

Homework 12.

Problem 1. 15 points. Oscillating external force.

For a linear one-dimensional motion of particle in an storage ring with circumference C consider an oscillating force applied to the particle:

$$\frac{d}{ds} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1(s) & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ f(s)\cos\omega t \end{bmatrix};$$

$$K_1(s+C) = K_1(s); f(s+C) = f(s); t = \frac{s}{v_o}; \mu_e = C \frac{\omega}{v_o} = 2\pi Q_e.$$

Betatron tune and the eigen vectors are known function

$$\mu_x = 2\pi Q_x; Y(s) = Y(s+C) = \begin{bmatrix} w(s) \\ w'(s) + \frac{i}{w(s)} \end{bmatrix}$$

are considered to be known.

(a) Find solution in a form

$$x = x_e(s) + x_o(s)$$

where is $x_o(s)$ well know free oscillations and forced oscillations

$$x_e(s) = b(s)\cos(\omega t + \varphi); b(s+C) = b(s)$$

Find expression for $b(s)$ in a form of integral over the ring circumference.

Hint: use class notes for a general case and apply it to 1D

(b) Find and write down resonant conditions, when amplitude of oscillation is unlimited.

Solution: Let's write this as:

$$X = \begin{bmatrix} x \\ p \end{bmatrix} \Rightarrow X' = DX + \text{Re } F; F = \begin{bmatrix} 0 \\ f(s) \end{bmatrix} e^{iks}; k = \frac{\omega}{v_o};$$

$$X = \text{Re } Z; Z' = DZ + F$$

$$Z = \tilde{U}A'; \tilde{U}' = D\tilde{U} \rightarrow \tilde{U}A' = F$$

$$A = A_o + \int_o^s \tilde{U}^{-1}(\xi)F(\xi)d\xi$$

$$Z(s) = c \left(A_o + \int_o^s \tilde{U}^{-1}(\xi)F(\xi)d\xi \right)$$

and

$$Z(s+C) = \tilde{U}(s+C) \left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right) =$$

$$\tilde{U}(s) \Lambda \left(A_o + \int_0^s \tilde{U}^{-1}(\xi) F(\xi) d\xi + \int_s^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right)$$

Since the external force is oscillating with a given frequency, let's try to find a particular solution which has both: a complex amplitudes periodic with s and oscillating with the external frequency. Since the general solution of non-homogeneous linear OD equation is a combination of a particular solution of the inhomogeneous equation plus a general solution of homogenous equation. Thus, let's set

$$\tilde{U}(s+C) \left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right) = \tilde{U}(s) \left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right) e^{ikC};$$

$$\tilde{U}(s+C) = \tilde{U}(s) \Lambda; \Lambda = \begin{bmatrix} e^{i\mu} & 0 \\ 0 & e^{i\mu} \end{bmatrix} \rightarrow$$

$$\Lambda \int_s^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi = (\mathbf{I} e^{ikC} - \Lambda) \left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right) =$$

with a simple solution of

$$\left(A_o + \int_0^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi \right) = (\mathbf{I} e^{ikC} - \Lambda)^{-1} \Lambda \int_s^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi;$$

$$\tilde{U}(s) = U(s) \Phi(s); \Phi(s) = \begin{bmatrix} e^{i\psi(s)} & 0 \\ 0 & e^{-i\psi(s)} \end{bmatrix};$$

$$Z = \tilde{U}(s) (\mathbf{I} e^{ikC} - \Lambda)^{-1} \Lambda \int_s^{s+C} \tilde{U}^{-1}(\xi) F(\xi) d\xi = e^{iks} g(s)$$

$$g(s) = U(s) (\mathbf{I} - \Lambda e^{-ikC})^{-1} \int_{s-C}^s \Phi(s-\xi) U^{-1}(\xi) \begin{bmatrix} 0 \\ f(\xi) \end{bmatrix} e^{-ik(s-\xi)} d\xi.$$

It is easy to see that is periodic function of s : $g(s+C) = g(s)$. Thus we found the solution

$$(\mathbf{I} - \Lambda e^{-ikC})^{-1} = \begin{bmatrix} \frac{1}{1 - e^{i(\mu - kC)}} & 0 \\ 0 & \frac{1}{1 - e^{-i(\mu + kC)}} \end{bmatrix}; \quad \Phi(s - \xi) = \begin{bmatrix} e^{i(\psi(s) - \psi(\xi))} & 0 \\ 0 & e^{-i(\psi(s) - \psi(\xi))} \end{bmatrix};$$

$$e^{-i(\psi(s) - \psi(\xi)) - 1}(\xi) \begin{bmatrix} 0 \\ f(\xi) \end{bmatrix} = -\frac{1}{2i} \begin{bmatrix} w' - \frac{i}{w} & -w \\ -w' - \frac{i}{w} & w \end{bmatrix} \begin{bmatrix} 0 \\ f(\xi) \end{bmatrix} = \frac{f(\xi)w(\xi)}{2i} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(\mathbf{I} - \Lambda e^{-ikC})^{-1} \Phi(s - \xi) U^{-1}(\xi) \begin{bmatrix} 0 \\ f(\xi) \end{bmatrix} = \frac{f(\xi)w(\xi)}{2i} \begin{bmatrix} \frac{e^{i(\psi(s) - \psi(\xi))}}{1 - e^{i(\mu - kC)}} \\ e^{-i(\psi(s) - \psi(\xi))} \\ -\frac{1}{1 - e^{-i(\mu + kC)}} \end{bmatrix}$$

$$z(s) = \frac{e^{iks} w(s)}{2i} \left\{ \int_s^{s+C} f(\xi) w(\xi) \left(\frac{e^{i(\psi(s) - \psi(\xi))}}{1 - e^{i(\mu - kC)}} - \frac{e^{-i(\psi(s) - \psi(\xi))}}{1 - e^{-i(\mu + kC)}} \right) e^{-ik(s - \xi)} d\xi \right\}$$

and

$$x(s) = \text{Re} \left[\frac{e^{iks} w(s)}{2i} \left\{ \begin{aligned} & \frac{1}{1 - e^{i(\mu - kC)}} \int_s^{s+C} d\xi f(\xi) w(\xi) e^{i(\psi(s) - \psi(\xi))} e^{-ik(s - \xi)} - \\ & - \frac{1}{1 - e^{-i(\mu + kC)}} \int_s^{s+C} d\xi f(\xi) w(\xi) e^{-i(\psi(s) - \psi(\xi))} e^{-ik(s - \xi)} \end{aligned} \right\} \right]$$

and corresponding derivative.

Second part: let's look onto the denominators $1 - e^{i(\mu - kC)}$ and $1 - e^{-i(\mu + kC)}$ - they turned into zeros (e.g. amplitude of excited oscillation grow to infinity – in linear approximation) when

$$\mu \pm kC = 2N\pi \rightarrow Q_{exc} = N \pm Q; \quad Q_{exc} = \frac{kC}{2\pi}; \quad Q = \frac{k\mu}{2\pi}.$$

where N is an arbitrary integer. It is called a resonant excitation of oscillations – it allows to measure the non-integer part of the tune. But note, it can not distinguish between $\pm Q$.

Finally, for such a system located at s_o : $f(s) = f_o \delta(s - s_o)$

$$x(s) = \text{Re} \left[\frac{e^{iks_o} w(s) w(s_o) f_o}{2i} \left\{ \frac{e^{i(\psi(s) - \psi(s_o))}}{1 - e^{i(\mu - kC)}} - \frac{e^{-i(\psi(s) - \psi(s_o))}}{1 - e^{-i(\mu + kC)}} \right\} \right]$$