PHY 564
Advanced Accelerator Physics
Lectures 18

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Solutions of standard accelerator problems

Q: Why we need parameterization? -> A: To comfortably solve typical accelerator problems

Lecture 17

Applications of parameterization to standard problems

Complete parameterization developed in previous lecture can be used to solve most (if not all) of standard problems in accelerator. Incomplete list is given below:

1. Dispersion
2. Orbit distortions
3. AC dipole (periodic excitation)
4. Tune change with quadrupole (magnets) changes
5. Chromaticity
6. Beta-beat
7. Weak coupling
8. Synchro-betatron coupling
9. ……..

We do not plan to go through all these examples while focusing on general methodology and use selected examples to demonstrate power of the symplectic linear parameterization.

Sample I. Let's start from simplest problems such as dispersion and closed orbit. We found a general form of parameterization of linear motion in Hamiltonian system, which is solution of homogeneous linear equations, where ***B*** is constant vector:

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X;\ X = \tilde{\mathbf{U}}(s)\cdot B \tag{17-1}$$

A standards problems is a solution of inhomogeneous equations:

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + F(s); \tag{17-2}$$

It can be done analytically by varying the constant ***B:***

$$X = \tilde{\mathbf{U}}(s)B(s) \Rightarrow \tilde{\mathbf{U}}\cdot B' = F(s) \Rightarrow B' = \tilde{\mathbf{U}}^{-1}(s)F(s) \Rightarrow B(s) = B_o + \int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

A general solution is a specific solution of inhomogeneous equation plus arbitrary solution of the homogeneous – result you expect in linear ordinary differential equations (in this case with s-depended coefficients):

$$X(s) = \tilde{\mathbf{U}}(s)A_o + \tilde{\mathbf{U}}(s)\int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi;\ \ \tilde{\mathbf{U}}^{-1} = \frac{i}{2}\mathbf{S}\cdot\tilde{\mathbf{U}}^{T}\cdot\mathbf{S} \tag{17-3}$$

For a periodic force (orbit distortions, dispersion function) $F(s+C)=F(s)$ one can fine periodic solution $X(s+C)=X(s)$:

$$\tilde{\mathbf{U}}^{-1}(s)\times\left\{\tilde{\mathbf{U}}(s)A_o+\tilde{\mathbf{U}}(s)\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi=\tilde{\mathbf{U}}(s+C)A_o+\tilde{\mathbf{U}}(s+C)\int_{s_o}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\right\}$$

$$A_o(\mathbf{I}-\Lambda)=\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\equiv\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\Rightarrow A_o=(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi \qquad (17\text{-}4)$$

$$X(s)=\tilde{\mathbf{U}}(s)(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

It is easy to see that $X(s+C)=X(s)$ exists if none of the eigen values is not equal 1 – otherwise matrix $(\mathbf{I}-\Lambda)$ would have zero determinant and can not be inverted!

It is called integer resonance – closed orbit does not exist!

Specific examples: Orbit distortions caused by the field errors, transverse dispersion.

When the conditions for the equilibrium particle and the reference trajectory are slightly violated:

$$X^T = \{x, P_1, y, P_3, \tau, \delta\}; F^T = \left\{0, \frac{e}{c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right), 0, \frac{e}{c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right), 0, 0\right\}$$

$$K(s) \equiv \frac{1}{\rho(s)} - \frac{e}{p_o c}\left(B_y\Big|_{ref} + \frac{E_o}{p_o c}E_x\Big|_{ref}\right) - f_x; \quad f_x = \frac{e}{p_o c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right); \quad . \quad (17\text{-}5)$$

$$\frac{e}{p_o c}B_x\Bigg|\left(_{ref} - \frac{E_o}{p_o c}E_y\Big|_{ref}\right) = -f_y = \frac{e}{p_o c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right)$$

Plugging (17-5) into (17-4) will give one the periodic closed orbit for such a case. For finding reduces to

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + g_x x\delta + g_y y\delta$$

with

$$F = S\frac{\partial H}{\partial X} = \left\{0, -g_x, 0, -g_y, 0, -\frac{m^2c^2}{p_o^3}\right\}^T . \quad (17\text{-}6)$$

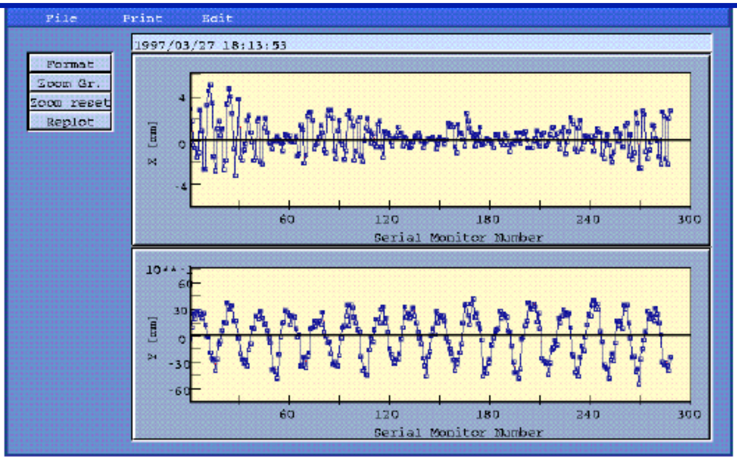


Fig. 10 Obseved closed orbit distortion.

1D ACCELERATOR

$$X(s)=\begin{bmatrix} \mathrm{w(s)} & \mathrm{w(s)} \\ \mathrm{w'(s)}+i/\mathrm{w(s)} & \mathrm{w'(s)}-i/\mathrm{w(s)} \end{bmatrix}\cdot \int_{s-C}^{s}\begin{bmatrix} \left(\mathrm{w}'(\xi)-i/\mathrm{w}(\xi)\right)e^{i\psi(s)-i\psi(\xi)}\left(1-e^{i\mu}\right)^{-1} & \text{-}\mathrm{w}(\xi)e^{i\psi(s)-i\psi(\xi)}\left(1-e^{i\mu}\right)^{-1} \\ \left(\text{-}\mathrm{w}'(\xi)+i/\mathrm{w}(\xi)\right)e^{i\psi(\xi)-i\psi(s)}\left(1-e^{i\mu}\right)^{-1} & \mathrm{w}(\xi)e^{i\psi(\xi)-i\psi(s)}\left(1-e^{i\mu}\right)^{-1} \end{bmatrix}\frac{iF(\xi)}{2}d\xi \qquad (17\text{-}7)$$

$X^T=\{x,x'\};F^T=\dfrac{e}{p_o c}\delta B_y\{0,1\}$ - orbit; $F^T=K(s)\{0,1\}$ for dispersion, i.e. $F^T=f(s)\{0,1\}$

$$X(s)=\int_{s-C}^{s}\left[\begin{matrix}\operatorname{Re}\left(\mathrm{w(s)}\mathrm{w}(\xi)e^{i(\psi(\mathrm{s})-\psi(\xi)-\mu/2)}\left(\dfrac{e^{-i\mu/2}-e^{i\mu/2}}{-i}\right)^{-1}\right) \\ \hline \dots\dots \end{matrix}\right]f(\xi)d\xi$$

i.e. $$x(s)=\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C f(\xi)\mathrm{w}(\xi)\cos(\psi(\mathrm{s})-\psi(\xi)-\mu/2)d\xi \qquad (17\text{-}8)$$

First example: orbit distortion

$$f_x(s) = -\frac{e\delta B_y(s)}{p_o c}; \quad f_y(s) = \frac{e\delta B_x(s)}{p_o c}$$

$$\delta x(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_y}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi$$

$$\delta y(s) = \frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_x}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \qquad (17\text{-}9)$$

but this is not the end of the story for horizontal motion! (what about change of the orbiting time?)

Second example: Dispersion

$$f_x(s) = K_o(s)\pi_l = K_o(s)\pi_\tau/\beta_o;$$

$$x(s) = D(s)\cdot\pi_l = D(s)\cdot\pi_\tau/\beta_o;$$

$$D(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C K_o(\xi)\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \qquad (17\text{-}10)$$

Sample II: Beta-beat – 1D case

It is simple fact that any solution can be expanded upon the eigen vectors of periodic system (FOD cell repeated again and again is an example). Let 's consider that at azimuth $s=s_o$ initial value of "injected" eigen vector **V** being different from the periodic solution **Y**. We expand it as

$$V(s_o)=aY_k(s_o)+bY_k^*(s_o)=\begin{bmatrix} v_o \\ v'_o+\dfrac{i}{v_o}\end{bmatrix};Y_k=\begin{bmatrix} w_o \\ w'_o+\dfrac{i}{w_o}\end{bmatrix}$$

$$a=\frac{1}{2i}\;Y_k^{*T}(s_o)SV(s_o)\;;\;b=\frac{1}{-2i}\;Y_k^{T}(s_o)SV(s_o) \qquad (17\text{-}11)$$

$$a=\frac{1}{2i}\left\{v_o w'_o-w_o v'_o+i\left(\frac{v_o}{w_o}+\frac{w_o}{v_o}\right)\right\};b=-\frac{1}{2i}\left\{v_o w'_o-w_o v'_o+i\left(\frac{v_o}{w_o}-\frac{w_o}{v_o}\right)\right\};$$

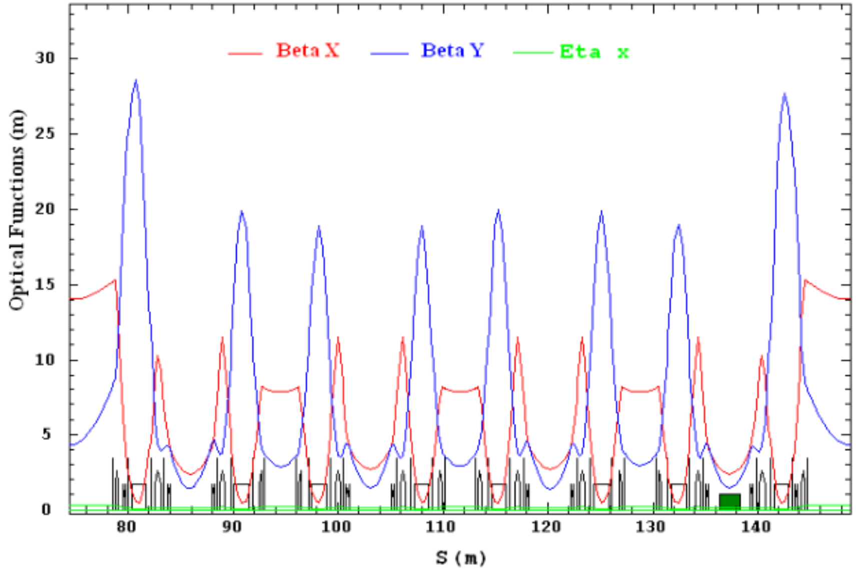
$$\frac{d}{ds}\tilde{Y}(s)=\mathbf{D}(s)\cdot\tilde{Y}(s);\quad \tilde{Y}(s)=Y(s)e^{i\psi(s)};Y(s+C)=Y(s)$$

It is self-evident that

$$\tilde{V}' = D\tilde{V};\quad \tilde{V}(s) = a\tilde{Y}_k(s) + b\tilde{Y}_k^*(s) = \begin{bmatrix} v \\ v' + \dfrac{i}{v} \end{bmatrix} e^{i\varphi} = Y_k = a\begin{bmatrix} w \\ w' + \dfrac{i}{w} \end{bmatrix} e^{i\psi} + b\begin{bmatrix} w \\ w' - \dfrac{i}{w} \end{bmatrix} e^{-i\psi} \qquad (17\text{-}12)$$

$$|v|^2 = \frac{|w|^2}{4}\left|ae^{i\psi} + be^{-i\psi}\right|^2 = \frac{|w|^2}{4}\left(|a|^2 + |b|^2 - 2\,\mathrm{Re}\left(ab^* e^{2i\psi}\right)\right)$$

i.e. beta-function will beat with double of the betatron phase.



Sample III: Perturbation theory (ala quantum mechanics)

Small variation of the linear Hamiltonian terms (including coupling)

$$\frac{dX}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right) \cdot X = \left(\mathbf{SH}(s) + \varepsilon \mathbf{SH}_1(s)\right) \cdot X$$

$$\frac{d\tilde{Y}_k(s)}{ds} = \mathbf{D}(s)\tilde{Y}_k(s); k = 1,..,n. \qquad (18\text{-}13)$$

Assuming that changes are very small we can express the changes in the eigen vectors using basis of (15):

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j \neq k} \left(a_{kj} \tilde{Y}_j + b_{kj} \tilde{Y}^*_j \right) + O(\varepsilon^2);\ k = 1,...,n$$

$$\tilde{Y}_{1k}^{\,*} = \tilde{Y}_k^{\,*} e^{-i\delta\phi_k} + \varepsilon c^*_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j \neq k} \left(a^*_{kj} \tilde{Y}^*_j + b^*_{kj} \tilde{Y}_j \right) + O(\varepsilon^2); \qquad (18\text{-}14)$$

$$\frac{d\tilde{Y}_{1k}}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right) \cdot \tilde{Y}_{1k} + o(\varepsilon^2);$$

We need substitute the expansion of the new eigen vectors into the differential equation and to keep first order term of ε

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}^{*}_j \right) + O(\varepsilon^2);\ k = 1,...,n$$

$$\tilde{Y}'_k e^{i\delta\phi_k} + \delta\phi'_k \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c'_k \tilde{Y}_k^{\,*} + \varepsilon c_k \tilde{Y}_k'^{*} + \varepsilon \sum_{j\neq k} \left(a'_{kj}\tilde{Y}_j + b'_{kj}\tilde{Y}^{*}_j \right) + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}'_j + b_{kj}\tilde{Y}'^{*}_j \right) =$$

$$\mathbf{D}\left(\tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}^{*}_j \right) \right) + \varepsilon \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} + O(\varepsilon^2)$$

$$\tilde{Y}'_j = \mathbf{D}\tilde{Y}_k; \tilde{Y}'^{*}_j = \mathbf{D}\tilde{Y}^{*}_j.$$

and all terms in red cancel each other leaving us with

$$\delta\phi'_k \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c'_k \tilde{Y}_k^{\,*} + \varepsilon \sum_{j\neq k} \left(a'_{kj}\tilde{Y}_j + b'_{kj}\tilde{Y}^{*}_j \right) = \varepsilon \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k}$$

which we can split into individual equations for each component using symplectic orthogonality of the eigen vectors

$$\tilde{Y}_k^{\,*} S \tilde{Y}_j = -\tilde{Y}_k S \tilde{Y}^{*}_j = 2i\delta_{ik};\ \tilde{Y}_k S \tilde{Y}_j = \tilde{Y}_k^{\,*} S \tilde{Y}_j^{\,*} = 0$$

Multiplying by $\tilde{Y}_m^{\ *}S$ or $\tilde{Y}_m S$ from the left yields:

$$-2\delta\phi_k' = \varepsilon \tilde{Y}_k^{*}\mathbf{SD}_1(s)\tilde{Y}_k \to \delta\phi' = \frac{\varepsilon}{2} Y_k^{\ *T}\mathbf{H}_1(s)Y_k; \quad \mathbf{SD}_1 = -\mathbf{H}_1;$$

$$-2ic' = \tilde{Y}_k^{T} S\mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \to c' = \frac{1}{2i} Y_k^{\ T}\mathbf{H}_1(s)Y_k e^{i(2\psi_k+\delta\phi_k)} \cong \frac{1}{2i} Y^T{}_k \mathbf{H}_1 Y_k e^{2i\psi_k}$$

$$2ia_{kj}' = \tilde{Y}_j^{*}\mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \to a_{kj}' = \frac{-1}{2i} Y_j^{\ *T}\mathbf{H}_1(s)Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)} \cong \frac{-1}{2i} Y_j^{\ *T}\mathbf{H}_1(s)Y_k e^{i(\psi_k-\psi_j)}; j \neq k$$

$$-2ib_{kj}' = \tilde{Y}_j^{*}\mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \to b_{kj}' = \frac{1}{2i} Y_j^{\ T}\mathbf{H}_1(s)Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)} \cong \frac{1}{2i} Y_j^{\ T}\mathbf{H}_1(s)Y_k e^{i(\psi_k+\psi_j)}; j \neq k.$$

with solutons in form of integrals:

$$\delta\phi(s) = \phi_o + \frac{\varepsilon}{2}\int_0^s Y_k^{\ *T}\mathbf{H}_1 Y_k\, d\xi;\ c(s) = c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\ T}\mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)};$$

$$a_{kj} = a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\ *T}\mathbf{H}_1 Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)}; b_{kj} = b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\ T}\mathbf{H}_1 Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)};$$

$$\tilde{Y}_{1k} e^{-i(\psi_k+\delta\phi_k)} = Y_k + \varepsilon c_k Y_k^{\ *} e^{-i(2\psi_k+\delta\phi_k)}\left(c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\ T}\mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)}\right) +$$

$$\varepsilon \sum_{j\neq k}\left(\begin{array}{l} Y_j e^{-i(\psi_k-\psi_j+\delta\phi_k)}\left(a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\ *T}\mathbf{H}_1 Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)}\right) + \\ Y^*{}_j e^{-i(\psi_k+\psi_j+\delta\phi_k)}\left(b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\ T}\mathbf{H}_1 Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)}\right) \end{array}\right) + O(\varepsilon^2)$$

Now we want to have periodic eigen vectors, e.g.

$$\tilde{Y}_{1k}(s+C) = \tilde{Y}_{1k}(s)e^{i\mu_{1k}}; \mu_{1k} = \mu_k + \frac{\varepsilon}{2}\int_0^C Y_k^{*T}\mathbf{H}_1 Y_k \, d\xi;$$

we need to choose the initial conditions to make a coefficient looking like:

$$d(s) = e^{-i\theta(s)}\left(d_o - \frac{1}{2i}\int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right); \; f(\xi+C) = f(\xi).$$

into periodic functions,

$$e^{-i\theta(s+C)}\left(d_o + \int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)}\right) = e^{-i\theta(s)}\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right);$$

$$\int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)} = \left(e^{i\Delta\theta(C)} - 1\right)\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right);$$

$$\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right) = \frac{1}{e^{i\Delta\theta(C)} - 1}\int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)}.$$

to get final

$$\tilde{Y}_{1k} e^{-i(\psi_k+\delta\phi_k)} = Y_k + \varepsilon \frac{Y_k^*}{2i\left(1-e^{i(2\mu_k+\delta\mu_k)}\right)} \int_s^{s+C} d\xi Y_k^T \mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)} +$$

$$\varepsilon \sum_{j\neq k} \left(\begin{array}{l} -\dfrac{Y_j}{2i\left(1-e^{i(\mu_k-\mu_j+\delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^{*T} \mathbf{H}_1 Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)} + \\ \dfrac{Y_j^*}{2i\left(1-e^{i(\mu_k+\mu_k+\delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^T \mathbf{H}_1 Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)} \end{array} \right) + O(\varepsilon^2) \qquad (18\text{-}15)$$

We should note, that while it was easy to keep $\delta\mu_k, \delta\phi_k$ in the final expression (18-15), it belongs to the next order correction and generally speaking should be dropped.

One should be aware of the resonant case $e^{i(\mu_k-\mu_i)}=1$, including parametric resonance $e^{2i\mu_k}=1$, when one should solve self-consistently the set of (18-14). It is well known case well described in weak coupling resonance case or in the case of parametric resonance.

Sample IV: small variation of the gradient. It can come from errors in quadrupoles or from a deviation of the energy from the reference value. In 1D case (reduced) it is simple addition to the Hamiltonian: (including sextupole term!)

$$H_1 = \delta K_1 \frac{z^2}{2};\ z = \{x, y\};$$

$$\pi_l = p/p_o - 1 \qquad (17\text{-}16)$$

$$\delta K_{1\ x,y} = \mp\delta\left(\frac{e}{pc}\frac{\partial B_y}{\partial x}\right) = \mp\left(\frac{e}{pc}\delta\frac{\partial B_y}{\partial x}\right) - K_1\pi_l \mp \left(\frac{e}{pc}\frac{\partial^2 B_y}{\partial x^2}D_x\right)\pi_l + o\left(\pi_l^{\ 2}\right)$$

Plugging our parameterization into the residual Hamiltonian we get:

$$z = w(s)\sqrt{2I}\cos\left(\psi(s) + \varphi\right)$$

$$H_1 = \delta K_1(s)\cdot w^2(s)\cdot I\cdot\cos^2\left(\psi(s)+\varphi\right) \qquad (17\text{-}17)$$

The easiest way is to average the Hamiltonian (on the phase of fast betatron oscillation – our change is small! And does not effect them strongly) to have a well-know fact that the beta-function is also a Green function (modulo 4π) of the tune response on the variation of the focusing strength.

$$\langle H_1 \rangle = \frac{\langle \delta K_1(s) \cdot w^2(s) \rangle}{2} \cdot I \equiv \frac{\langle \delta K_1(s) \cdot \beta(s) \rangle}{2} \cdot I$$

$$\langle \varphi' \rangle = \frac{\partial \langle H_1 \rangle}{\partial I} = \frac{\langle \delta K_1(s) \cdot \beta(s) \rangle}{2}; \qquad (17\text{-}18)$$

$$\Delta\varphi = \frac{1}{2} \oint \delta K_1(s) \cdot \beta(s) ds; \quad \Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi} \oint \delta K_1(s) \cdot \beta(s) ds;$$

Direct way will be to put it into the equations (43) and to find just the same, that <I'>=0 and the above result.

Finally, putting a weak thin lens as a perturbation gives a classical relation:

$$\delta K_1(s) = \frac{1}{f} \delta(s - s_o)$$

$$\Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi} \frac{\beta_o(s)}{f} \qquad (17\text{-}19)$$

In general case of change in Hamiltonian of linear motion

$$H = \frac{1}{2} X^T \left(\mathbf{H}_o + \mathbf{H}_1\right) X;\;\; X \to \left\{\varphi_k, I_k\right\} \to H_1\left(\varphi_k, I_k, s\right);$$

$$\Delta\mu_k = \frac{\partial}{\partial I} \int_o^C \left\langle H_1\left(\varphi_k, I_k, s\right)\right\rangle_{\varphi_k} ds. \qquad (17\text{-}20)$$

$$H_1\left(\varphi_k, I_k, s\right) = \frac{1}{2} A^T \tilde{U}^T \delta\mathbf{H}_1 \tilde{U} A =$$

$$\frac{1}{8}\left\{\sum_{k=1}^{n} \sqrt{2I_k}\left(Y_k e^{i(\psi_k+\varphi_k)} + Y_k^* e^{i(\psi_k+\varphi_k)}\right)\right\}^T \delta\mathbf{H}_1 \left\{\sum_{k=1}^{n} \sqrt{2I_k}\left(Y_k e^{i(\psi_k+\varphi_k)} + Y_k^* e^{i(\psi_k+\varphi_k)}\right)\right\}$$

$$\left\langle H_1\left(\varphi_k, I_k, s\right)\right\rangle_{\varphi} = \frac{1}{2}\sum_{k=1}^{n} I_k \operatorname{Re}\left(Y_k^* \delta\mathbf{H}_1(s) \tilde{Y}_k\right); \frac{d\varphi_k}{ds} = \frac{\partial\langle H_1\rangle}{\partial I_k} = \frac{1}{2}\operatorname{Re}\left(Y_k^* \delta\mathbf{H}_1(s)\tilde{Y}_k\right); \qquad (17\text{-}22)$$

or

$$\frac{d\varphi_k}{ds} = \frac{\partial H_1}{\partial I_k} = \frac{1}{4}\left(\tilde{Y}_k e^{i(\psi_k+\varphi_k)} + \tilde{Y}_k e^{i(\psi_k+\varphi_k)}\right)\delta\mathbf{H}_1 \left\{\sum_{k=1}^{n} \sqrt{2I_k}\left(\tilde{Y}_k e^{i(\psi_k+\varphi_k)} + \tilde{Y}_k e^{i(\psi_k+\varphi_k)}\right)\right\}$$

$$\left\langle \frac{d\varphi_k}{ds}\right\rangle = \left\langle \frac{\partial H_1}{\partial I_k}\right\rangle = \frac{1}{2}\operatorname{Re}\left(Y_k^* \delta\mathbf{H}_1(s)\tilde{Y}_k\right).$$

with

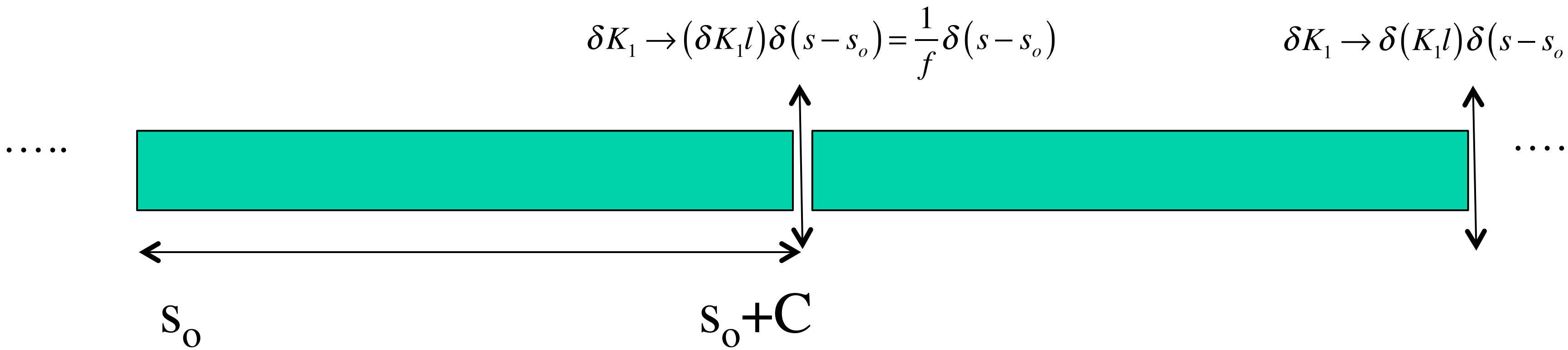
$$\Delta Q_k = \frac{\Delta\mu_k}{2\pi} = \frac{1}{4\pi}\int_0^c \operatorname{Re}\left(Y_k^*(s)\delta\mathbf{H}_1(s)\tilde{Y}_k(s)\right) ds \qquad (17\text{-}23)$$

Just to drive it home: 1D case

$$\Delta Q_k = \frac{\Delta\mu_k}{2\pi} = \frac{1}{4\pi}\int_0^c \mathrm{Re}\left(\begin{bmatrix} w & w' + \frac{i}{w} \end{bmatrix}\begin{bmatrix} \delta K_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w \\ w' + \frac{i}{w} \end{bmatrix}\right) ds = \tag{17-24}$$

$$\frac{1}{4\pi}\int_0^c w^2\,\delta K_1 ds = \frac{1}{4\pi}\int_0^c \beta(s)\,\delta K_1(s) ds$$

Traditional method



$$T(s_o) = I\cos\mu + J\sin\mu = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu + \alpha\sin\mu \end{bmatrix}$$

$$T^{'} = \begin{bmatrix} 1 & 0 \\ -\delta K_1 l & 1 \end{bmatrix} T = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ ..... & \cos\mu + \alpha\sin\mu - \beta K_1 l\sin\mu \end{bmatrix}$$

$$TraceT^{'} = TraceT - \beta K_1 l\sin\mu;$$

$$\cos\mu' = \cos\mu - \frac{\beta\delta K_1 l}{2}\sin\mu; \beta K_1 l << 1 \to \mu' = \mu + \delta\mu$$

$$\cos(\mu + \delta\mu) \cong \cos\mu - \delta\mu\sin\mu \to \delta\mu = \frac{\beta\delta K_1 l}{2} \equiv \frac{\beta}{2f}; \delta Q = \frac{\beta}{4\pi f}.$$

Sample V: Going beyond Hamiltonian system – taking dissipation into account

Let's consider that an additional linear term is no longer a Hamiltonian

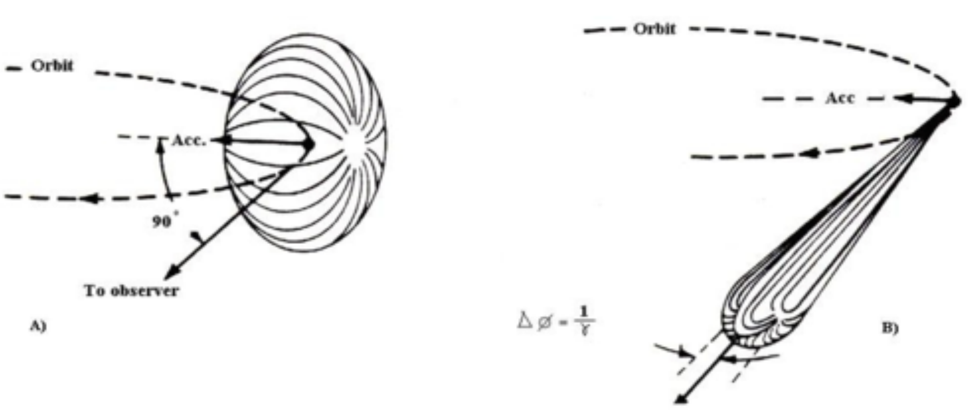
$$\frac{dX}{ds} = (\mathbf{D}(s) + \varepsilon \mathbf{d}(s)) \cdot X;\ \mathbf{D} = \mathbf{SH}; Trace[\mathbf{D}] = 0; Trace[\mathbf{d}] \neq 0 \qquad (18\text{-}25)$$

e.g. the overall motion is no longer symplectic

$$X(s) = \mathbf{R}(s) X_o \rightarrow \frac{d\mathbf{R}}{ds} = (\mathbf{D} + \varepsilon \mathbf{d})\mathbf{R} \rightarrow \frac{d \det[\mathbf{R}(s)]}{ds} = Trace[\mathbf{d}(s)]$$

$$\det[\mathbf{R}(s)] = \varepsilon \int_o^s Trace[\mathbf{d}(\xi)] d\xi; \qquad (18\text{-}26)$$

Such contributions can come form natural dissipative (or anti-dissipative) processes such as radiation reaction (synchrotron radiation damping), ionization cooling or from special accelerator systems, such as electron or stochastic cooling. Here we are not specifying what is the source of the non-Hamiltonian force and only assume that it is linear.



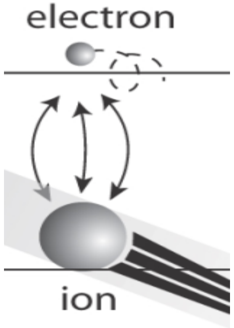
Similarly to regular parameterization, we can assume that motion can be expanded as a set of eigen modes

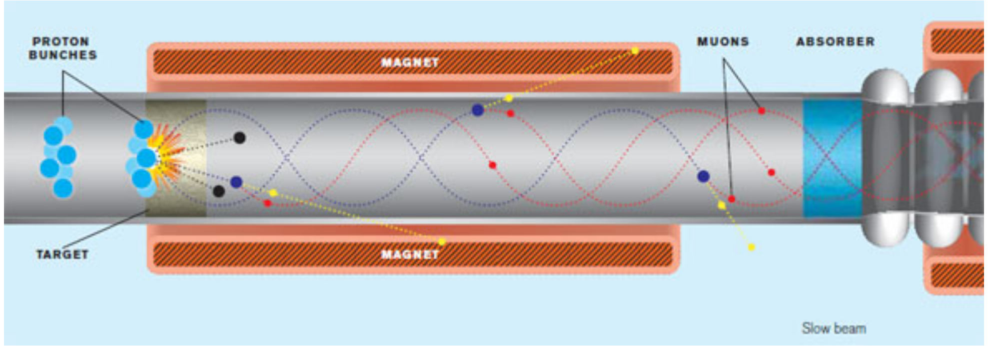
$$X(s)=\tilde{V}(s)\chi(s)\cdot B=\sum_{k=1}^{n}\tilde{V}_k(s)e^{\chi_k(s)}b_k;\det\tilde{V}(s)=1; \qquad (18\text{-}27)$$

than (18-26)

$$\begin{gathered}\frac{d}{ds}\sum_{k=1}^{2n}\chi_k(s)=\varepsilon Trace[\mathbf{d}(s)]\\ \sum_{k=1}^{2n}\chi_k(s)=\varepsilon\int_o^s Trace[\mathbf{d}(\xi)]d\xi;\end{gathered} \qquad (18\text{-}27)$$

which is commonly know as a the sum of decrements theorem: sum of the decrements (or increments!) of all eigen modes is equal to the integral of the trace of the dissipative matrix. This is to a degree the most trivial and well known relation for ordinary differential equation.





What is more interesting is to find decrements (increments) of the amplitudes of individual modes. Rewriting already established expansion (18-1)

$$X(s) = \frac{1}{2}\tilde{\mathbf{U}}(s)\cdot A(s) = \mathrm{Re}\sum_{k=1}^{n} Y_k(s) e^{i(\psi_k(s)+\varphi_k)} a_k(s); \frac{d}{ds}\tilde{\mathbf{U}}(s) = \mathbf{D}(s)\cdot\tilde{\mathbf{U}}(s) \tag{18-28}$$

$$\mathrm{Re}\sum_{k=1}^{n} Y_k(s) e^{i(\psi_k+\varphi_k)} \frac{da_k}{ds} = \varepsilon\mathbf{d}\cdot\mathrm{Re}\sum_{m=1}^{n} Y_m e^{i(\psi_m+\varphi_m)} a_m;$$

Using symplectic orthogonality of the eigen vectors we get equations of the evolution for individual amplitudes:

$$\frac{da_k}{ds} = \frac{\varepsilon}{2i}\cdot e^{-i(\psi_k+\varphi_k)}\left(\sum_{m=1}^{n} Y_k^{*T}(\mathbf{Sd}) Y_m e^{i(\psi_m+\varphi_m)} a_m + Y_k^{*T}(\mathbf{Sd}) Y_m^{*} e^{-i(\psi_m+\varphi_m)} a_m^{*}\right); \tag{18-29}$$

Hence, the perturbation can slightly change the eigen modes (as we discussed above in *ala quantum* perturbation) and phase of oscillations – the right side is not necessarily a real number.

But the main effect of-interest is in change of the amplitude of the oscillations, which comes from a simple averaging of (18-29). Since

$$\begin{aligned}\Delta\psi_k &= \psi_k(s+C) - \psi_k(s) = \mu_k\\ \Delta(\psi_k \pm \psi_m) &= \mu_k \pm \mu_m\end{aligned};$$

the only non-oscillating term in (18-29) is $Y_k^{*T}(\mathbf{Sd})Y_k$ and averaging yields

$$\begin{aligned}\left\langle \frac{da_k}{ds} \right\rangle &= \frac{\varepsilon}{2i} Y_k^{*T}(s)(\mathbf{Sd}(s))Y_m(s)\langle a_k \rangle;\\ \langle a_k \rangle(s) &= \langle a_k \rangle_o \exp\left[-\frac{\varepsilon}{2i} \int_0^s Y_k^{*T}(\xi)\cdot\mathbf{S}\cdot\mathbf{d}(\xi)\cdot Y_m(\xi)\,d\xi \right];\end{aligned} \qquad (18\text{-}30)$$

At no surprise, we arrived to an equation nearly identical to (18-23) with only exception that we did not assumed that motion is Hamiltonian. Indeed, if

$$\varepsilon\mathbf{d}(s) = \mathbf{S}\delta\mathbf{H}_1$$

$$\langle a_k \rangle(s) = \langle a_k \rangle_o \exp\left[\frac{1}{2i} \int_0^s Y_k^{*T}(\xi)\,\delta\mathbf{H}_1 Y_m(\xi)\,d\xi \right];$$

$$\Delta\varphi = \frac{1}{2} \int_0^s Y_k^{*T}(\xi)\,\delta\mathbf{H}_1 Y_m(\xi)\,d\xi$$

It should not be surprising – we are solving more or less the same problem using more or less the same method of varying constants.

The most useful form of (18-30) is calculation of dumping (or anti-damping) coefficients

$$\left|a_k\right| \cong \left|a_{k0}\right| e^{-\frac{\xi_k s}{C}}$$
$$\xi_k = -\frac{\varepsilon}{2}\int_0^C \operatorname{Im}\left(Y_k^{*T}(s)\left(\mathbf{Sd}(s)\right)Y_m(s)\right)ds; \qquad (18\text{-}31)$$

Naturally, the sum of the decrements is determined by the trace of the matrix. What is non-trivial is that we can re-distribute some (if not all) decrements between various modes of oscillations using coupling between them.

As indicated above, we combine the real and imaginary parts:

$$a_k e^{i\varphi} \cong a_{k0}\cdot e^{\frac{s}{C}\left(i\Delta\mu - \xi_k\right)}$$
$$i\Delta\mu - \xi_k = \frac{\varepsilon}{2}\int_0^C \left(Y_k^{*T}(s)\left(\mathbf{Sd}(s)\right)Y_m(s)\right)ds; \qquad (18\text{-}32)$$

We will use this expression now and again.

Again 1D case

It gives us know fact that damping of the amplitude of the oscillation is ½ of the dissipative term in $x'' - \xi_o x' + K_1(s)x = 0$:

$$\varepsilon \mathbf{d} = \begin{bmatrix} 0 & 0 \\ 0 & -\xi_o \end{bmatrix}$$

$$\xi_x = -\frac{1}{2}\operatorname{Im}\begin{bmatrix} \mathrm{w} & \mathrm{w}' - \frac{i}{\mathrm{w}} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -\xi_o \end{bmatrix}\begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + \frac{i}{\mathrm{w}} \end{bmatrix} = \frac{\xi_o}{2}\operatorname{Im}\begin{bmatrix} -\mathrm{w}' + \frac{i}{\mathrm{w}} & w \end{bmatrix}\begin{bmatrix} 0 \\ \mathrm{w}' + \frac{i}{\mathrm{w}} \end{bmatrix} = \frac{\xi_o}{2}.$$

By the way, the real part of the expression gives

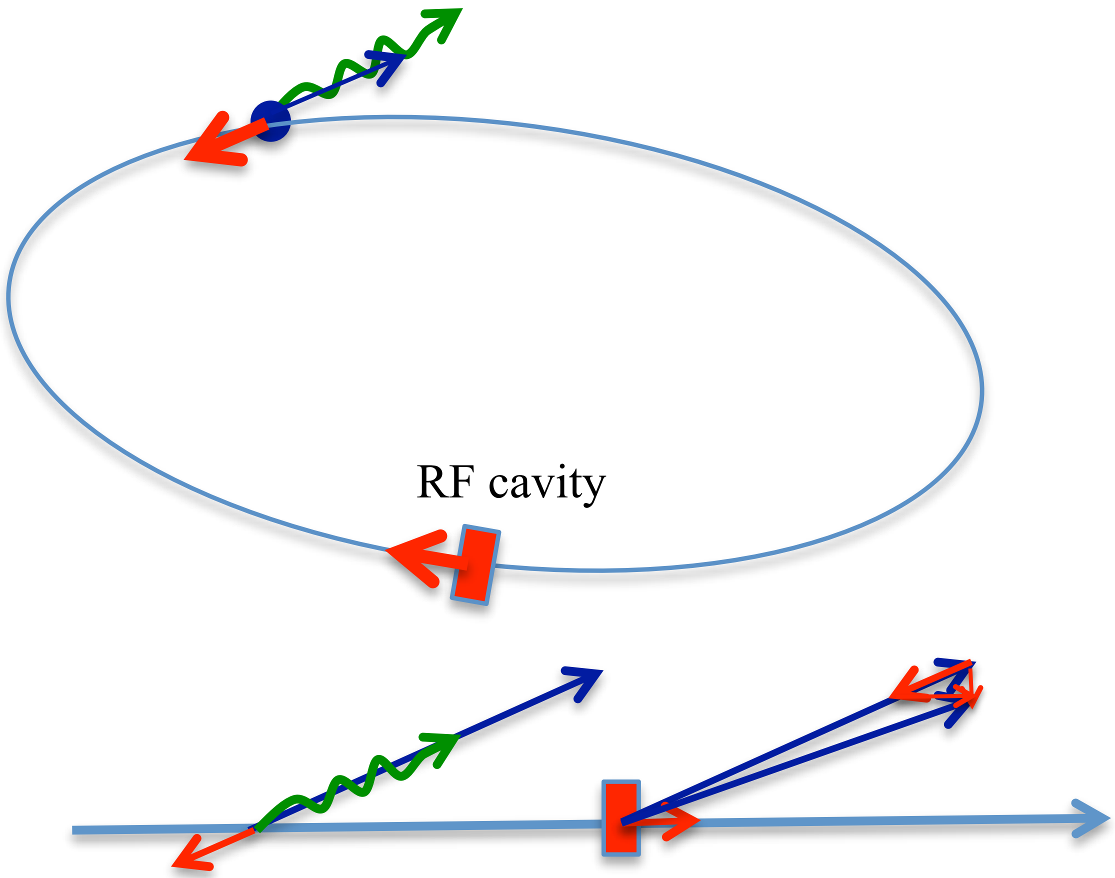
$$\varphi'_x = \frac{1}{2}\mathrm{w}'_x \mathrm{w}_x \frac{\xi_o}{2}$$

while being interesting academically, it does not play too much role in the accelerators.

We will return to damping when considering synchrotron radiation effects in accelerators.

How radiation cools beam in a storage ring: vertical motion

Particle radiate in the direction of the motion and RF cavity restores only longitudinal part of the momentum



Sample VI: Going beyond Hamiltonian system – random kicks

Particle in accelerators frequently experience a sudden events, which change their momenta essentially in instance. Naturally, there are no sudden changes of position – it would require not infinite force, but also a finite time to change position.

Examples of such processes include: radiation of a photon (so called quantum fluctuation of radiation), scattering on residual gas or on other particles inside the beam. The later is called intra-beam scattering and is one of limiting factors in attaining small beam emittances.

Again, let's just add an additional term in our equation of motion (18-1):

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + DP(s); DP(s) = \sum_a \delta P_a \cdot \delta(s - s_a) \qquad (18\text{-}33)$$

which has similar appearance as (18-2) but has very different nature – it represents a random process, not a regular continuous force. Nevertheless, we can find directly the change of the oscillation amplitude and phase at each random kick:

$$\sum_{k=1}^{n} e^{i\psi_k(s_a)} Y_k(s_a)\delta\left(a_k e^{i\varphi}\right)_{s_a} = \delta P_a \to \delta\left(a_k e^{i\varphi}\right)_{s_a} = e^{-i\psi_k(s_a)}\frac{1}{2i}Y_k^{T*}(s_a)\mathbf{S}\delta P_a;$$
$$a_k(s)e^{i\varphi} = a_{ok} + \sum_{s_a<s} e^{-i\psi_k(s_a)}\frac{1}{2i}Y_k^{T*}(s_a)\mathbf{S}\delta P_a; \qquad (18\text{-}34)$$

Naturally, the exact result depends of a realization of the random process. But statistically we can write the average change if the actions:

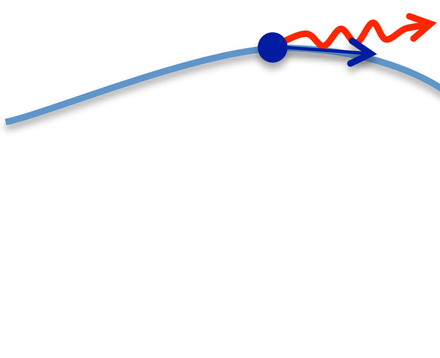
$$J_k = \frac{a_k^2}{2} \to \delta J_k = \frac{\left(a_k + \delta a_k\right)^2 - a_k^2}{2} = 2a_k \delta a_k + \left(\delta a_k\right)^2 \tag{18-35}$$

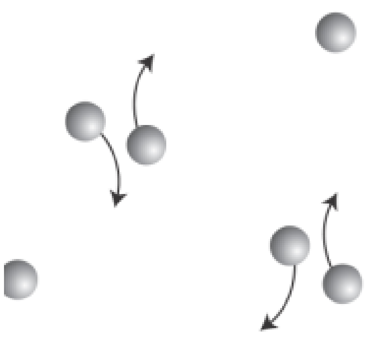
Now we need to look on the average picture again:

$$\tilde{a}_k = a_k e^{i\varphi}; \delta\tilde{a}_k = e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a$$

$$\delta\left|\tilde{a}_k\right|^2 \to \left(\tilde{a}_k + \delta\tilde{a}_k\right)\left(\tilde{a}_k^* + \delta\tilde{a}_k^*\right) - \tilde{a}_k \tilde{a}_k^* = \left|\delta\tilde{a}_k\right|^2 + 2\,\mathrm{Re}\,\tilde{a}_k^* \delta\tilde{a}_k \tag{18-34}$$

$$\tilde{a}_k^* \delta\tilde{a}_k = a_k e^{-i\varphi} e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a$$





Hence,

$$\left\langle \tilde{a}_k^* \delta \tilde{a}_k \right\rangle = \left\langle a_k e^{-i\varphi} e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a \right\rangle = 0 \qquad (18\text{-}35)$$

and

$$\left\langle \delta J_k \right\rangle = \left\langle \frac{\delta a_k^2}{2} \right\rangle = \frac{1}{2} \left\langle \left| \delta \tilde{a}_k \right|^2 \right\rangle = \frac{1}{8} \left| Y_k^{T*}(s_a) \mathbf{S} \delta P_a \right|^2 \qquad (18\text{-}36)$$

Now we need to introduce probability of the random kick δP at azimuth s to write an statistical average growth of the oscillation amplitude:

$$\left\langle \frac{dJ_k}{ds} \right\rangle = \frac{1}{8} \left\langle \left| Y_k^{T*}(s) \mathbf{S} \delta P \right|^2 \cdot \phi\left(s, \delta P\right) \right\rangle = D_k(s) \qquad (18\text{-}37)$$

This growth is called diffusion (or random walk). It has interesting characteristic that amplitude of oscillations growth proportionally to the square root of time – e.g. the action grows linearly.

Again, we will discuss values for specific processes later. What is interesting now is to combine damping and diffusion. To do this we need to tone that without diffusion

$$\frac{dJ_k}{ds} = \frac{1}{2}\frac{da_k^2}{ds} = a_k \frac{da_k}{ds} = -2\xi_k J_k \tag{18-38}$$

and adding diffusion we get to

$$\frac{d\langle J_k\rangle}{ds} = -2\xi_k(s)\langle J_k\rangle + D_k(s);$$
$$\langle J_k(s)\rangle = J_{ok} e^{-2\int_o^s \xi_k(z)dz} + \int_0^s e^{-2\int_z^s \xi_k(u)du} D_k(z)dz; \tag{18-39}$$

In storage rings it is frequently that the processes are very slow and you can average the damping and the diffusion over the circumference

$$\langle D_k\rangle = \langle D_k(s)\rangle_C ; \langle \xi_k\rangle = \langle \xi_k(s)\rangle_C$$
$$\langle J_k(s)\rangle = e^{-2\langle\xi_k\rangle s}\left(J_{ok} + \langle D_k\rangle \int_0^s e^{2\langle\xi_k\rangle z} dz \right) = J_{ok} e^{-2\langle\xi_k\rangle s} + \frac{\langle D_k\rangle}{2\langle\xi_k\rangle}\left(1 - e^{-2\langle\xi_k\rangle s}\right); \tag{18-40}$$

and stationary action at large s (many turns) being

$$\langle J_k(s)\rangle \to \frac{\langle D_k\rangle}{2\langle\xi_k\rangle} \tag{18-41}$$

This formula is very useful for both calculating and estimating the beam emittances in presence of diffusion and dimpling.

Note, that an anti-damping $\langle \xi_k \rangle < 0$ will cause exponential growth of the oscillating amplitude and is almost is bad and instability of periodic Hamiltonian motion. Hence, this is important for accelerators where damping plays significant role in the beam dynamics, e.g. damping (anti-damping) time is much smaller or compatible with the beam life-time in the accelerator.

Remarkably, I know about one storage ring (VEPP-4 in Novosibirsk), which was initially built for proton-antiproton collisions but then will turned into electron-position collider. Since protons do not radiate any significant part of radiation, synchrotron radiation decrements were not important and neglected during design. When the switch to electrons and positrons, which have damping times of millisecond, did occurred, it turned out that synchrotron radiation will damp one degree of freedom and anti-damp the other… It was required to add an additional radiation device into the lattice (a strong wiggler) to solve this important problem.

It is not all… But already, not too shabby for a single parameterization

$$X(s) = \frac{1}{2}\tilde{\mathbf{U}}(s)\cdot A(s) = \operatorname{Re}\sum_{k=1}^{n} Y_k(s) e^{i(\psi_k(s)+\varphi_k)} a_k(s);$$

$$\frac{d}{ds}\tilde{\mathbf{U}}(s) = \mathbf{D}(s)\cdot\tilde{\mathbf{U}}(s); \tilde{\mathbf{U}} = \left[...Y_k e^{i\psi_k}, Y_k^* e^{-i\psi_k}, ...\right]; k = 1,..,n$$

$$\tilde{\mathbf{U}}^T \mathbf{S} \tilde{\mathbf{U}} = 2i\mathbf{S}.$$