PHY 564
Advanced Accelerator Physics
Lecture 12
Synchrotron oscillations in a storage ring

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Energy Stability in Circular Accelerators: Intuitive approach

$$V_{rf} = V_o \cdot \sin(2\pi \cdot f_{rf} \cdot t); \quad f_{rf} = h / T_o$$

Revolution Time T = Circumference/velocity

Synchronous particle: n is just a turn number

$$2\pi \cdot f_{rf} \cdot t_s(n) = N\pi \quad \rightarrow \sin(2\pi \cdot f_{rf} \cdot t_s(n)) = 0$$

$$C = C(\mathrm{E}); \ \mathrm{v} = c \cdot \sqrt{1 - \left(\frac{mc^2}{E}\right)^2}$$

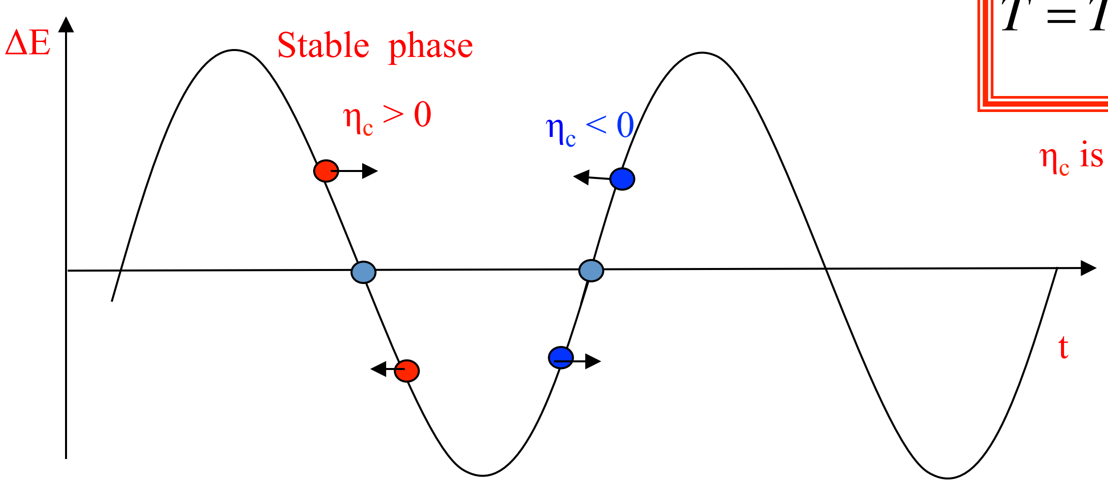
$$T = \frac{C(\mathrm{E})}{c \cdot \sqrt{1 - \left(\frac{mc^2}{E}\right)^2}}$$

$$t(n) = t_s(n) + \tau(n)$$

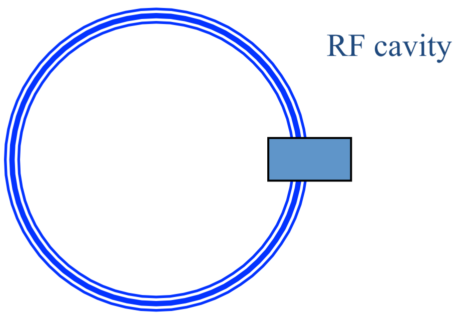
$$\tau(n+1) = \tau(n) + \eta_c T_o \frac{\Delta \mathbf{E}(n)}{\mathbf{E}_o};$$

$$\Delta \mathbf{E}(n+1) = \Delta \mathbf{E}(n) \pm qV_o \cdot \sin\left(f_{rf} \cdot \tau(n+1)\right)$$

$$T = T_o \cdot \left(1 + \eta_c \frac{\Delta \mathbf{E}}{\mathbf{E}_o} + \ldots.\right); \ \Delta \mathbf{E} \equiv \mathbf{E} - \mathbf{E}_o$$



η_c is a function of the accelerator lattice



η_c= 0 is a special case and called a transition

Synchrotron oscillations.

Continue considering momentum of particle to be a constant, we switch to a geometrical variables using p_o for normalization $\pi_\tau = \frac{\delta}{p_o}; \pi_{x,y} = \frac{P_{1,3}}{p_o}$. We finished last class with establishing periodic transverse motion (orbits) for particles with constant energy $\pi_\tau = \frac{\delta}{p_o}$. In addition particles will execute transverse betatron oscillation with respect to this orbit:

$$Z = Z_\beta + \pi_\tau \cdot \eta(s); \eta(s+C) = \eta(s); \eta' = D\eta + C;$$
$$Z'_\beta = DZ'_\beta; Z_\beta = \operatorname{Re} \sum_{k=1}^{2} a_k Y_{k\beta}(s) e^{i(\psi_k(s)+\varphi_k)}; \qquad (12\text{-}01)$$

where are periodic eigen vectors of the transverse oscillations:

$$T_{4x4} Y_{k\beta} = e^{i\mu_k} Y_{k\beta}.$$

In addition, we found that particles with energy deviation are slipping in time as follows:

$$\tau(s) = \pi_\tau\left(\eta_\tau \cdot s + \chi_\tau(s)\right) + \tau_\beta(s); \chi_\tau(s+C) = \chi_\tau(s)$$
$$\eta_\tau \cdot s + \chi_\tau(s) = \left(\frac{mc}{p_o}\right)^2 \cdot s + \int_0^s \left(g_x(\xi)\eta_x(\xi) + g_y(\xi)\eta_y(\xi)\right) d\xi; \qquad (12\text{-}02)$$
$$\eta_\tau = \frac{1}{C}\int_0^C \left(g_x\eta_x + g_y\eta_y\right) ds + \left(\frac{mc}{p_o}\right)^2;$$

with τ_β is the contribution from the betatron motion. To be exact, we just separated two parts of the linear motion using the fact that solution o linear differential equation are additive (linear combination of solution is a solution) and that there no time dependence.

Now, let's find the full set of eigen vectors for 3D motion using $\mathbf{T_{6x6}}$ one turn transport matrix.

Let's start from obvious eigen vector:

$$Y_\tau = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; T_{6x6} Y_\tau = Y_\tau; \lambda_\tau = 1. \tag{12-03}$$

nothing depends on the time shift! A particle following the reference particle with some time delay follow the same trajectory but with the given time delay. Next eigen vector is not a simple vector but a root vector:

$$Y_\delta = \begin{bmatrix} \eta_x \\ \eta_{px} \\ \eta_y \\ \eta_{py} \\ \chi_\tau \\ 1 \end{bmatrix} = \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix}; \quad T_{6x6} Y_\delta = Y_\delta + \eta_\tau Y_\tau; \quad \lambda_\delta = 1. \tag{12-04}$$

Note, this is clearly degenerated case when matrix $\mathbf{T_{6x6}}$ can not be diagonalized and we have to use root vectors, but the symplectic product

$$Y_\tau^T S Y_\delta = 1 \tag{12-05}$$

is well behaving.

What it left is to define the structure of 6-component betatron eigen vectors. Again, since energy is constant, it does not depend on the transverse motion, e.g. the corresponding element is simply zero:

$$Y_k = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + \dfrac{iq_k}{\mathrm{w}_{kx}} \right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + \dfrac{i(1-q_k)}{\mathrm{w}_{ky}} \right) e^{i\chi_{ky}} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix} \tag{12-06}$$

which is generally not true for the time component. While it can be calculated directly and after long manipulations brought to the form we derive easily using symplectic orthogonality of eigen vector of symplectic matrix T:

$$\begin{gathered} Y_i^T \left(T^T S T \right) Y_k = Y_i^T S Y_k \to \lambda_i \lambda_k \cdot Y_i^T S Y_k = Y_i^T S Y_k; \\ \left(\lambda_i \lambda_k - 1 \right) Y_i^T S Y_k = 0. \end{gathered} . \tag{12-07}$$

With root vectors is just a bit different, but still trivial. Note that betatron eigen vectors is a regular are symplectic-orthogonal to Y_τ is a regular eigen vector with eigen value of 1 and, naturally,

$$Y_\tau^T S Y_k = 0; \; k = 1,2. \tag{12-08}$$

You can check directly that this is true using explicit expressions (12-06) and (12-03). Note, that this is also requirement is equivalent to requirement that 6[th] component of betatron eigen vectors Y_k is equal zero. It takes one extra step to prove that for root eigen vector Y_δ:

$$\begin{gathered}
Y_k^{\ T}\left(T^T S T\right) Y_\delta = Y_k^{\ T} S Y_\delta;\ \ T Y_\delta = Y_\delta + \eta_\tau Y_\tau \\
\lambda_k \cdot \left(Y_k^{\ T} S Y_\delta + \eta_\tau Y_k^{\ T} S Y_\tau\right) = Y_k^{\ T} S Y_\delta;\ \ Y_k^{\ T} S Y_\tau = 0;. \\
\left(\lambda_k - 1\right) Y_k^{\ T} S Y_\delta = 0 \rightarrow Y_k^{\ T} S Y_\delta = 0.
\end{gathered} \tag{12-09}$$

This gives us automatically explicit expression for 5[th] component of the betatron eigen vectors:

$$\begin{gathered}
Y_k^{\ T} S Y_\delta = 0 \rightarrow \begin{bmatrix} Y_{k\beta} \\ \mathrm{y}_{k\tau} \\ 0 \end{bmatrix}^T \begin{bmatrix} S_{4x4} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \chi_\tau \\ 1 \end{bmatrix} = Y_{k\beta}^{\ T} S \eta + \mathrm{y}_{k\tau} = 0; \\
\mathrm{y}_{k\tau} = -Y_{k\beta}^{\ T} S \eta = \eta^T S Y_{k\beta} = \qquad . \\
\left(\eta_x\left(\mathrm{v}_{kx} + \frac{i q_k}{\mathrm{w}_{kx}}\right) - \eta_{px}\mathrm{w}_{kx}\right) e^{i\chi_{kx}} + \left(\eta_y\left(\mathrm{v}_{ky} + \frac{i(1-q_k)}{\mathrm{w}_{ky}}\right) - \eta_{py}\mathrm{w}_{ky}\right) e^{i\chi_{ky}}.
\end{gathered} \tag{12-10}$$

This equation makes explicit the dependence of the arrival time on amplitudes and phases of betatron oscillation. In locations where dispersion is zero (called achromatic), this dependence vanishes

$$\eta = 0 \Leftrightarrow \mathrm{y}_{k\tau} = 0. \tag{12-11}$$

Separating betatron oscillations. Note that in accelerator jargon we call fast transverse motions (e.g. not related to energy of particles) "betatron" oscillations. Naming is historical and related to oscillations studied in one of the accelerator types – betatron. Let's formally separate energy dependent motion from transverse "betatron" oscillations using a Canonical transformation:

$$\tilde{H}\left(X_\beta\right) = H\left(\tilde{X} + X_\delta\right) - \frac{\partial F}{\partial s} = H\left(\tilde{X} + X_\delta\right) + \\ + \eta_x' \tilde{\pi}_\tau \left(\tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau\right) - \eta_{px}' \tilde{\pi}_\tau \tilde{x} + \eta_y' \tilde{\pi}_\tau \left(\tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau\right) - \eta_{py}' \tilde{\pi}_\tau \tilde{y} \qquad (12\text{-}12)$$

while we can prove that matrix of such transformation is symplectic, it is also very easy to do using a generation function noticing that $\tilde{\pi}_\tau = \pi_\tau$ is not changing during the transformation

$$F\left(q,\tilde{P}\right) = \left(x - \eta_x \tilde{\pi}_\tau\right)\left(\tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau\right) + \left(y - \eta_y \tilde{\pi}_\tau\right)\left(\tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau\right) \\ + \tau\tilde{\pi}_\tau - \left(\eta_x\eta_{px} + \eta_y\eta_{py}\right)\frac{\tilde{\pi}_\tau^{\ 2}}{2}$$

$$\pi_\tau = \frac{\partial F}{\partial \tau} = \tilde{\pi}_\tau; \ \ \tilde{\tau} = \frac{\partial F}{\partial \tilde{\pi}_\tau} = \tau - \eta_x \tilde{\pi}_x + \eta_{px}\tilde{x} - \eta_y \tilde{\pi}_y + \eta_{py}\tilde{y}; \qquad (12\text{-}13)$$

$$x_\beta = \tilde{x} = \frac{\partial F}{\partial \tilde{\pi}_x} = x - \eta_x \tilde{\pi}_\tau; y_\beta = \tilde{y} = \frac{\partial F}{\partial \tilde{\pi}_y} = y - \eta_y \tilde{\pi}_\tau;$$

$$\pi_x = \frac{\partial F}{\partial x} = \tilde{\pi}_x + \eta_{px}\tilde{\pi}_\tau; \ \ \pi_y = \frac{\partial F}{\partial y} = \tilde{\pi}_y + \eta_{py}\tilde{\pi}_\tau.$$

We should note that transformation (12-13) is linear:

$$\tilde{X} == \mathbf{L} \cdot X = \mathbf{I} \cdot X - \pi_\tau \begin{bmatrix} \eta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} O \\ -\eta_x\left(\pi_x - \eta_{px}\pi_\tau\right) + \eta_{px}\left(x - \eta_x\pi_x\right) - \eta_y\left(\pi_y - \eta_{py}\pi_\tau\right) + \eta_{py}\left(y - \eta_y\pi_\tau\right) \\ 0 \end{bmatrix} \tag{12-14}$$

with matrix L naturally being symplectic – you can directly check that it is correct. We can follow a direct way of finding form of Hamiltonian in the new variables (e.g. transforming coefficients in Hamiltonian (11-21) and adding *s*-derivative of the transfer function), but since the transformation is simply linear we can use a short-cut rewriting (2-14) as

$$\tilde{X} = \begin{bmatrix} \tilde{Z} \\ \tilde{\tau} \\ \tilde{\pi}_\tau \end{bmatrix} = X - \begin{bmatrix} \pi_\tau \eta \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} O \\ \eta^T S \tilde{Z} \\ 0 \end{bmatrix}; X = \begin{bmatrix} Z \\ \tau \\ \pi_\tau \end{bmatrix}; \tilde{Z} = Z - \pi_\tau \eta;$$
$$X = \tilde{X} + \begin{bmatrix} \tilde{\pi}_\tau \eta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} O \\ \eta^T S \tilde{Z} \\ 0 \end{bmatrix}; \tag{12-15}$$

where we used that $\eta^T S \eta = 0$.

Now it is relatively straight-forward to write equations of motion is new variables:

$$\frac{dX}{ds} = \mathbf{D}X; \frac{d\tilde{X}}{ds} = \tilde{\mathbf{D}}\tilde{X}; \mathbf{D} = \begin{bmatrix} D_{4x4} & 0 \quad C \\ .... & .... \end{bmatrix}; \frac{dZ}{ds} = D_{4x4}Z + C\pi_\tau; \eta' = D_{4x4}\eta + C;$$

$$\frac{d\tilde{X}}{ds} = \frac{dX}{ds} - \pi_\tau \begin{bmatrix} \eta' \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} O \\ \frac{d}{ds}\left(\eta^T S\tilde{Z}\right) \\ 0 \end{bmatrix} = \tag{12-16}$$

$$\begin{bmatrix} Z' - \pi_\tau \eta' \\ \frac{d}{ds}\left(\tau - \eta^T S\tilde{Z}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} D_{4x4}\left(Z - \pi_\tau \eta\right) \\ \frac{d}{ds}\tilde{\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} D_{4x4}\tilde{Z} \\ \tilde{\tau}' \\ 0 \end{bmatrix}.$$

It means that the transverse part of the Hamiltonian remain the same since $D_{4x4} = S_{4x4} \cdot H_{4x4}$

$$\frac{d\tilde{Z}}{ds} = D_{4x4}\tilde{Z}; \quad \tilde{D}_{4x4} \equiv D_{4x4}; \tag{12-17}$$

but the C components, as expected, vanish. It means that in new Hamiltonian mixed components between longitudinal $\{\tilde{\tau}, \tilde{\pi}_\tau\}$ and $\tilde{Z}^T = \{x_\beta, \pi_{x\beta}, y_\beta, \pi_{y\beta}\}$ disappear. A non-zero component of type $(a\tilde{\tau} + b\tilde{\pi}_\tau)\tilde{z}_i$ in the new Hamiltonian will generate non-zero additional component in (12-17):

$$\frac{d\tilde{z}_k}{ds} = S_{ki}\frac{\partial \tilde{H}}{\partial \tilde{z}_i} = S_{ki}\left(a\tilde{\tau} + b\tilde{\pi}_\tau\right)$$

which contradict the findings.

Hence, both the Hamiltonian and new D matrix have a block diagonal form:

$$\tilde{\mathbf{H}} = \begin{bmatrix} H_{4x4} & O_{4x2} \\ O_{4x4} & \tilde{H}_{l2x2} \end{bmatrix}; \tilde{\mathbf{D}} = \mathbf{S} \cdot \tilde{\mathbf{H}} = \begin{bmatrix} D_{4x4} & O_{4x2} \\ O_{4x4} & \tilde{D}_{l2x2} \end{bmatrix}; \ \tilde{H}_{4x4} = H_{4x4} \tag{12-18}$$

It is also possible to prove it explicitly by considering in detail the only remaining equation in (12-16) for $\tilde{\tau}'$. This also allows us to find explicitly expression for the longitudinal Hamiltonian $\tilde{H}_l$.

$$\frac{d\tilde{\tau}}{ds} = \frac{d}{ds}\left(\tau - \eta^T S Z\right) = \frac{d}{ds}\left(\tau - \eta^T S \tilde{Z}\right)$$

$$\frac{d\tau}{ds} == \left(g_x x + g_y y + \left(\frac{mc}{p_o}\right)^2 \right)\pi_\tau = \pi_\tau \left(\frac{mc}{p_o}\right)^2 + C^T S\left(\tilde{Z} + \tilde{\pi}_\tau \eta^T\right); \tag{12-19}$$

$$\left(\eta^T S \tilde{Z}\right)' = \left(\eta^T D^T{}_{4x4} + C^T\right) S\tilde{Z} + \eta^T S D^T{}_{4x4} \tilde{Z} = C^T S \tilde{Z}; \ \rightarrow \frac{d\tilde{\tau}}{ds} = \tilde{\pi}_\tau \left(\left(\frac{mc}{p_o}\right)^3 + C^T S \eta^T \right)$$

with $C^T = \begin{bmatrix} 0 & -g_x & 0 & -g_y \end{bmatrix}$ and we used obvious

$$\eta^T D^T{}_{4x4} S\tilde{Z} + \eta^T S D_{4x4} \tilde{Z} = \eta^T H_{4x4} \tilde{Z} - \eta^T H_{4x4} \tilde{Z} = 0 .$$

We can re-write (21019) for $\tilde{\tau}'$ explicitly as

$$\frac{d\tilde{\tau}}{ds} = \tilde{\pi}_\tau \left(\left(\frac{mc}{p_o} \right)^2 + \eta_x g_x + \eta_y g_y \right); \tag{12-20}$$

and the new Hamiltonian as

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta;$$

$$\mathcal{H}_\beta = \frac{\pi_{x\beta}{}^2 + \pi_{y\beta}{}^2}{2} + \frac{F}{p_o c}\frac{x_\beta{}^2}{2} + \frac{N}{p_o c} x_\beta y_\beta + \frac{G}{p_o c}\frac{y_\beta{}^2}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$

$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o} \right)^2 + g_x \eta_x + g_y \eta_y \right) \frac{\pi_\tau{}^2}{2} = c_\tau \frac{\pi_\tau{}^2}{2} \tag{12-21}$$

$$\frac{dX_\beta}{ds} = S \frac{\partial \mathcal{H}_\beta}{\partial X_\beta} = D_\beta X_\beta; \quad \frac{d}{ds}\begin{bmatrix} \tilde{\tau} \\ \pi_\tau \end{bmatrix} = \begin{bmatrix} c_\tau \\ 0 \end{bmatrix} \pi_\tau; \rightarrow \tilde{\tau} = \pi_\tau \int_o^s c_\tau(\xi) d\xi$$

where I dropped tilde for compactness. It is important to remember that in this new variables

$$\tilde{\tau} = \frac{\partial F}{\partial \tilde{\pi}_\tau} = \tau - \eta_x \pi_{x\beta} + \eta_{px} x_\beta - \eta_y \tilde{\pi}_y + \eta_{py} y_\beta;$$

$$\tau = c\left(t_o(s) - t\right). \tag{11-22}$$

It means that arrival time dependence on transverse oscillations is well hidden in

$$t = t_o(s) - \frac{\tau}{c} = t_o(s) - \frac{\tilde{\tau}}{c} - \frac{\eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta}{c} \qquad (11\text{-}25)$$

Since, without time dependent components in the Hamiltonian, the betatron and longitudinal motions are fully decoupled. It also means that in new variables our eigen vectors become:

$$Y_{k\beta} = \begin{bmatrix} Y_{k\beta} \\ 0 \\ 0 \end{bmatrix}; \; Y_{\tilde{\tau}} = \begin{bmatrix} O \\ 1 \\ 0 \end{bmatrix}; Y_{\tilde{\delta}} = \begin{bmatrix} O \\ 0 \\ 1 \end{bmatrix}; \qquad (12\text{-}26)$$

The reason for disappearance of the transverse component in $Y_{\tilde{\delta}}$: it is caused by measuring the transverse orbit from the closed orbit for deviated energy. Similarly, disappearance time component in betatron eigen vectors is caused by its explicit inclusion into the new time variable.

Adding RF fields. Finally we are ready to move to synchrotron oscillations. Let's consider that we adding alternating (AC) longitudinal electric field on the beam axis

$$\frac{d\mathrm{E}}{ds} = -eE_s(s,t) \tag{12-27}$$

For a moment we do not need to pick any specific form of this field, as far it does supports our assumption that the reference particle's trajectory in time, space and momentum exist. Specifically it means that we request that

$$\frac{d\mathrm{E}_o}{ds} = -eE_s\left(s,t_o(s)\right) = 0 \tag{12-28}$$

e.g. that alternating electric field crosses zero at the time of the passing of the reference particle. For a storage ring (a periodic system) the accelerating field has to be is periodic. The AC field (called RF field in the accelerators) has to satisfy the same condition.

$$\begin{gathered} E_s(s+C,t) = E_s(s+C,t);\ E_s\left(s+nC,t_o(s+nC)\right) = 0; \\ t_o(s+nC) = t_o(s) + nT_o;\ T_o = \frac{C}{\mathrm{v}_o}. \end{gathered} \tag{12-29}$$

where T_o is called revolution period in the storage ring.

In practice, the alternating EM fields are generated in resonant cavities and have a sine-wave time dependence:

$$E_s(s,t) = \sum_n \operatorname{Re} E_n(s) e^{i\omega_n t} = \sum_n |E_n(s)| \sin(-\omega_n t + \phi_n(s)); \qquad (12\text{-}30)$$

where we simply numerated various RF frequencies ω_n, which frequently can be just a single frequency. In combination with (12-29) it yields requirement that all RF frequencies have to be harmonic of the revolution frequency:

$$T_o \omega_n = 2\pi h_n; h_n - inger; \;\; \omega_n = h_n \omega_o$$
$$f_{RFn} = \frac{\omega_n}{2\pi} = h_n f_{rev}; \;\; f_{rev} = \frac{\omega_o}{2\pi} = \frac{1}{T_o} = \frac{\mathrm{v}_o}{C}. \qquad (12\text{-}31)$$

with the field on axis of:

$$E_s(s,t) = \sum_n \operatorname{Re} E_n(s) e^{i\omega_n t} = e \sum_n |E_n(s)| \sin\left(h_n \omega_o (t - t_o(s)) - \phi_n(s)\right);$$
$$\sum_n |E_n(s)| \sin(\phi_n(s)) = 0.$$

For a single harmonic RF (12-32) becomes $\phi_n(s) = \pm\pi$ and

$$E_s(s,t) = eE_{rf}(s) \sin\left(h_{rf} \omega_o (t - t_o(s))\right)$$

Note that the sign of E_{rf} depends on what node of the sin-wave we choose. We can add the term corresponding to this longitudinal field using our full accelerator Hamiltonian (which doable but not necessary), or by noticing that

$$\frac{d\pi_\tau}{ds} = \frac{1}{p_o c}\frac{d(\mathrm{E}\text{-}\mathrm{E}_o)}{ds} = \frac{e}{p_o c}\sum_n \left|E_n(s)\right| \sin\left(h_n \omega_o t - \phi_n(s)\right) \qquad (12\text{-}32)$$

corresponds to a term in Hamiltonian of

$$\delta H = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right|\cos\left(h_n k_o\left(\tilde{\tau}+\tau_{add}\right)+\phi_n\right)}{h_n k_o};\; k_o = \frac{\omega_o}{c} = \frac{2\pi}{C}\frac{\mathrm{v}_o}{c};$$
$$\tau_{add} = \eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta; \qquad (12\text{-}33)$$
$$\frac{d\pi_\tau}{ds} = -\frac{\partial(\delta H)}{\partial \tilde{\tau}} = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right|\sin\left(h_n k_o\left(\tilde{\tau}+\tau_{add}\right)+\phi_n\right)}{h_n k_o};$$

Thus, we can write a generic Hamiltonian without expansion in time domain:

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta + \delta\mathcal{H}$$
$$\mathcal{H}_\beta = \frac{mc}{p_o}\cdot\frac{\pi_{x\beta}^2 + \pi_{y\beta}^2}{2} + \frac{F}{p_o}\frac{x_\beta^2}{2} + \frac{N}{mc}x_\beta y_\beta + \frac{G}{p_o}\frac{y_\beta^2}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$
$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o}\right)^2 + g_x \eta_x + g_y \eta_y\right)\frac{\pi_\tau^2}{2} = c_\tau \frac{\pi_\tau^2}{2} \qquad (12\text{-}34)$$
$$\delta\mathcal{H} = \frac{e}{p_o c}\sum_n \frac{\left|E_n\right|\cos\left(h_n k_o\left(\tilde{\tau}+\tau_{add}\right)+\phi_n\right)}{h_n k_o}$$

Linearized part of the additional Hamiltonian term is

$$\delta\mathcal{H}_{\tau} = -\frac{e(\tilde{\tau}+\tau_{add})2}{p_o c}\sum_n h_n k_o |E_n| \cos(\phi_n). \tag{12-35}$$

While looking simpler than original Hamiltonian, adding RF fields made the Hamiltonian fully 3D coupled through τ_{add}. Hence, next step – let's consider case without betatron oscillations $\tilde{\tau}=\tau$:

$$\mathcal{H}_s = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} + \frac{e}{p_o c}\sum_n \frac{|E_n(s)|\cos(h_n k_o \tau + \phi_n)}{h_n k_o} \tag{12-37}$$

or in linear case

$$\mathcal{H}_{sL} = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} - \frac{\tau^2}{2}\frac{e}{p_o c}\sum_n h_n k_o |E_n(s)|\cos(\phi_n) \tag{12-38}$$

Coefficients in both Hamiltonians are s-dependent and the Hamiltonians are not constants of motion. Naturally the (12-38) linear system, when stable, can be solved using 1D parameterization

$$\begin{gathered}\tau = a_s \mathrm{w}_s(s)\cos(\psi_s(s)+\varphi_s);\ \ \psi_s' = \frac{1}{\mathrm{w}_s};\\ \pi_\tau = \left\{a_s \mathrm{w}_s'(s)\cos(\psi_s(s)+\varphi_s) - \frac{1}{\mathrm{w}_s(s)}\sin(\psi_s(s)+\varphi_s)\right\}.\end{gathered} \tag{12-39}$$

This said, in majority of the storage rings, synchrotron oscillations are very slow and it takes from hundreds to tens of thousands turns to complete a single synchrotron oscillations. In this case small variations during one pass around a ring can be averaged. The easiest way is just to average the Hamiltonians (12-37) and (12-38): Beware, this is an approximation which brakes if synchrotron tune is relatively larger (let's say ~ 0.1). Still, it is easy way to get something useful – lets' do it:

$$\left\langle \mathcal{H}_s \right\rangle = \left(\eta_\tau \frac{{\pi_\tau}^2}{2} + \frac{eU_{RF}(\tilde{\tau})}{p_o c} \right); U'_{RF}(0) = 0;$$

$$U_{RF}(\tilde{\tau}) = \frac{1}{C} \cdot \frac{e}{mc} \sum_n \frac{V_n}{h_n k_o} \cos\left(h_n k_o \tilde{\tau} + \bar{\phi}_n\right); \ \eta_\tau = \left(\frac{mc}{p_o}\right)^3 + \left\langle g_x \eta_x + g_y \eta_y \right\rangle; \qquad (12\text{-}40)$$

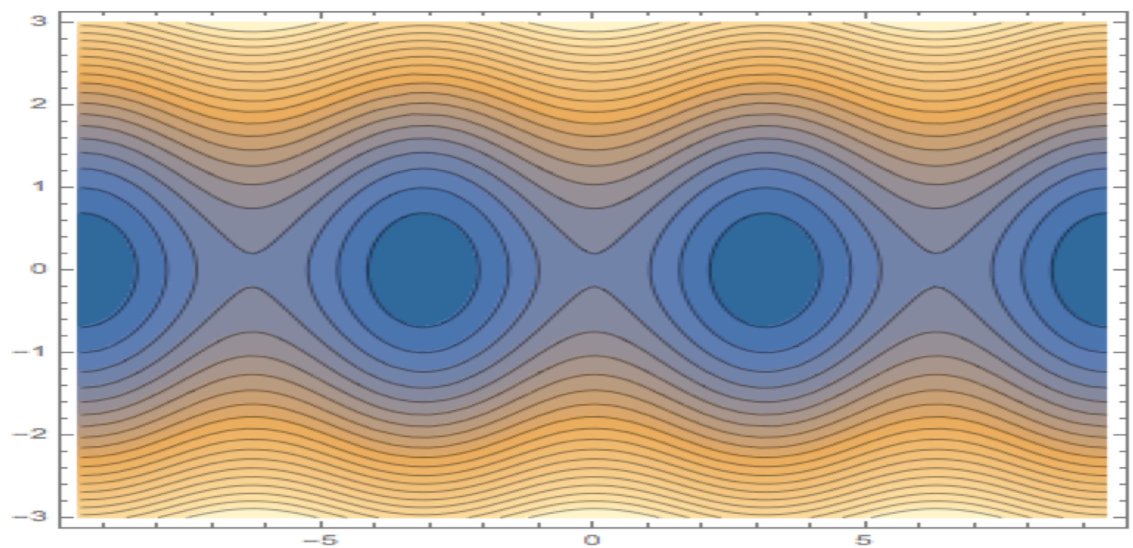
$$V_n \cos\left(\theta + \bar{\phi}_n\right) = \frac{\cos\theta}{h_n k_o} \int_o^C \left|E_n(s)\right| \cos\phi_n(s) - \frac{\sin\theta}{h_n k_o} \int_o^C \left|E_n(s)\right| \sin\phi_n(s).$$

The averaged Hamiltonian does not depend on s and is invariant of motion. Thus we can say that

$$\langle \mathcal{H}_s \rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{eU_{RF}(\tilde{\tau})}{mc} = \mathcal{H}_o; \qquad (12\text{-}41)$$

are equivalent to trajectories in the phase space of τ, π. Let's consider a single frequency RF – a traditional single frequency RF – well know pendulum equation:

$$\langle \mathcal{H}_s \rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos(k_o h_{rf}\tau)}{k_o h_{rf}} = \mathcal{H}_o;$$
$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin(k_o h_{rf}\tau); \qquad (12\text{-}42)$$



Plot of the equipotential for Hamiltonian (12-42) – stable motion occurs around the zero or 180 degrees, depending on the relative sign of $eU_{RF}\eta_\tau$.

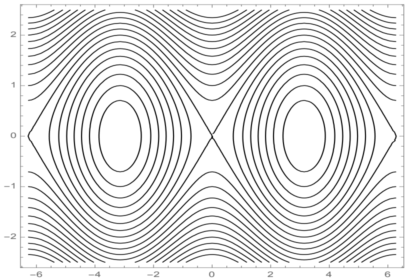
Stationary points are

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau = 0; \rightarrow \pi_\tau = 0; \ \frac{d\pi_\tau}{ds} = 0; \rightarrow \phi_o = k_o h_{rf}\tau = N\pi; \qquad (12\text{-}43)$$

Expanding around the stationary point we

$$\left\langle \mathcal{H}_s \right\rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} - \frac{1}{C}\frac{eV_{RF}}{p_o c} k_o h_{rf} \frac{\cos(\tau)}{k_o h_{rf}} \frac{\tau^2}{2} \cos(\phi_o);$$

$$\Omega_s^{\ 2} = -\eta_\tau k_o h_{rf} \frac{eV_{RF}}{mc} \cos(\phi_o);\ \cos(\phi_o) = \pm 1;\ Ck_o = 2\pi\beta_o$$



$$\Omega_s = \sqrt{\left| \frac{\eta_\tau k_o h_{rf}}{C} \frac{eV_{RF}}{p_o c} \right|} = \sqrt{\left| \left\langle \left(\frac{mc}{p_o} \right)^3 + g_x \eta_x + g_y \eta_y \right\rangle k_o h_{rf} \frac{eV_{RF}}{mc} \right|}; \qquad (12\text{-}44)$$

$$\Omega_s = \sqrt{\left| \eta_\tau k_o h_{rf} \frac{eV_{RF}}{mc} \right|}; \mu_s = \Omega_s C = \sqrt{\left| 2\pi \eta_{\tau o} h_{rf} \frac{eV_{RF}}{E_o} \right|};$$

$$Q_s = \frac{\mu_s}{2\pi} = \sqrt{\left| \frac{\eta_{\tau o} h_{rf}}{2\pi} \frac{eV_{RF}}{E_o} \right|}$$

Stable points are

$$-\eta_\tau eV_{RF} \cos(\phi_o) > 0; \Rightarrow \ \phi_s = 2N\pi; \qquad \eta_\tau eV_{RF} < 0;$$

$$\phi_s = (2N+1)\pi;\ \ \eta_\tau eV_{RF} > 0;$$

As we discussed during last class η_τ determines the sign on the longitudinal mass. When it is negative, not minima but maxima of the potential correspond to stable points.

Decoupling synchrotron oscillations. There is one clear case when synchrotron oscillations can be decoupled, while using exact treatment for linearized transverse motion. It is the case when all component of transverse dispersion are equal zero in the location of RF system, or in other term arrival time of particles is no longer depend on betatron oscillations and only on energy

$$t = t_o(s) - \frac{\tau}{c} = t_o(s) - \frac{\tilde{\tau}}{c}; \left(\frac{\eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta}{c} \right) \to 0! \qquad (12\text{-}45)$$

It is important to note that we shall assume that RF system is installed in a straight section if accelerator to satisfy achromatic condition (12-45). Otherwise, curvature will generate on-zero dispersion function η :

$$\frac{d\eta}{ds} = D \cdot \eta + \begin{bmatrix} 0 \\ -g_x \\ 0 \\ -g_y \end{bmatrix}; \quad g_x = \left(\frac{mc}{p_o} \right)^2 \frac{eE_x}{p_o c} - \frac{mc^2}{p_o \mathrm{v}_o} K; \; g_y = \left(\frac{mc}{p_o} \right)^2 \frac{eE_y}{p_o c} \qquad (12\text{-}46)$$

It is indeed a practical solution, since combining bending field with RF cavity is at least cumbersome, if at all possible. Hence, we stick with a simple straight trajectory through our RF system. Let's consider a single harmonic cavity (or a linac) with electric field on axis being:

$$\mathbf{E}_s(s,t) = \mathbf{E_o}(s) \cos(\omega_0 t + \varphi_o) \qquad (12\text{-}47)$$

which gives the energy change to a particle with charge e:

$$\Delta E = e \int_{-L}^{L} \mathbf{E_o}(s) \cos(\omega_0 t(s) + \varphi_0) ds = e \int_{-L}^{L} \mathbf{E_o}(s) \cos\left(\omega_0 \left(t_o(s) - \frac{\tau}{c} \right) + \varphi_0 \right) ds \qquad (12\text{-}48)$$

Even we assume that reference particle in the storage ring traveling with $t_o(s)$ does not change its energy, or

$$\Delta E_o = e\int_{-L}^{L} \mathbf{E_o}(s)\cos\left(\omega_0 t_o(s)+\varphi_0\right)ds = 0 \qquad (12\text{-}49)$$

analytical solution for an arbitrary s-dependent **E**-field:

$$\mathcal{H}_s = \left(\frac{mc}{p_o}\right)^2 \frac{\pi_\tau^{\ 2}}{2} + \frac{e}{p_o c}\frac{\mathbf{E_o}(s)\cos\left(hk_o\tau+\varphi_o\right)}{hk_o} \qquad (12\text{-}37')$$

or in linear case

$$\mathcal{H}_{sL} = \left(\frac{mc}{p_o}\right)^2 \frac{\pi_\tau^{\ 2}}{2} - hk_o\frac{\tau^2}{2}\frac{e\mathbf{E_o}(s)}{p_o c}\cos\left(hk_o\tau+\varphi_o\right) \qquad (12\text{-}38')$$

does not exist. Hence, we will use a typical assumption, valid for ultra-relativistic particles (when as well as in the case when the relative energy/momentum change in the RF system is negligibly small, e.g. assumption that particle moves with constant velocity. In this approximation, the RF system is described by a single parameter – the energy/momentum change:

$$\Delta\pi_\tau = \frac{\Delta E}{p_o c} = \frac{\omega_0\tau}{p_o c^2}\cdot eV_{rf}\cos\left(\omega_o t+\varphi_o\right);\ V_{RF} = \sqrt{V_s^{\ 2}+V_c^{\ 2}};\tan\left(\varphi_o\right) = \frac{V_c}{V_s};$$

$$V_c = \int_{-\infty}^{\infty}\mathbf{E_o}(s)\cos\left(\omega_0\frac{s}{\mathrm{v}}\right)ds;\ V_s = \int_{-\infty}^{\infty}\mathbf{E_o}(s)\sin\left(\omega_0\frac{s}{\mathrm{v}}\right)ds \qquad (12\text{-}50)$$

Linear expansion of (12-50) about the reference particle gives:

$$\Delta\pi_\tau = \frac{\Delta E}{p_o c} = -sign\,\frac{\omega_0 \tau}{p_o c^2}\cdot eV_{rf};\, sign = sign\,[\sin(\omega_o t + \varphi_o)];\ \cos(\omega_o t_o + \varphi_o) = 0; \qquad (12\text{-}51)$$

corresponding to a short-lens transport matrix:

$$\begin{bmatrix} \tau \\ \pi_\tau \end{bmatrix}_+ = \begin{bmatrix} 1 & 0 \\ -u & 1 \end{bmatrix} = \frac{eV_{rf}\omega_0}{p_o c^2}\cdot sign\,[\sin(\omega_o t_o + \varphi_o)]; \qquad (12\text{-}51)$$

which in combination with the round trip matrix for $X_\tau^T = [\tau, \pi_\tau]$ of

$$T_{\tau acc} = \begin{bmatrix} 1 & \eta_c \\ 0 & 1 \end{bmatrix} \rightarrow$$

$$T_\tau(s|s+C) = \begin{bmatrix} 1 & 0 \\ -u & 1 \end{bmatrix}\begin{bmatrix} 1 & \eta_c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \eta_c \\ -u & 1 - u\eta_c \end{bmatrix}; \qquad (12\text{-}52)$$

$$\cos\mu_s = \frac{Tr[T_\tau]}{2} = 1 - \frac{u\eta_c}{2}$$

with obvious stability criteria

$$0 < u\eta_c < 2 \qquad (12\text{-}53)$$

With eigen mode parameters at the exit of the RD system:

$$\sin\mu_s = \sqrt{1-\cos^2\mu_s} = \sqrt{u\eta_c - \frac{(u\eta_c)^2}{4}}; \beta_\tau = abs\left(\frac{\eta_c}{\sin\mu_s}\right); \alpha_\tau = \frac{1-\cos\mu_s}{\sin\mu_s};$$
$$w_\tau = \sqrt{abs\left(\frac{\eta_c}{\sin\mu_s}\right)};\ w_\tau{}' = -\frac{1-\cos\mu_s}{w_\tau \sin\mu_s}. \tag{12-54}$$

This solution naturally reduces to "oscillatory" solution when RF "focusing" is weak we have the answers identical to "smooth" oscillatory solution (12-44):

$$0 < u\eta_c << 1;\ \mu_s{}^2 \cong u\eta_c;\ \Omega_s = \frac{\mu_s}{C}. \tag{12-55}$$

with

$$\sin\mu_s \approx \mu_s \approx \sqrt{|u\eta_c|} << 1;\ \mu_s \equiv 2\pi Q_s;\ \beta_\tau = \sqrt{\left|\frac{\eta_c}{u}\right|};\ \alpha_\tau = \frac{\sqrt{|u\eta_c|}}{2} << 1;$$
$$w_\tau = \sqrt[4]{\left|\frac{\eta_c}{u}\right|};\ w_\tau{}' = -\frac{\sqrt{|u\eta_c|}}{2w_\tau} << \frac{1}{w_\tau};$$
$$Y_\tau \cong \begin{bmatrix} w_\tau \\ \dfrac{i}{w_\tau} \end{bmatrix}; \psi_\tau = \frac{\sqrt{|u\eta_c|}}{C} s.$$

which are closely describe slow synchrotron oscillations.

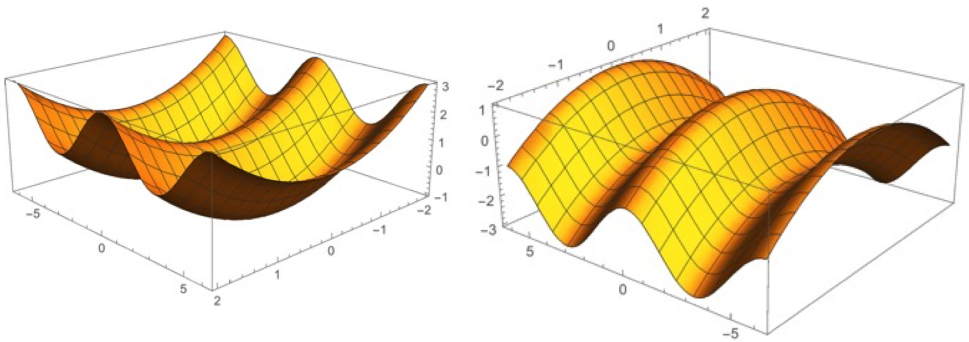
It also means that for a storage ring with RF system located in dispersion free straight section and Q_s<<1 we can use approximate description of the synchrotron oscillation, including the non-linear part of the motion, using s-independent Hamiltonian:

$$\langle \mathcal{H}_s \rangle = \eta_\tau \frac{\pi_\tau^{\ 2}}{2} + \frac{1}{C}\sum_{h_{rf}} \frac{eV_{RF}}{p_o c}\frac{\cos\left(k_o h_{rf}\tau\right)}{k_o h_{rf}};$$
$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin\left(k_o h_{rf}\tau\right); \tag{12-56}$$

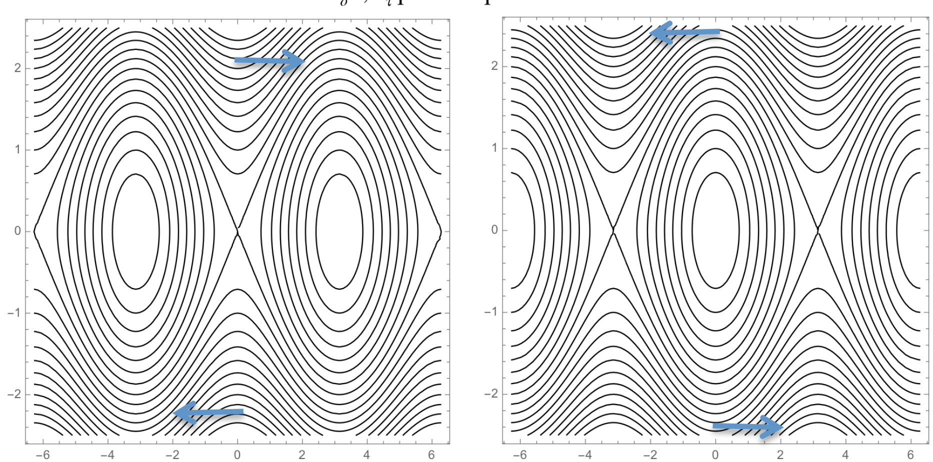
Since the Hamiltonian is constant, the energy level of Hamiltonian represent trajectories in τ,π_τ phase space. For a single harmonic RF, sign of η_τ determines stationary phase of RF (for the reference particle!)

$\eta_\tau > 0$ $\qquad\qquad$ $\eta_\tau < 0$

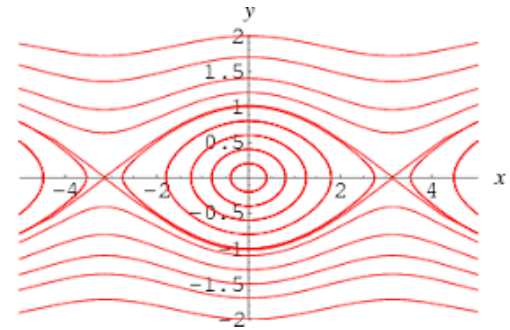
Hamiltonian $\mathbf{H}(\tau,\pi_\tau)$

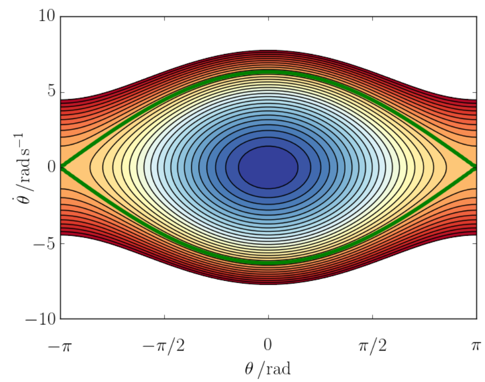


$k_o\tau,\pi_\tau$ phase l=plots



What is important to notice that for "negative" mass case $\eta_\tau < 0$, stable phase points correspond to the maximum of the potential.





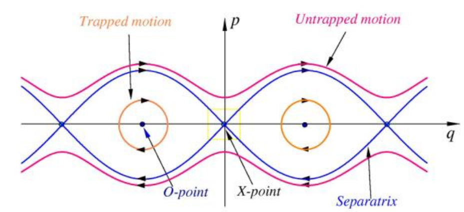
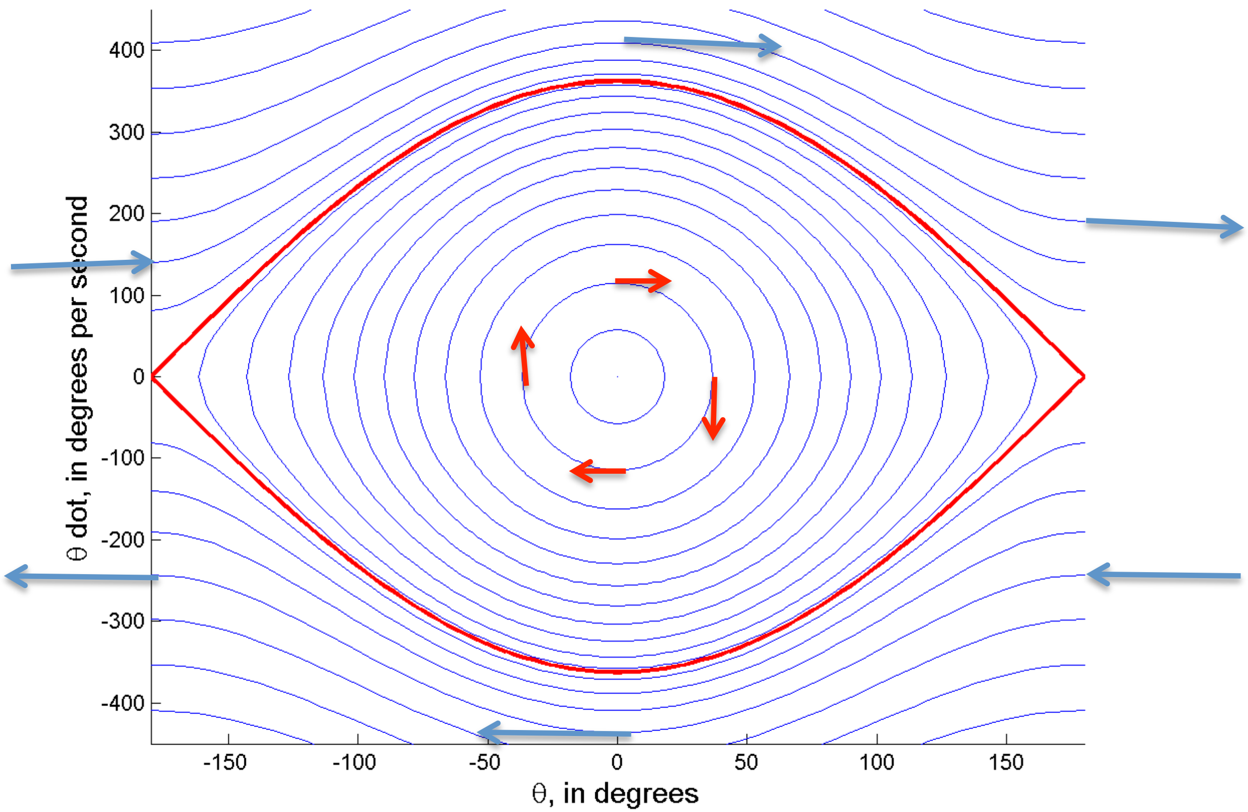


FIG. 4. Phase space (q, p) of the pendulum.

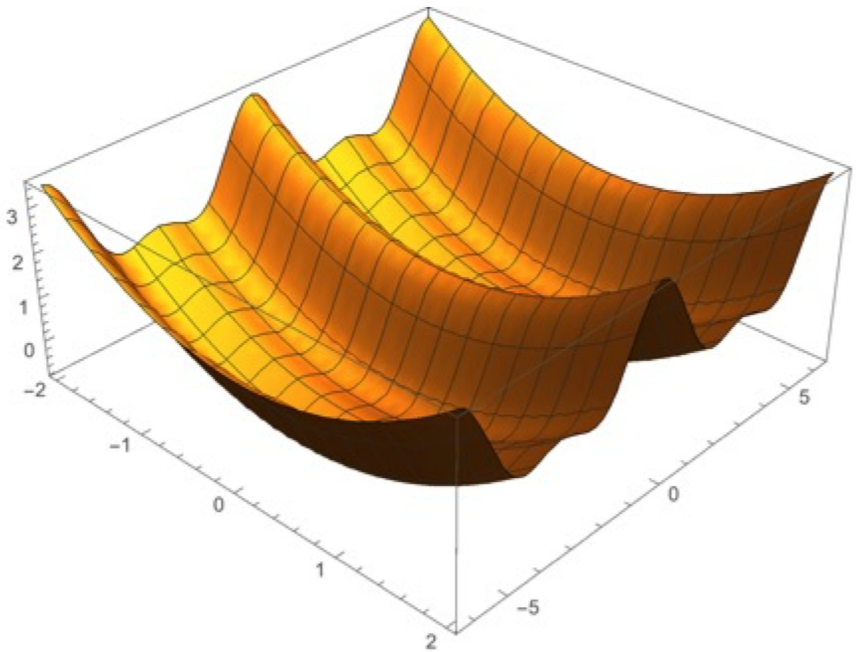
The phase space is separated to zone of confined motion around stable points and unstable motion, when particles slip continuously in their arrival time. The trajectory which separates stable (confined) and unstable (unconfined) motion of particles is called separatrix. It corresponds to the trajectory originating in unstable stationary point (saddle point in the Hamiltonian profile). We can easily find the value of Hamiltonian corresponding to this point

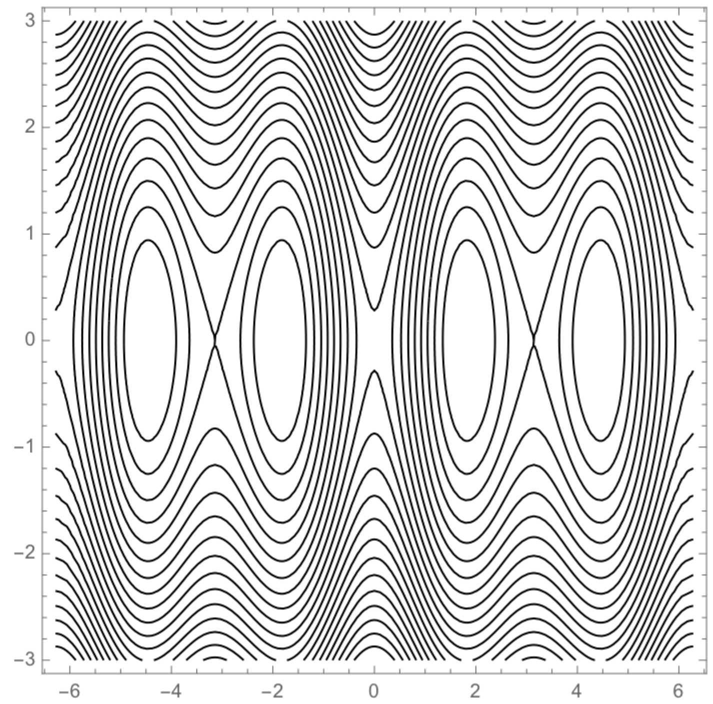
$$\langle \mathcal{H}_s \rangle = \eta_\tau \frac{\pi_\tau^2}{2} + \frac{1}{C}\frac{eV_{RF}}{p_o c}\frac{\cos(k_o h_{rf}\tau)}{k_o h_{rf}} \rightarrow \mathcal{H}_{sep} == sign(\eta_\tau)\left|\frac{1}{C}\frac{eV_{RF}}{p_o c}\right|$$
$$\left(\frac{\pi_\tau^2}{2}\right)_{\max} = \left|\frac{2}{C\eta_\tau k_o h_{rf}}\frac{eV_{RF}}{p_o c}\right| \rightarrow \pi_{\tau sep} = \sqrt{2\left|\frac{2}{C\eta_\tau k_o h_{rf}}\frac{eV_{RF}}{p_o c}\right|} \qquad (12\text{-}57)$$

which indicated energy acceptance of RF separatrix (jargon word frequently used: RF bucket).



Adding harmonics of the RF frequency makes "RF buckets" more interesting: as an example bellow is the phase space when we add to the main RF harmonic its second harmonic RF with twice the amplitude and the same phase. The phase space develops separatrices inside main separatrix.





What we learned today

- In general case, full rigorous analytical description of synchrotron (energy-time) oscillations, which involve time-dependent electric field, possible only in parametric form.
- Time-and-s-dependent electric field is changing particle's energy and makes coefficients in Hamiltonian *s*-dependent. While problem is solvable with help of computers, analytical treatment of propagation through an RF system is possible in approximation of constant particle velocity
- In this case RF system is described by energy change. Linearized matrix is equivalent to that of a short lens approximation for a quadrupole
- We can rigorously separate transverse (betatron oscillations) motion from longitudinal motion when energy of particles is constant using dispersion function. All coupling between transverse and longitudinal motion is reduced to arrival time dependence on betatron oscillations.
- This contribution disappears in location where dispersion (all 4 components) are zero. We can than – *using thin lens RF system matrix* - can calculate one turn matrix for longitudinal oscillations, stability condition and corresponding optics function
- In case of slow synchrotron oscillations, linear oscillation are reduced to harmonic oscillator and one-turn-averaging of exact Hamiltonian provides description identical to pendulum equations
- In this approximation, we can define stable and unstable stationary points as well as separatrix (RF bucker), which divides confined motion from unconfined. We also can define energy of acceptance of "RF bucket "