

Homework 12. Due October 21

Problem 1. 10 points. Matrix Gymnastics.

For a one-dimensional motion consider parameterization via the eigen vectors and their propagation along s :

$$Y(s) = \begin{bmatrix} w(s) \\ w'(s) + \frac{i}{w(s)} \end{bmatrix}; Y^*(s) = \begin{bmatrix} w(s) \\ w'(s) - \frac{i}{w(s)} \end{bmatrix};$$

$$\beta(s) \equiv w^2(s); \quad \alpha(s) \equiv -\frac{\beta'(s)}{2} \equiv -w(s)w'(s); \quad \frac{d\psi}{ds} = \frac{1}{w^2(s)} \equiv \frac{1}{\beta(s)};$$

$$Y(s_2)e^{i\Delta\psi} = \mathbf{M}(s_1|s_2)Y(s_1) \Leftrightarrow Y^*(s_2)e^{-i\Delta\psi} = \mathbf{M}(s_1|s_2)Y^*(s_1); \Delta\psi = \psi(s_2) - \psi(s_1).$$

or in other form as:

$$\begin{bmatrix} Y(s_2)e^{i\Delta\psi} \\ Y^*(s_2)e^{-i\Delta\psi} \end{bmatrix} = \mathbf{M}(s_1|s_2) \begin{bmatrix} Y(s_1) \\ Y^*(s_1) \end{bmatrix};$$

$$\tilde{W}(s) = \begin{bmatrix} Y(s)e^{i\psi(s)} \\ Y^*(s)e^{-i\psi(s)} \end{bmatrix}; \tilde{W}(s_2) = \mathbf{M}(s_1|s_2)\tilde{W}(s_1);$$

Show that then matrix can be expressed through the values of envelopes w , and its derivatives w' and the “betatron” phase advance $\Delta\psi = \psi(s_2) - \psi(s_1)$ as:

$$\mathbf{M}(s_1|s_2) = \begin{bmatrix} \frac{w_2}{w_1} \cos \Delta\psi - w_1' w_2 \sin \Delta\psi & w_1 w_2 \sin \Delta\psi \\ -\sin \Delta\psi \left(\frac{1}{w_1 w_2} + w_1' w_2' \right) - \cos \Delta\psi \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1} \right) & \frac{w_1}{w_2} \cos \Delta\psi + w_2' w_1 \sin \Delta\psi \end{bmatrix}$$

or in traditional terms:

$$\mathbf{M}(s_1|s_2) = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \Delta\psi + \alpha_1 \sin \Delta\psi) & \sqrt{\beta_1 \beta_2} \sin \Delta\psi \\ -\frac{\sin \Delta\psi (1 + \alpha_1 \alpha_2) + \cos \Delta\psi (\alpha_2 - \alpha_1)}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \Delta\psi - \alpha_2 \sin \Delta\psi) \end{bmatrix}$$

Hint: if you are using $\tilde{W}(s)$, than use that $\tilde{W}^T S W = -2iS$. If you hate complex variables, than construct $\tilde{U}(s) = \begin{bmatrix} \tilde{R}(s) \\ \tilde{Q}(s) \end{bmatrix}; \tilde{R}(s) = \text{Re} \tilde{Y}(s); \tilde{Q}(s) = \text{Im} \tilde{Y}(s)$; show that it is symplectic, and define how it transformed with s .

Solution: While it is possible to do the exercise in a complex form like

$$\begin{aligned}\tilde{U}(s_2) &= \mathbf{M}\tilde{U}(s_1) \rightarrow \mathbf{M} = \tilde{U}(s_1)\tilde{U}^{-1}(s_2) \\ \tilde{U}^{-1} &= -\frac{1}{2i}S\tilde{U}^T S \rightarrow \mathbf{M} = -\frac{1}{2i}\tilde{U}(s_1)S\tilde{U}^T(s_2)S \\ \mathbf{M} &= \frac{1}{2i}\tilde{U}(s_1)S(\tilde{U}^T(s_2)S)^T\end{aligned}\quad (1)$$

$$\tilde{U}(s) = \begin{bmatrix} w(s) & w(s) \\ w'(s) + \frac{i}{w(s)} & w'(s) - \frac{i}{w(s)} \end{bmatrix} \begin{bmatrix} e^{i\psi(s)} & 0 \\ 0 & e^{-i\psi(s)} \end{bmatrix}$$

and just multiply the matrices. It is relatively easy but boring exercise to show that \mathbf{M} in (1) is a real matrix. It takes few steps:

$$\tilde{U}^* = \tilde{U} \cdot \mathbf{P}; \mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \mathbf{P}\mathbf{S} = -\mathbf{S}\mathbf{P}; \mathbf{P}^T = \mathbf{P}; \tilde{U}^{*T} = \mathbf{P}\tilde{U}^*; \mathbf{P}^2 = \mathbf{I};$$

$$\mathbf{M}^* = \left(-\frac{1}{2i}\tilde{U}_1\mathbf{S}\tilde{U}_2^T\mathbf{S}\right)^* = \frac{1}{2i}\tilde{U}_1^*\mathbf{S}\tilde{U}_2^T\mathbf{S} = \frac{1}{2i}\tilde{U}_1 \cdot \mathbf{P}\mathbf{S}\mathbf{P} \cdot \tilde{U}_2^T\mathbf{S} = -\frac{1}{2i}\tilde{U}_1\mathbf{S}\tilde{U}_2^T\mathbf{S} = \mathbf{M}$$

It is interesting to use real form of the “eigen” vector relations:

$$\begin{aligned}Y_k &= R_k + iQ_k; Y_k^* = R_k - iQ_k; \\ \mathbf{T} \cdot Y_k &= e^{i\mu_k}Y_k; \mathbf{T} \cdot Y_k^* = e^{-i\mu_k}Y_k^* \rightarrow \mathbf{T} \cdot R_k = R_k \cos \mu_k - Q_k \sin \mu_k; \mathbf{T} \cdot Q_k = R_k \sin \mu_k + Q_k \cos \mu_k; \\ V^T S V &= S; \\ V &= [R_1, Q_1 \dots R_n, Q_n]; \mathbf{T} \cdot V = V \cdot \Omega(C); \Omega(C) = \begin{bmatrix} \Omega_1 & 0 & \dots & 0 \\ 0 & \Omega_2 & \dots & \dots \\ \dots & \dots & \dots & 0 \\ \dots & \dots & 0 & \Omega_n \end{bmatrix} \Omega_k(C) = \begin{bmatrix} \cos \mu_k & -\sin \mu_k \\ \sin \mu_k & \cos \mu_k \end{bmatrix} \\ \tilde{V}(s_2) &= \mathbf{M}(s_1|s_2)\tilde{V}(s_1); \tilde{V}(s) = V(s)\Omega(s); V(s+C) = V(s); \Omega_k(s) = \begin{bmatrix} \cos \psi_k & -\sin \psi_k \\ \sin \psi_k & \cos \psi_k \end{bmatrix}\end{aligned}$$

with obvious expression for matrix but already in real form:

$$\begin{aligned}\mathbf{M}(s_1|s_2) &= \tilde{V}(s_1)\tilde{V}^{-1}(s_2) = V(s_1)\Omega(s_1)\Omega(s_2)V^{-1}(s_2) \\ V^{-1} &= -SV^T S \rightarrow \mathbf{M}(s_1|s_2) = V(s_1)\Omega(s_1 - s_2)SV(s_2)^T S\end{aligned}$$

In 1D case takes few multiplications

$$\begin{aligned}
 V(s) &= \begin{bmatrix} w(s) & 0 \\ w'(s) & \frac{1}{w(s)} \end{bmatrix}; \Omega(s) = \begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix}; V^{-1}(s) = \begin{bmatrix} \frac{1}{w(s)} & 0 \\ -w'(s) & w(s) \end{bmatrix} \\
 M(s_1|s_2) &= \begin{bmatrix} w(s_1) & 0 \\ w'(s_1) & \frac{1}{w(s_1)} \end{bmatrix} \begin{bmatrix} \cos \Delta \psi & \sin \Delta \psi \\ -\sin \Delta \psi & \cos \Delta \psi \end{bmatrix} \begin{bmatrix} \frac{1}{w(s_2)} & 0 \\ -w'(s_2) & w(s_2) \end{bmatrix} = \\
 & \begin{bmatrix} w(s_1) \cos \Delta \psi & w(s_1) \sin \Delta \psi \\ w'(s_1) \cos \Delta \psi - \frac{\sin \Delta \psi}{w(s_1)} & w'(s_1) \sin \Delta \psi + \frac{\cos \Delta \psi}{w(s_1)} \end{bmatrix} \begin{bmatrix} \frac{1}{w(s_2)} & 0 \\ -w'(s_2) & w(s_2) \end{bmatrix} = \\
 & \begin{bmatrix} \frac{w(s_1) \cos \Delta \psi}{w(s_2)} - w'(s_2) w(s_1) \sin \Delta \psi & w(s_2) w(s_1) \sin \Delta \psi \\ \frac{ad-1}{c} & \frac{w(s_2) \cos \Delta \psi}{w(s_1)} - w'(s_1) w(s_2) \sin \Delta \psi \end{bmatrix}
 \end{aligned}$$

Substitution to α, β is trivial. #