

Homework 10

Problem 1. 3x5 points. Beam envelope in straight section.

For a one-dimensional motion consider beam propagating in a straight section starting as s_o and having length L . Let's eigen vector (beam envelope) at s_o is given by:

$$Y(s_o) = \begin{bmatrix} w_o \\ w'_o + \frac{i}{w_o} \end{bmatrix}; \beta_o \equiv w_o^2; \alpha_o = -\frac{\beta'_o}{2} \equiv -w_o w'_o; \quad (1)$$

(a) Propagate the eigen vector along the straight section. Show that β -function can be expressed as

$$\beta(s) = \beta^* + \frac{(s-s^*)^2}{\beta^*};$$

where β^*, s^* can be found from initial conditions (1). Hint, use derivative of β -function to find s^* . β^* is frequently used in colliders to describe the beam envelope in detectors.

(b) Calculate the (betatron) phase advance acquired in the straight section. Express it using β^*, s^* . Write expression for $x(s)$ and $x'(s)$. Show that $x^2 = \text{const}$.

(c) What is the maximum possible phase advance in a straight section (e.g. when s_o, L are unlimited)?

Solution: (a) Propagating the eigen vector through a drift is just multiplying it by the drifts transport matrix:

$$\tilde{Y}(s) = \begin{bmatrix} 1 & \Delta s \\ 0 & 1 \end{bmatrix} Y(s_o) = \begin{bmatrix} w_o + \Delta s \left(w'_o + \frac{i}{w_o} \right) \\ w'_o + \frac{i}{w_o} \end{bmatrix} = \begin{bmatrix} w(s) \\ w'(s) + \frac{i}{w(s)} \end{bmatrix} e^{i\Delta\psi}; \Delta s = s - s_o; \quad (2)$$

$$\beta(s) = w^2(s) = \left| w_o + \Delta s \left(w'_o + \frac{i}{w_o} \right) \right|^2 = (w_o + \Delta s w'_o)^2 + \frac{\Delta s^2}{w_o^2} = \beta_o - 2\alpha_o \Delta s + \frac{\Delta s^2}{\beta_o} (1 + \alpha_o^2)$$

It is clearly a positively defined parabola and we just should find where it has a minimum:

$$\beta'(s^*) = 2(w_o + \Delta s^* w'_o) w'_o + 2 \frac{\Delta s^*}{w_o^2} = 0 \rightarrow \Delta s^* = -w_o^2 \frac{w_o w'_o}{1 + (w_o w'_o)^2} = \frac{\alpha_o \beta_o}{1 + \alpha_o^2}$$

$$\beta^* = \beta(s^*) = \frac{\beta_o}{1 + \alpha_o^2}; w^* = \sqrt{\frac{\beta_o}{1 + \alpha_o^2}}; w'^* = 0;$$

Now we need just to apply (2) again with $s_o = s^*$:

$$\beta(s) = w^2(s) = \left| w^* + \frac{i(s-s^*)}{w^*} \right|^2 = \beta^* + \frac{(s-s^*)^2}{\beta^*} \#.$$

(b) Using (2) again we have:

$$w(s)e^{i\psi(s)} = w_o + (s - s_o)s \left(w'_o + \frac{i}{w_o} \right) = w^* + i \frac{s - s^*}{w^*} = w^* \left(1 + i \frac{s - s^*}{\beta^*} \right);$$

$$\psi(s) = \tan^{-1} \left(\frac{s - s^*}{\beta^*} \right) \rightarrow \psi(s_2) - \psi(s_1) = \tan^{-1} \left(\frac{s_2 - s^*}{\beta^*} \right) - \tan^{-1} \left(\frac{s_1 - s^*}{\beta^*} \right).$$

Trajectory:

$$x(z) = a\sqrt{\beta(z)} \cos(\psi(z) + \varphi); \beta(z) = \beta^* + \frac{z^2}{\beta^*}; \tan \psi(z) = \frac{z}{\beta^*};$$

$$x'(z) = a \left(\frac{\beta'(z)}{2\sqrt{\beta(z)}} \cos(\psi(z) + \varphi) - \frac{1}{\sqrt{\beta(z)}} \sin(\psi(z) + \varphi) \right)$$

We should note that:

$$\frac{\beta(z)}{\beta^*} = 1 + \frac{z^2}{\beta^{*2}} = 1 + \tan^2 \psi = \frac{1}{\cos^2 \psi}; \tan \psi(s) = \frac{z}{\beta^*};$$

$$x(z) = a\sqrt{\beta(z)} (\cos \psi \cos \varphi - \sin \psi \sin \varphi) = \frac{a\sqrt{\beta^*}}{\cos \psi} (\cos \psi \cos \varphi - \sin \psi \sin \varphi)$$

$$x(z) = a\sqrt{\beta^*} (\cos \varphi - \tan \psi \sin \varphi) = a\sqrt{\beta^*} \left(\cos \varphi - \frac{z}{\beta^*} \sin \varphi \right)$$

e.g. the trajectory is a straight line with constant

$$x'(z) = -\frac{a}{\sqrt{\beta^*}} \sin \varphi$$

(c) Assuming an very long drift

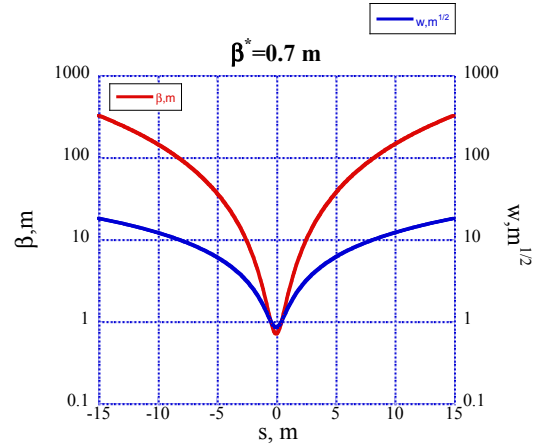
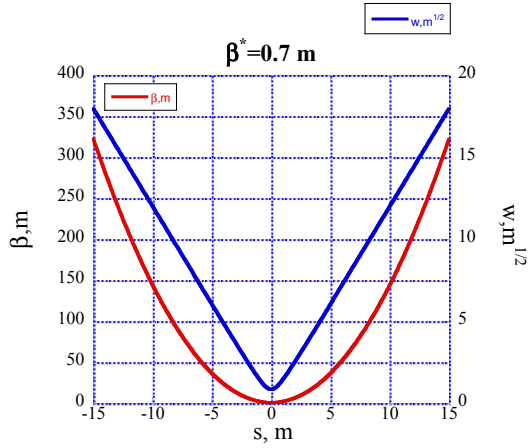
$$s_1 \rightarrow -\infty; s_2 \rightarrow +\infty$$

$$\psi(s_2) - \psi(s_1) \rightarrow \tan^{-1} \left(\frac{\rightarrow +\infty}{\beta^*} \right) - \tan^{-1} \left(\frac{\rightarrow -\infty}{\beta^*} \right) = \pi$$

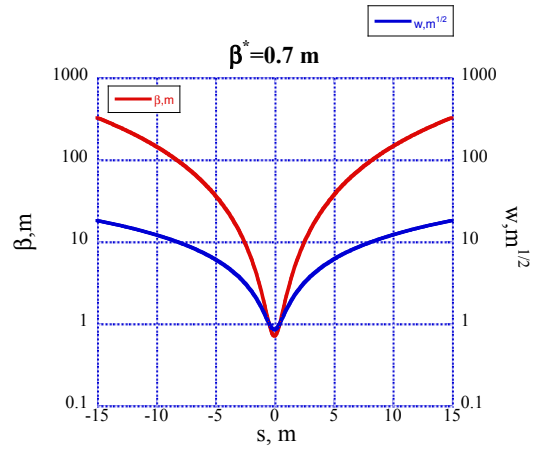
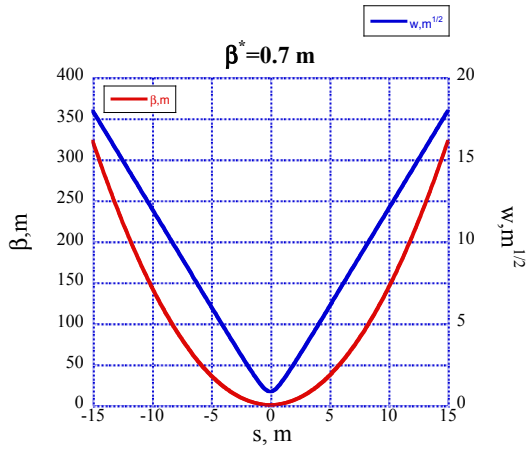
Naturally, you can get exactly the same result by integrating the phase advance using

$$\frac{d\psi}{ds} = \frac{1}{\beta(s)} \rightarrow \psi(s_2) - \psi(s_1) = \int_{s_1}^{s_2} \frac{ds}{\beta^* + \frac{(s-s^*)^2}{\beta^*}} =$$

$$\tan^{-1} \left(\frac{s_2 - s^*}{\beta^*} \right) - \tan^{-1} \left(\frac{s_1 - s^*}{\beta^*} \right)$$



Plot of beta-function and beam envelope in 30-m long straight section with $\beta^*=0.7 \text{ m}$ – typical for RHIC interaction region.



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