Homework I, Feb 10, 2014 **Due Feb, 17, 2014**

HW 1 (1 point): Very-very large accelerator

With current superconducting magnet technology, e.g. dipole magnets operating at 8.4 T at LHC, what will be maximum energy of the circular particle accelerator possible to built on the Earth surface. Assume that dipole magnets fill 91.56% of the ring circumference. Radius of the Earth is 6,371 km. Compare this energy with projected maximum energy in LHC of 7 TeV.

Solution:

Average bending field is 0.9156*8.4 T and radius 6,371 km, hence

$$B\rho = 6371 \cdot 0.9156 \cdot 8.4 \text{ T km}$$
$$E \cong pc[TeV] \cong 0.3 \cdot B\rho[T\ \ km] = 14,700\ TeV = 14.7\, PeV$$

which 2,100 higher then 7 TeV.

HW 2 (1 point): Microtron: note the typo in the problem you received!

For a classical microtron having energy gain per pass of 1.022 MeV and magnetic field of 0.5 T (5 kGs) find set of possible operational RF frequencies (Hint: use k=1,2,3,… as multiplier). What will be radius of first orbit in the microtron?

Solution: First lets fine the relativistic factor of electron on consecutive passes through the cavity:

$$E_N = mc^2 + N \cdot \Delta E_{rf}; \Delta\gamma = \frac{\Delta E_{rf}}{mc^2} = \frac{1.022\ MeV}{0.511\ MeV} = 2; \quad \gamma_N = 2N+1$$

Then from eq. (2.6) in Lecture 2 we get

$$\frac{2\pi\gamma_N mc}{eB} = \frac{M}{f_{rf}} \Rightarrow f_{rf} = \frac{M}{2N+1} \cdot \frac{eB}{2\pi mc}$$
$$\frac{M}{2N+1} = k = 1,2.3....; \quad f_{rf} = k \cdot f_0; \quad f_0 = \frac{eB}{2\pi mc} = \frac{c}{2\pi}\frac{eB}{mc^2}$$
$$mc^2 = 0.511 MeV \Rightarrow \frac{mc^2}{e} = \frac{0.511}{0.3} kGs\, cm; c = 3 \cdot 10^{10} cm/\sec$$
$$f_0 = \frac{eB}{2\pi mc} = \frac{3 \cdot 10^{10} cm/\sec}{2\pi} \frac{0.3 \cdot 5kGs}{0.511 kGs\, cm} =$$

Calculating the velocity of the electron on the first turn:

$$\gamma_1 = 3 \rightarrow \mathrm{v} = c\sqrt{1-\gamma_1^{-2}} = 0.942 \cdot 3 \cdot 10^8 \frac{m}{\sec} = 2.83 \cdot 10^8 \frac{m}{\sec}$$

The time of flight for the first turn is

$$R_1 = \frac{C_1}{2\pi} = \frac{T_1 \mathrm{v}}{2\pi} = 0.97 \cdot 10^{-2} m \sim\ 1\ cm$$

Discussion: while possible on paper, radius of 1 cm is too small to be practical and the RF frequency of 14 GHz and above is too high to be practical too. Hence, using a filed of 0.05 T will result is something much more practical: 10 cm radius of trajectory and 1.4 a GHz-scale RF frequency.

HW 3 (2 points): 300 MeV betatron

Assume that the steel in the central yoke reaches maximum average field of 1.6 T (16 kGs) in the betatron when beam reaches energy of 300 MeV.

(a) find the radius of the beam orbit in this betatron;
(b) Assuming that the coil of the magnet has 10 turns and the rated to maximum voltage of 100 kV and that the coil intercepts with 1.25 the flux of that intercepted by the beam orbit, estimate the shortest acceleration time for an electron from 1 MeV (the total energy, kinetic energy 0.489 MeV) to 300 MeV?

Hints: (i) neglect the initial momentum; (ii) note that the field is related to the relativistic momentum, not the energy of the particle.

Solution:

(a) At 300 MeV beam energy electrons are already ultra relativistic, e.g.

$$\gamma = \frac{E}{mc^2} \sim 600; 1 - \frac{\mathrm{v}}{c} \sim 10^{-6} \Rightarrow pc \cong E$$

Thus the radius of the orbit can be easily calculated using (2.3) the fact that the guiding field is only ½ of the average field:

$$R = \frac{p_{\max} c}{eB_{o\max}} = \beta \frac{E_{\max}}{eB_{o\max}} \cong \frac{300 MeV}{0.3(MeV/(kGs\ cm) \cdot 8 kGs} = 125\ cm;$$

$$1 - \beta \propto \gamma^{-2}/2 \sim 10^{-6} << 1$$

(b) According to induction law the integral of the electric field (e.g. the voltage) induced along the electron beam trajectory is equal to

$$e\oint \vec{E} \cdot d\vec{l} = -\frac{e}{c}\frac{d}{dt}\left(\int \vec{H} \cdot d\vec{s}\right)$$

As per the problem, the magnetic flux passing through the coil is 1.25-fold higher – it means that the induced voltage per one turn of coil (remember transformer?) is

$$V_{turn} = 1.25 \frac{\Delta \mathrm{E}_{turn}}{e}; \ \frac{\Delta \mathrm{E}_{turn}}{e} \oint \vec{E}\, d\vec{l}$$

The coil has 10 turns and the total induced voltage is 10-fold of that for one turn:

$$V_{coil} = 12.5 \frac{\Delta \mathrm{E}_{turn}}{e} \leq 100 kV$$

Let's assume that the betatron power supply operates at maximum, e.g. at 100 kV. It means that the integral of the electric field along the trajectory is

$$2\pi R \cdot E_{acc} = \frac{1}{1.25} \cdot \frac{100 kV}{10\ turns} = 8 kV$$

$$E_{acc} = 1.02\ \ V/cm$$

Particle momentum changes as in eq. (2.9):

$$\frac{dpc}{dt} = eEc \cong 3\cdot 10^{10}\,\frac{eV}{\sec}$$

The change of the total momentum is ~ 300 MeV/c (exact number is: at 300 MeV energy $pc = 299.99913\,MeV$ and at 1 MeV $pc = 0.8595\,MeV$ - thus the total change of the momentum will be $\Delta pc = 299.14\,MeV$) and time required for this to happen is

$$T = eEc \cong \frac{3\cdot 10^{8}\,eV}{3\cdot 10^{10}\,\dfrac{eV}{\sec}} = 10^{-2}\sec$$

One can estimate that electron will travel for about 0.38 millions turns.

HW 4 (1 point): C.M. energy

Find available energy from a head-on collision of two particles with different masses and energies: $p^i_{1,2} = \gamma_{1.2}\left(m_{1.2}c, \pm\hat{z}m\mathrm{v}_{1.2}\right)$. Express the leading term for $\gamma_{1.2} >> 1$.

Solution:

$$p^i_1 = \gamma_1 m_1 c\left(1, \vec{\beta}_1\right); p^i_2 = \gamma_2 m_2 c\left(1, \vec{\beta}_2\right);$$

$$p^i_1 p_{1i} = \left(m_1 c\right)^2; p^i_2 p_{2i} = \left(m_2 c\right)^2$$

$$M = \frac{\sqrt{P_i P^i}}{c}; P_i P^i = \left(p^i_1 + p^i_2\right)\left(p_{1i} + p_{2i}\right) = p^i_1 p_{1i} + 2p^i_1 p_{2i} + p^i_2 p_{2i}$$

$$P_i P^i = \left(m_1 c\right)^2 + \left(m_2 c\right)^2 + 2\gamma_1\gamma_2 m_1 m_2 c^2\left(1 - \vec{\beta}_1\vec{\beta}_2\right)$$

For head-on collision $\vec{\beta}_1\vec{\beta}_2 = -\left|\beta_1\right|\left|\beta_2\right| = -\sqrt{1-\gamma_1^{-2}}\sqrt{1-\gamma_2^{-2}}$

Hence:

$$P_i P^i = \left(m_1 c\right)^2 + \left(m_2 c\right)^2 + 2\gamma_1\gamma_2 m_1 m_2 c^2\left(1 + \sqrt{1-\gamma_1^{-2}}\sqrt{1-\gamma_2^{-2}}\right)$$

$$M = \frac{\sqrt{P_i P^i}}{c} = \sqrt{m_1^2 + m_2^2 + 2\gamma_1\gamma_2 m_1 m_2\left(1 + \sqrt{1-\gamma_1^{-2}}\sqrt{1-\gamma_2^{-2}}\right)}$$

$$\gamma_{1.2} >> 1 \Rightarrow Mc^2 \cong 2\sqrt{E_1 E_2};\ M \cong 2\sqrt{\gamma_1\gamma_2 m_1 m_2}$$

Thus, for ultra-relativistic particles the available c.m. energy an about twice the square root of the product of the energies of the colliding particles inde. For example, colliding 20 GeV electrons with 250 GeV proton will give c.m. energy of about 141 GeV.

HW 5 (1 point): Luminosity with a target.

Calculate luminosity of CEBAF facility delivering 10 microamperes of electron beam on a 1 cm deep liquid H_2O target at room temperature.

Solution:

Density of the water is 1 gram per cm^3, e.g. electrons penetrate though the 1 cm of water see matter of 1 gram/ cm^2. H_2O mole number is 18, it means that 18 grams of water contains Avogadro number of H_2O molecules: $N_A = 6.02\times10^{23}$. Let remember that H contains one proton and one electron and $^8O_{16}$ contains 8 electrons and protons and 8 neutrons. It means that the target contains

$$n_{H_2O} = \frac{N_A}{18} \approx 3.34\cdot10^{22}\,cm^{-2}$$

$$n_e = n_p = 10n_{H_2O} = 3.34\cdot10^{23}\,cm^{-2};$$

$$n_n = 8n_{H_2O} = 2.68\cdot10^{23}\,cm^{-2};$$

molecules, electron, protons and neutrons per unit transverse area. If one interested in transverse density of nucleons (e.g. neutrons and protons together) – it is simply the Avogadro number in this case. Now we need to calculate the flux of electrons per second coming from CEBAF accelerator, which is

$$\dot{N}_e = \frac{I}{e} = \frac{10^{-5}\,C/\sec}{1.6\cdot10^{-19}} = 6.25\cdot10^{13}\sec^{-1}$$

The luminosity for various collisions will be:

$$L_{e+H_2O} = \dot{N}_e n_{H_2O} \approx 2\cdot10^{36}\,cm^{-2}\sec^{-1}$$

$$L_{e+e} = L_{e+p} = \dot{N}_e n_{e,p} \approx 2\cdot10^{37}\,cm^{-2}\sec^{-1}$$

$$L_{e+n} = \dot{N}_e n_n \approx 1.7\cdot10^{37}\,cm^{-2}\sec^{-1}$$

$$L_{e+nucleon} = L_{e+p} + L_{e+n} \approx 3.7\cdot10^{37}\,cm^{-2}\sec^{-1}$$

By the way, CEBAF operates with 100 microamperes of electron beam and with typical luminosities ~ 10^{38} $cm^{-2}sec^{-1}$. Best modern colliders operating with amms of circulating beams currents are reaching into fewx10^{35} $cm^{-2}sec^{-1}$ range of luminosities – still it is too hard to bit the Avagadro number! http://www.slac.stanford.edu/cgi-wrap/getdoc/slac-pub-11726.pdf