 0. 1. 0                                                        ! Consrain Y_final=Y0.
'MATRIX'
1 11  PRINT       ! PRINT: log computation outcome data to zgoubi.MATRIX.out, for further plotting.
'REBELOTE'
25 0.1 0 1                                   ! Scan parameter 35 (particle 1's D) in OBJT.
1
OBJET 35 1:5.00639
'SYSTEM'
1
gnuplot < ./gnuplot_MATRIX_Qxy.gnu
'MARKER'   ProbMATRIX_E
'END'
```


A gnuplot script to obtain Fig. 3.43:

```
# gnuplot_MATRIX_Qxy.gnu
set xlab "kin. E [MeV]";set ylab "{/Symbol n}_x,   ({/Symbol n}_x^2+{/Symbol n}_y^2)^{1/2}";set y2label "{/Symbol n}_y"
set key t l maxrow 1; set xtics; set ytics nomirror; set y2tics nomirror
BORO = 64.62444403717985; am = 938.27203e6; c = 2.99792458e8; BrhoRef = BORO *1e-3; eV2MeV = 1e-6
plot  "zgoubi.MATRIX.out" u ((sqrt(($47*BrhoRef*c)**2 + am*am)-am)*eV2MeV):($56) w lp pt 5 lt 1 lw .5 lc rgb "red" \
  tit "{/Symbol n}_x " , \
  "zgoubi.MATRIX.out" u ((sqrt(($47*BrhoRef*c)**2 + am*am)-am)*eV2MeV):($57) axes x1y2 w lp \
  pt 6 lt 3 lw .5 lc rgb "blue" tit "{/Symbol n}_y " ,\
  "zgoubi.MATRIX.out" u ((sqrt(($47*BrhoRef*c)**2 + am*am)-am)*eV2MeV):(sqrt($56**2+$57**2)) \
  w lp pt 7 lt 1 lw .5 lc rgb "black" t "     ({/Symbol n}_x^2+{/Symbol n}_y^2)^{1/2}";pause 1
```


Fig. 3.43 A scan of the energy dependence of the horizontal and vertical wave numbers. Markers are from raytracing, solid lines are from theory (Eq. 3.18). The figure also shows that the raytracing yields $\nu_R^2 + \nu_y^2 = 1$, $\forall E$, as expected
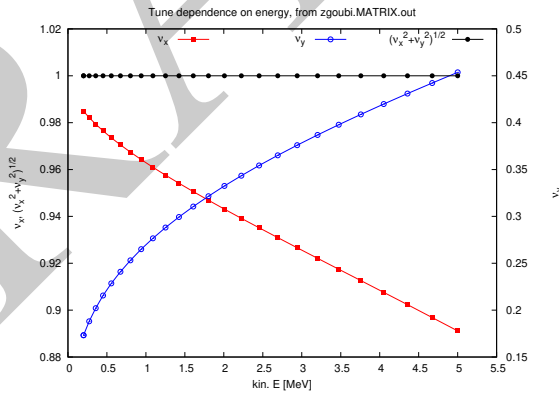
²⁶⁴¹ **3.9 RF Phase at the Accelerating Gap**

²⁶⁴² According to Sect. 3.2.3 (Fig. 3.13), the RF is taken about half-way of the ac-
²⁶⁴³ celerating range, namely, referring to Fig. 3.39, $T_{rev} = 0.131\,\mu s$ and $f_{rf} = 1/T_{rev} =$
²⁶⁴⁴ 7.633 MHz.

²⁶⁴⁵ An input data file for this simulation is given in Tab. 3.27.

²⁶⁴⁶ In a similar way to the diagrams in Fig. 3.13, the resulting $B(R)$ curve is given in
²⁶⁴⁷ Fig. 3.44, the resulting $\Delta W(\phi)$ curve in Fig. 3.45.

Table 3.27 Simulation input data file: accelerating a proton to get the evolution of RF phase The
[#S_60degSectorWIdx:#E_60degSectorWIdx] segment of Tab. 3.18 is INCLUDEd, here

```
Cyclotron, classical. Acceleration to 6.02 MeV.
'MARKER'  ProbRFPhase_S                                        ! Just for edition purposes.
 'OBJET'
64.62444403717985                                             ! Reference: 200keV (assuming proton).
2
1 1                                                           ! Just one ion.
12.629892  0.  0.   0.   0.  1. 'm'                           ! Injection energy 200keV, proton.
1
'PARTICUL'                                        ! Particle data are necessary as CAVITE is used,
PROTON                                  ! otherwise, by default \zgoubi\ only requires rigidity.
'INCLUDE'
1
3* sectorWithIndex.inc[#S_60degSectorWIdx:#E_60degSectorWIdx] ! Three 60 deg sectors. R0=50cm, k=-0.03.
'FAISTORE'                                              ! Log particle coordinates at each turn.
zgoubi.fai
1
'CAVITE'   GAP1
7  PRINT                               ! PRINT: log CAVITE computational dat to zgoubi.CAVITE.out.
0.00   7.63358778626e6                 ! f_rf= 1/T_rev, T_rev at about middle of acceleration range.
100e3    -1.57079632679                            ! Peak voltage;,  relative phase of 1st cavity.
'INCLUDE'
1
3* sectorWithIndex.inc[#S_60degSectorWIdx:#E_60degSectorWIdx] ! Three 60 deg sectors. R0=50cm, k=-0.03.
'CAVITE'   GAP1
7  PRINT                               ! PRINT: log CAVITE computational dat to zgoubi.CAVITE.out.
0.00   7.63358778626e6                 ! f_rf= 1/T_rev, T_rev at about middle of acceleration range.
100e3    +1.57079632679                            ! Peak voltage;,  relative phase of 1st cavity.
'FAISCEAU'                                         ! Local particle coordinates logged in zgoubi.res.
'REBELOTE'                       ! K = 99 : coordinates at end of previous pass are used as initial
42  1.1 99                       ! coordinates for the next pass ; idem for spin components.
'SYSTEM'
1                                                             ! 1 SYSTEM command follows.
/usr/bin/gnuplot < ./gnuplot_CAVITE.gnu &             ! Plot phase, as read from zgoubi.CAVITE.out.
'MARKER'  ProbRFPhase_E                                       ! Just for edition purposes.
'END'
```

A gnuplot to obtain the accelerated orbit of Fig. 3.45:

```
# gnuplot_CAVITE.gnu
set xlabel "RF phase [rad]" ; set ylabel "{/Symbol D}W [MeV]"; set xtics ;set ytics mirror
plot 'zgoubi.CAVITE.Out' u ($11):($2 - ($6-50)/10000.) w lp notit ; pause 2
```

²⁶⁴⁸ More turns are performed by changing the arguments under REBELOTE in the
²⁶⁴⁹ input data file (Tab. 3.27), from 42 to 75 in the present case. The resulting energy gain
²⁶⁵⁰ of the proton as a function of RF phase is shown Fig. 3.46. A first graph in Fig. 3.47
²⁶⁵¹ shows the evolution of its relative rigidity, namely D-1 as a function of distance, with
²⁶⁵² $D = B\rho(s)/BORO$ and BORO=64.624444 kG cm the reference rigidity as defined
²⁶⁵³ under OBJET; a second graph shows its orbital radius as a function of distance.

**Fig. 3.44** Radial dependence of the magnetic field over the acceleration range. The field is 0.5 T at a reference radius $R_0 = 0.5$ m, the slope results from the index $k = -0.03$. A graph obtained using zpop: menu 7; 1/1 to open zgoubi.plt; 2/[2,32] for B_Z versus Y; 7 to plot
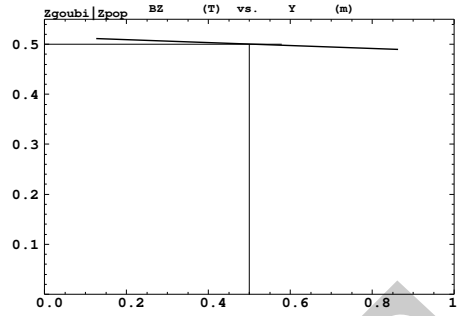
Fig. 3.45 Span in phase of the energy gain $\Delta W = q\hat{V}\sin\phi$ over the acceleration range 200 keV to 5 MeV. The vertical separation of the two $\Delta W(\phi)$ branches on the left ($\Delta\phi < 0$ above and $\Delta\phi > 0$ underneath) is artificial (a "-($6-50)/10000." "trick" in the gnuplot script of Tab. 3.27), for the sake of clarity - they actually superimpose
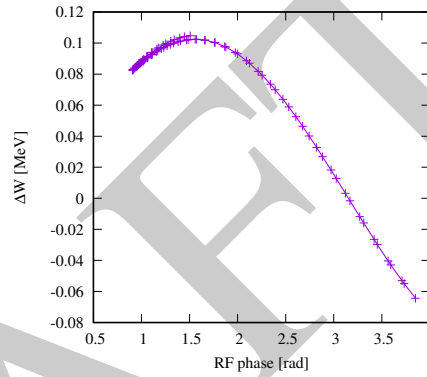
Fig. 3.46 Span in phase of the energy gain $\Delta W = q\hat{V}\sin\phi$ over an acceleration and deceleration cycle, starting from 200 keV. The vertical separation of $\Delta W(\phi)$ branches at the left and right ends is artificial
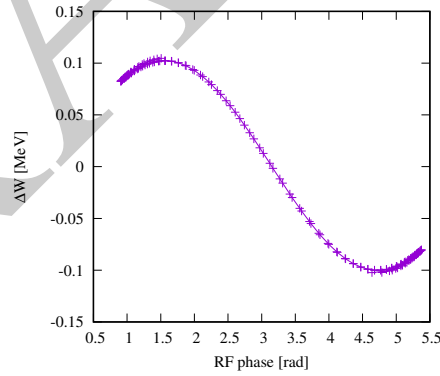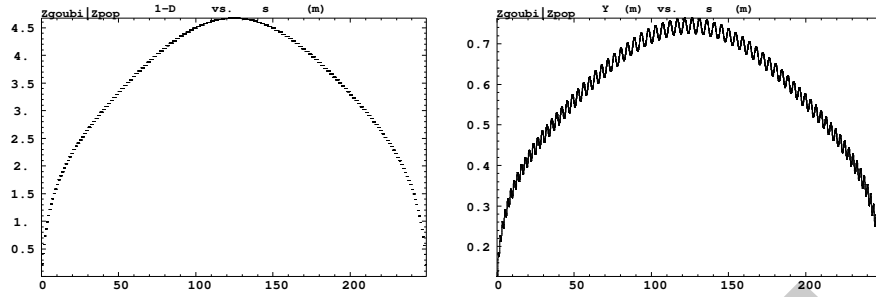
Fig. 3.47 Left: relative rigidity offset of the proton as a function of distance around the ring, accelerating over half the path, and subsequently decelerating back to the initial energy, under the effect of the cumulated phase-shift. Right: increase first and decrease next of the orbital radius as a function of azimuthal distance

3.10 The Cyclotron Equation

Cyclotron model settings of exercise 3.3 are considered in questions (a) to (c), first: two dees, double accelerating gap, uniform field $B = 0.5$ T. The analytical field modeling DIPOLE [16, *lookup* INDEX] is used.

(a) Simulation data file.

Acceleration is over the energy range $[0.2, 20]$ MeV, the maximum of $\cos(\phi)$ (Fig. 3.14) is placed at $E_k = E_{k,m} = 10$ MeV.

The cyclotron equation (Eq. 3.26) can be written under the form

$$\cos \phi = \cos \phi_i - \frac{\pi}{q\hat{V}} \left[\frac{\omega_{rf}}{2M\omega_{\text{rev}}} (E^2 - E_i^2) - (E - E_i) \right] \tag{3.41}$$

where the index i denotes injection parameters, ϕ is the phase of the RF at particle arrival at the accelerating gap, $\hat{V}$ is the peak gap voltage, $E = E_k + M$ is the total energy with M the rest mass. The value of E_k at the maximum of $\cos \phi$ is drawn from $d(\cos \phi)/dE_k = 0$, namely

$$E_{k,m} = \left(\frac{\omega_{\text{rev}}}{\omega_{rf}} - 1 \right) M \tag{3.42}$$

Taking $E_{k,m} = 10$ MeV one gets

$$\frac{\omega_{\text{rev}}}{\omega_{rf}} \approx 1 + 0.106578, \qquad f_{rf} \approx 0.989454 \omega_{\text{rev}}/2\pi = 7.542209 \text{ MHz}$$

The corresponding input data file is given in Tab. 3.28. Figure 3.48 shows the case of two particles accelerated at a rate of 400 kV per turn, one resulting from an initial phase at the gap of $\phi_i = \pi/2$ and reaches 20 MeV in about 52 turns, the other resulting from an initial phase $\phi_i = 3\pi/4$ and reaches 20 MeV in about 64 turns. In the latter case, the $\pi/4$ phase shift results from an initial path length offset

Table 3.28 Simulation input data file: the cyclotron equation (Eq. 3.26). This requires a uniform field, for that the [#S_60degSectorUnifB:#E_60degSectorUnifB] segment of Tab. 3.6 is INCLUDEd, here. Note the PRINT instruction under CAVITE: it causes a print out of CAVITE computational data in zgoubi.CAVITE.out, during the ray tracing, including RF phase and ion energy which can then be plotted (gnuplot script below, called by SYSTEM, and Fig. 3.48). The second particle under OBJET is launched on the closed orbit, its initial phase at the voltage gap is $\pi/2$. The first and third particles leave with an initial longitudinal shift $\delta s = \mp 10.26647$ cm at OBJET resulting in $\pi/4$ and $3\pi/4$ initial phase at the voltage gap

```
Cyclotron, classical. Acceleration to 6.02 MeV.
 'MARKER'  ProbCycloEq_S                                       ! Just for edition purposes.
  'OBJET'
64.62444403717985                                  ! Reference: 200keV (assuming proton).
2
3 1                                                                  ! A single particle.
12.924888 0. 0. 0 -10.266476 1. '2' ! Path length offset +pi/4, initial phase at gap: phi_rf=pi/2-pi/4.
12.924888 0. 0. 0. 0.            1. '1'              ! Initial phase at gap is phi_rf=pi/2.
12.924888 0. 0. 0. 10.266476 1. '2' ! Path length offset +pi/4, initial phase at gap: phi_rf=pi/2+pi/4.
1 1 1
 'PARTICUL'                                  ! Particle data are necessary as CAVITE is used,
PROTON                                       ! otherwise, by default \zgoubi\ only requires rigidity.
 'INCLUDE'
1
3 *./60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]            ! Uniform field, no index.
 'CAVITE'   GAP1
7  PRINT                            ! PRINT: log CAVITE computational dat to zgoubi.CAVITE.out.
0.00  7.54220925334568e6            ! f_rf= 1/T_rev, T_rev at about middle of acceleration range.
200e3   -1.57079632679             ! Peak voltage;,  relative phase of 1st cavity.
 'INCLUDE'
1
3 *./60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]            ! Uniform field, no index.
 'CAVITE'   GAP1
7  PRINT                            ! PRINT: log CAVITE computational dat to zgoubi.CAVITE.out.
0.00  7.54220925334568e6            ! f_rf= 1/T_rev, T_rev at about middle of acceleration range.
200e3   +1.57079632679             ! Peak voltage;,  relative phase of 1st cavity.
 'FAISTORE'                                      ! Log particle coordinates at each turn.
zgoubi.fai
1
 'REBELOTE'                      ! K = 99 : coordinates at end of previous pass are used as initial
135 1.1 99                      ! coordinates for the next pass ; idem for spin components.
 'SYSTEM'
1                                                            ! 1 SYSTEM command follows.
/usr/bin/gnuplot < ./gnuplot_CAVITE.gnu &       ! Plot Ek versus phase, as read from zgoubi.CAVITE.out.
 'MARKER'  ProbCycloEq_E                                      ! Just for edition purposes.
 'END'
```

———

*A gnuplot script to obtained Fig.* 3.49:

```
# gnuplot_CAVITE.gnu
set xlabel "E_k  [MeV]" ; set ylabel "cos({/Symbol f})"; set xtics; set ytics mirror
pi = 4. * atan(1.); E0 = 938.2720813; qV=400e-3; Ei=0.2; E_km = 10 # locate max of cos(phi) at 10 MeV
omgR = 1. /  (1. + E_km/E0); mxTurn=80
plot [0.2:20] [-1.1:1.8]  for [i=2:1:-1] \
'zgoubi.CAVITE.Out' u ($5==i && $6<mxTurn? $10 :1/0):(cos($11)) w p pt i+4 ps .5 notit ,\
cos(pi/2.)  +pi*(1.-omgR *(1.+(x+Ei)/(2*E0))) *(x-Ei)/(.5*qV) w l lw 2 lc rgb "blue" \
tit "V/gap=200kV, {/Symbol f}_0={/Symbol p}/2" ,\
cos(3*pi/4.)+pi*(1.-omgR *(1.+(x+Ei)/(2*E0))) *(x-Ei)/(.5*qV) w l lw 2 lc rgb "red"  \
tit "           {/Symbol f}_0={/Symbol 3p}/4" ,\
```

$$\delta s = \beta c T_{\mathrm{rf}}/4 = 10.26647 \,\mathrm{cm}$$

as specified under OBJET ($\beta c = 0.020648c$ is the proton velocity at $E_i = 200$ keV), yielding $\delta\phi = \omega_{\mathrm{rf}}\delta s / \beta c = \pi/4$. A third curve in the figure is for to 200 kV voltage and initial phase at gap $\phi = \pi/2$, in that case $\cos(\phi)$ reaches the value of 1 at about 4 MeV, 32 turns, and the particle starts decelerating.

(b) Energy-phase relationship.

Fig. 3.48 Proton energy versus turn, case of (Tab. 3.28) voltage 400 kV/turn, two protons with initial phase respectively $\pi/2$ ($\delta s = 0$) and $3\pi/4$ ($\delta s = 10.26647$ cm), which make it up to 20 MeV and beyond. The third case case, voltage 200 kV/turn, initial phase $\pi/2$ ($\delta s = 0$), features a maximum energy of 4 MeV and deceleration from there on. A graph obtained using zpop: menu 7; 1/5 to read from zgoubi.fai; 2/[39,2] for Y versus turn
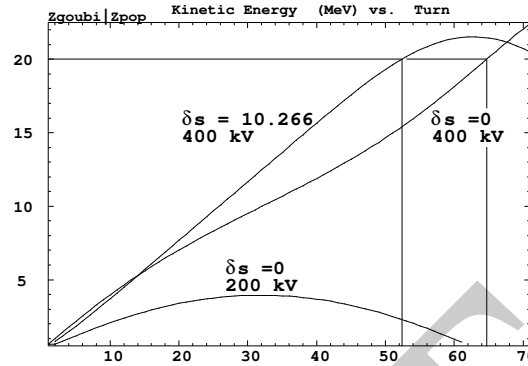
A graph of the energy-phase relationship obtained by ray tracing, for $\phi_i = \frac{3\pi}{4}$ and $\frac{\pi}{2}$ at the three different gap voltages $\hat{V} = 100,\ 200$ and 400 kV, is given in Fig. 3.49, together with theoretical expectations (Eq. 3.26).
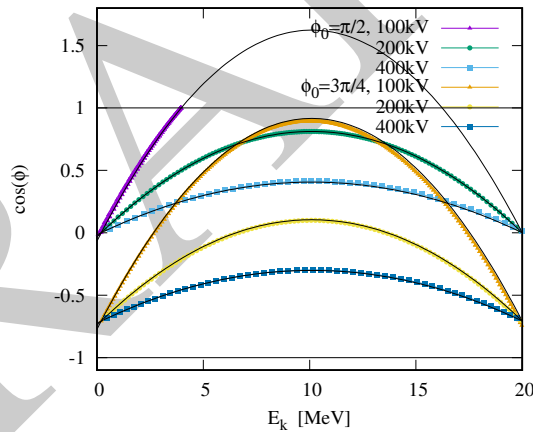
Fig. 3.49 A graph of the energy dependence of the arrival phase at the voltage gap, for a few different values of gap voltage $\hat{V}$ and initial phase ϕ_i. Markers are from raytracing, using the input data file of Tab. 3.28 repeatedly for the various values of $\hat{V}$ and ϕ_i. Superimposed solid lines are from theory (Eq. 3.26 and Fig. 3.14)

3.11 Cyclotron Extraction

(a) Distance between turns.

Simulation input data of exercise 3.3, Tab. 3.8, can be referred to as a guidance to build the present simulation file.

A proton is accelerated in 26 turns, in a uniform field $B_0 = 0.5$ T, from 20 keV (rigidity $BORO \times D = 0.064624444 \times 0.3162126 = 0.0204350634608$ T m, injection radius $Y_0 = BR/B_0 = 4.08701269216$ cm) to 5.02 MeV. The RF phase is ignored

²⁶⁸¹ thus CAVITE[IOPT=3] is used, with a 100 kV gap voltage. The input data file for
²⁶⁸² this simulation is given in Tab. 3.29.

Table 3.29 Simulation input data file: accelerating a proton to check evolution of $\Delta R/R$, in a dipole field with index. The #S_180degSectorUnifB to #E_180degSectorUnifB segment of Tab. 3.6 is INCLUDEd

```
Cyclotron extraction. Uniform field.
'MARKER'  ProbdRRUnifB_S                                          ! Just for edition purposes.
 'OBJET'
64.62444403717985                                                ! Reference: 200keV (assuming proton).
2
1 1                                                              ! A single particle.
4.08701 0. 0. 0. 0. 0.3162126 'o'        ! p[MeV/c]= 6.126277, Brho[kG.cm]=20.435, kin-E[MeV]=0.02.
1
! 4.003593 0. 0. 0. 0. 0.3162126 'o'   ! Brho[kG.cm]=20.435, kin-E[MeV]=0.02, case of field with index.
'PARTICUL'                                      ! Particle data are necessary as CAVITE is used,
PROTON                                          ! otherwise, by default \zgoubi\ only requires rigidity.
'INCLUDE'
1
./180degSector.inc[#S_180degSectorUnifB:#E_180degSectorUnifB]     ! one 180 deg sector, uniform field.

'FAISTORE'
zgoubi.fai                                      ! Log current particle coordinates, in zgoubi.fai.
1

'CAVITE'  cavity                                                  ! Accelerating gap.
3                                               ! In this option, dW = qVsin(phi_s), independent of time.
0. 0.
100e3  1.57079632679
'INCLUDE'
1
./180degSector.inc[#S_180degSectorUnifB:#E_180degSectorUnifB]     ! one 180 deg sector, uniform field.

'FAISCEAU'                                                       ! Particle coordinates before gap.
'CAVITE'  cavity                                                  ! Accelerating gap.
3                                               ! In this option, dW = qVsin(phi_s), independent of time.
0. 0.
100e3  1.57079632679

'REBELOTE'                      ! K = 99 : coordinates at end of previous pass are used as initial
25  1.1 99                      ! coordinates for the next pass ; idem for spin components.
'FAISCEAU'                                      ! Local particle coordinates logged in zgoubi.res.
'SYSTEM'
2                                                               ! 1 SYSTEM command follows.
/usr/bin/gnuplot < ./gnuplot_Zplt_UnifB.gnu &                   ! Plot accelerated orbits.
/usr/bin/gnuplot < ./gnuplot_Zfai_dRR.gnu &                      ! Plot delta_R(R).
'MARKER'  ProbdRRUnifB_E                         ! Just for edition purposes.
'END'
```

A gnuplot script to obtain the accelerated orbit of Fig. 3.50:

```
# gnuplot_Zplt_UnifB.gnu
set xtics ;  set ytics ; set xlabel "X_{Lab} [m]" ; set ylabel "Y_{Lab} [m]"
set size ratio 1 ; set polar ; cm2m = 0.01 ; pi = 4.*atan(1.)
set arrow from 0, 0 to 0.7, 0 nohead linecolor "red" lw 6; set arrow from 0, 0 to -0.65, 0 nohead linecolor "blue" lw 6
noel_1=5 ; noel_2=10  # 1st DIPOLE is element $42=noel_1; 4th DIPOLE is $42=noel_2. $42=column number in zgoubi.plt.
plot "zgoubi.plt" u ($42< noel_2? $22   +pi/3.*(($42-noel_1)/2) :1/0):($10 *cm2m) w p pt 5 ps .2 lc rgb "black"  notit ,\
"zgoubi.plt" u ($42>=noel_2? $22+pi+pi/3.*(($42-noel_2)/2) :1/0):($10 *cm2m) w p pt 5 ps .2 lc rgb "black"  notit; pause 1
```


A gnuplot script to obtain the turn separation curves of Fig. 3.50. In this script, zgoubi.fai2 is a copy of zgoubi.fai (see exercise 3.3) in which the first particle data line (particle data at the first pass) has been removed. This allows drawing the difference ΔR between two successive passes, using the "paste" command (see Tab. 3.8 for a similar 1-row shift using awk commands):

```
# gnuplot_Zfai_dRR.gnu
set xtics; set ytics mirror ; set key maxrow 2 ; set xlabel "R [cm]" ; set ylabel "{/Symbol D}R [cm]"
set key r c; set logscale y; unset colorbox
plot [8:65] "<paste zgoubi.fai2 zgoubi.fai" u ($10):($10-$63) w p pt 7 ps 1.5 lc rgb "black"  tit "num." ,\
"zgoubi.fai2" u ($10):($10/2./($38-1)) w l lc rgb "red"  tit "theory" ; pause 1
```

The accelerated orbit and the distance ΔR between turns are displayed in Fig. 3.50. Theoretical expectation (Eq. 3.27) in the case of slow acceleration (typically, the fixed energy closed orbit configuration of Fig. 3.21) is also displayed, for comparison.
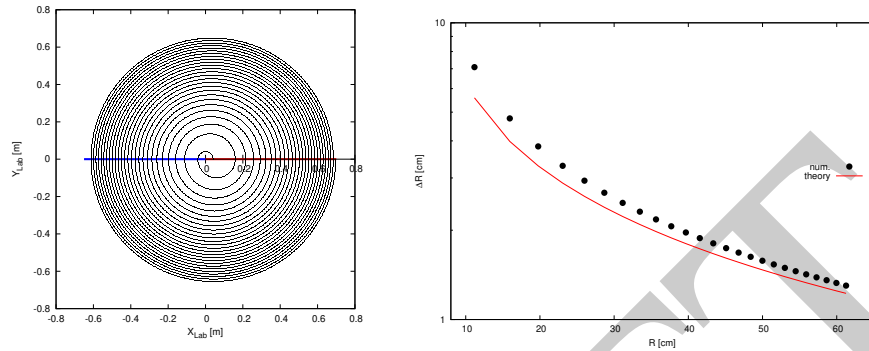
Fig. 3.50 Left: accelerated orbit from 20 keV to 5.02 MeV, at a rate of 200 keV per turn over 26 turns, in a uniform field. The thick horizontal line (colored) figures the accelerating gap. Right: the resulting dependence of orbit separation ΔR on radius, from raytracing (markers) and from theory (solid line); the theoretical curve assumes small dE (adiabatic acceleration, concentric orbits), which is not quite the case here with $\Delta E = 200$ keV/turn
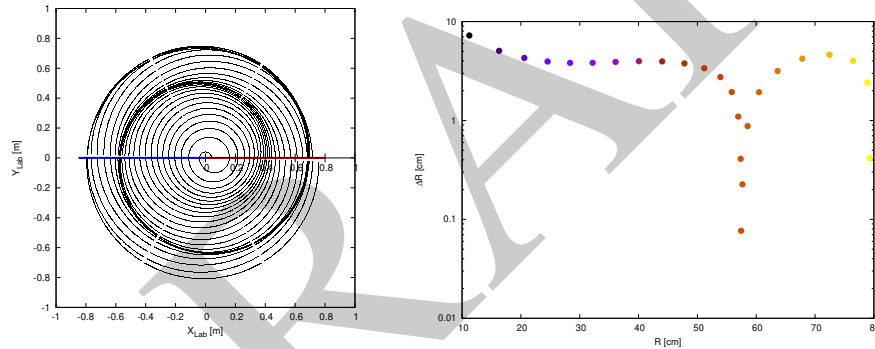
Fig. 3.51 Left: accelerated orbit from 20 keV to 5.02 MeV, at a rate of 200 keV per turn over 26 turns, in a dipole field with index. The thick horizontal line (colored) figures the accelerating gap. Right: the resulting dependence of orbit separation ΔR on radius, observed at the second gap

(b) Beam losses.

Indications to solve this exercise:

- a beam with Gaussian momentum distribution and *rms* momentum spread $\delta p/p = 10^{-3}$ can be defined using MCOBJET,

- use REBELOTE to accelerate over a given number of turns,

- an extraction septum placed half-way between two successive turns can be simulated using COLLIMA, placed after REBELOTE (the execution pointer will

2693 quietly continue beyond REBELOTE do-loop once the latter is completed). COL-
2694 LIMA counts particles stopped. FAISTORE (or FAISCNL) can be placed after
2695 COLLIMA, to log particle data: particles stopped by COLLIMA have their IEX tag
2696 set to IEX=-4 [16, *lookup* COLLIMA].

2697 Change the value of NPASS under REBELOTE for a different number of accel-
2698 erated turns, and COLLIMA positioning data accordingly.

2699 (c) Change the field index.

2700 The cyclotron model of Tab. 3.21 is used here, reference field $B_0 = 5$ kG on the
2701 200 keV orbit, and field index k=-0.03. A proton is accelerated over 26 turns, from
2702 20 keV to 5.02 MeV, as in question (a). The 20 keV closed orbit radius (taken as the
2703 injection radius) differs from question (a) due to the index k=-0.03, and can be found
2704 using a FIT procedure (Tab. 3.30); it comes out to be $Y_0 = 4.0040586$ cm.

2705 The input data file for this exercise is given in Tab. 3.31.

Table 3.30 Simulation input data file: finding the 20 keV injection radius in the presence of a non-zero index k, using FIT The INCLUDE segment is taken from Tab. 3.21

```
Cyclotron, classical. Find injection radius for k=-0.03
'OBJET'
64.62444403717985                                      ! Reference: 200keV (assuming proton).
2
1 1                                                    ! A single particle.
4.0040586 0. 0. 0. 0. 0.3162126 'o'        ! p[MeV/c]= 6.126277, Brho[kG.cm]=20.435, kin-E[MeV]=0.02.
1
! 4.003593 0. 0. 0. 0. 0.3162126 'o' ! Brho[kG.cm]=20.435, kin-E[MeV]=0.02, case of field with index.
'PARTICUL'
PROTON
'INCLUDE'
1
./60DegSectorR200.inc[#S_60DegSectorR200:#E_60DegSectorR200]        ! One 60 deg sectors with index.

'FIT'
1                                                                  ! 1 variable:
1 30 0 1.                 ! variable is Y0 (parameter 30) under OBJET (keyword 1 in the sequence).
2                                                                  ! 2 constraints:
3.1 1 2 #End 0. 1. 0                           ! constraint 1: final Y = Y0.
3   1 3 #End 0. 1. 0                           ! constraint 2: final T = 0.
'END'
```

2706 The resulting proton trajectory is displayed in Fig. 3.51 (the gnuplot script given in
2707 Tab. 3.31 is used). The greatly different accelerated orbit in this case, compared to the
2708 uniform field case in (a) (Fig. 3.50), results from a modulation of the distance between
2709 turns, which is an effect of the oscillation motion undergone by the accelerated orbit
2710 (around the local on-momentum half-circle orbit arc). This effect may be exploited
2711 to increase extraction efficiency, by causing such a radial modulation as to maximize
2712 turn separation at the location of the septum [17].

2713 (d) Optimize extraction.

2714 The modulation is minimized (or enhanced possibly, at the last turn, for minimized
2715 losses at extraction) by optimizing the injection conditions (x_0, x_0').

Table 3.31 Simulation input data file: accelerating a proton to check evolution of $\Delta R/R$, in a dipole field with index. The [#S_60DegSectorR200:#E_60DegSectorR200] segment of Tab. 3.21 is INCLUDEd

```
Cyclotron extraction. Field with index.
'MARKER'  ProbdRRIdx_S                                         ! Just for edition purposes.
 'OBJET'
64.62444403717985                                             ! Reference: 200keV (assuming proton).
2
1 1                                                          ! A single particle.
4.0040586 0. 0. 0. 0. 0.3162126 'o'       ! p[MeV/c]= 6.126277, Brho[kG.cm]=20.435, kin-E[MeV]=0.02.
1
! 4.003593 0. 0. 0. 0. 0.3162126 'o'   ! Brho[kG.cm]=20.435, kin-E[MeV]=0.02, case of field with index.
'PARTICUL'                                         ! Particle data are necessary as CAVITE is used,
PROTON                                             ! otherwise, by default \zgoubi\ only requires rigidity.
'INCLUDE'
1
3* ./60DegSectorR200.inc[#S_60DegSectorR200:#E_60DegSectorR200]       ! Three 60 deg sectors with index.

'FAISTORE'
zgoubi.fai                                         ! Log current particle coordinates, in zgoubi.fai.
1

'CAVITE'  cavity                                                   ! Accelerating gap.
3                                                  ! In this option, dW = qVsin(phi_s), independent of time.
0. 0.
100e3  1.57079632679
'INCLUDE'
1
3* ./60DegSectorR200.inc[#S_60DegSectorR200:#E_60DegSectorR200]       ! Three 60 deg sectors with index.

'FAISCEAU'                                                       ! Particle coordinates before gap.
'CAVITE'  cavity                                                   ! Accelerating gap.
3                                                  ! In this option, dW = qVsin(phi_s), independent of time.
0. 0.
100e3  1.57079632679

'REBELOTE'                            ! K = 99 : coordinates at end of previous pass are used as initial
25  1.1 99                            ! coordinates for the next pass ; idem for spin components.
'FAISCEAU'                                         ! Local particle coordinates logged in zgoubi.res.
'SYSTEM'
2                                                                  ! 1 SYSTEM command follows.
/usr/bin/gnuplot < ./gnuplot_Zplt.gnu &                            ! Plot accelerated orbits.
/usr/bin/gnuplot < ./gnuplot_Zfai_DR.gnu &                         ! Plot delta_R(R).
'MARKER'  ProbdRRIdx_E                                         ! Just for edition purposes.
'END'
```

A gnuplot script to obtain the accelerated orbit of Fig. 3.51:

```
# gnuplot_Zplt.gnu
set xtics ; set ytics ; set xlabel "X_{Lab} [m]" ; set ylabel "Y_{Lab} [m]"
set size ratio 1 ; set polar ; cm2m = 0.01 ; pi = 4.*atan(1.)
set arrow from 0, 0 to 0.8, 0 nohead linecolor "red" lw 6; set arrow from 0, 0 to -0.85, 0 nohead linecolor "blue" lw 6
noel_1=4 ; noel_2=12  # 1st DIPOLE is element $42=noel_1; 4th DIPOLE is $42=noel_2. $42=column number in zgoubi.plt.
plot "zgoubi.plt" u ($42< noel_2? $22   +pi/3.*(($42-noel_1)/2) :1/0):($10 *cm2m) w p pt 5 ps .2 lc rgb "black"  notit ,\
"zgoubi.plt" u ($42>=noel_2? $22+pi+pi/3.*(($42-noel_2)/2) :1/0):($10 *cm2m) w p pt 5 ps .2 lc rgb "black"  notit; pause 1
```

A gnuplot script to obtain the turn separation curves of Fig. 3.51. *In this script, zgoubi.fai*2 *is a copy of zgoubi.fai in which the first particle data line (particle data at the first pass) has been removed. This allows drawing the difference* ΔR *between two successive passes, using the "paste" command - see Tab.* 3.8 *for a similar* 1*-row shift using awk commands*:

```
# gnuplot_Zfai_DR.gnu
set xtics; set ytics mirror; set key maxrow 2 ; set xlabel "R [cm]" ; set ylabel "{/Symbol D}R [cm]"
set key r c; set logscale y; unset colorbox
plot "<paste zgoubi.fai2 zgoubi.fai" u ($10):($10-$63):($10) w p pt 7 ps 1.5 lw .1 lc palette  notit ; pause 1
```

3.12 Acceleration and Extraction of a 6-D Polarized Bunch

This simulation can be set up using material drawn from previous exercises. It is not fully developed here, guidelines are given.

The cyclotron simulation hypotheses of exercise 3.10-a are considered, the input data file for this exercise can be built from that of Tab. 3.28, with a few modifications, namely:

- downstream of REBELOTE, add a 1 meter DRIFT: an embryo of an "high energy line" into which the bunch is steered at extraction;

- that DRIFT is preceded by CHANGREF to center the current reference frame on the final coordinates Y and T of the accelerated orbit; the latter have to be determined by prior raytracing;

- add histograms (to be logged in zgoubi.res) for observation of transverse and longitudinal particle coordinate densities in the bunch at extraction. This uses HISTO, as many times as needed.

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