Homework 7. Due September 30

Problem 1. 7 points. For 1D motion consider a linear map

$$\begin{bmatrix} x' \\ p' \end{bmatrix} = M \begin{bmatrix} x \\ p \end{bmatrix}; M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(a) Find how a circle in {x,p} phase-plane is transformed into {x',p'} phase-plane? What is the area inside this figure?

(b) Find in what shape an unit square e.g. with corners at (0,0), (0,1), (1,0) and (1,1))is transformed? What is the area inside this figure?

Solution:

(a) Circles in x-p plane are

$$x^2 + p^2 = R^2$$

are transformed into x',p' plance using invers transformaton

$$\begin{bmatrix} x \\ p \end{bmatrix} = M^{-1} \begin{bmatrix} x' \\ p' \end{bmatrix}; M^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

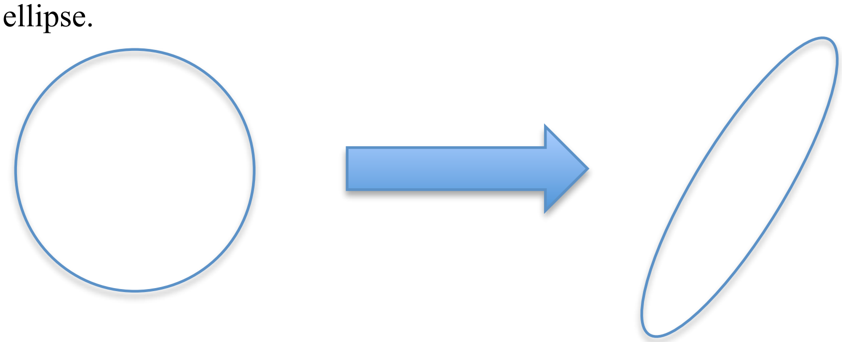
$$x = \frac{dx' - bp'}{ad-bc}; p = \frac{ap' - cx'}{ad-bc};$$

$$x^2 + p^2 = \left(\frac{dx' - bp'}{ad-bc} \right)^2 + \left(\frac{ap' - cx'}{ad-bc} \right)^2 = R^2$$

$$(dx' - bp')^2 + (ap' - cx')^2 = r^2 (ad-bc)^2$$

$$(c^2 + d^2) x'^2 - 2(ac + db) x'p' + (a^2 + b^2) p'^2 = R^2 (ad-bc)^2$$

It is a tilted ellipse.



The angle of tilt is

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2(ac+db)}{c^2 + d^2 - a^2 - b^2} \right)$$

and the area of the ellips is

$$\pi R^2 \cdot (ad-bc)$$

For symplectic map, the area is preserved: $\pi R^2 \to \pi R^2$.

In properly turned coordinate system the ellipse is described by

$$\frac{X^2}{A^2}+\frac{P^2}{B^2}=1$$

with A and B being the main axis. In arbitrary rotates coordinates it is:

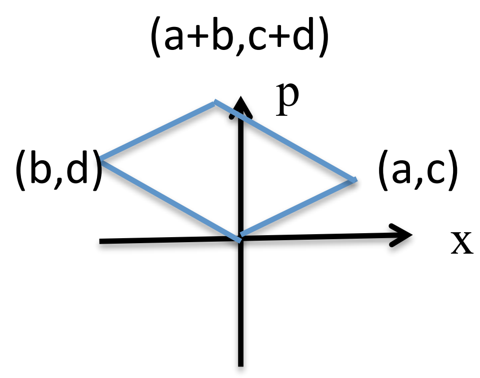
$$\begin{aligned}
&\alpha x^2+\beta xp+\gamma p^2=\delta\\
&\alpha=A^2\sin^2\theta+B^2\cos^2\theta\\
&\beta=\left(B^2-A^2\right)\sin\theta\cos\theta\\
&\gamma=A^2\cos^2\theta+B^2\sin^2\theta\\
&\delta=A^2B^2
\end{aligned}$$

Since the ellipse area is $\pi AB=\pi R^2\left(ad-bc\right)$ - which is easily expected

We also can get the main axes:

$$\begin{aligned}
&A^2B^2=R^4\left(ad-bc\right)^2;A^2+B^2=R^2\left(a^2+b^2+c^2+d^2\right)\\
&A^2\left(R^2\left(a^2+b^2+c^2+d^2\right)-A^2\right)=R^4\left(ad-bc\right)^2\\
&A^4-A^2R^2\left(a^2+b^2+c^2+d^2\right)+R^4\left(ad-bc\right)^2=0\\
&\frac{A^2}{R^2}=\frac{\left(a^2+b^2+c^2+d^2\right)\pm\sqrt{\left(a^2+b^2+c^2+d^2\right)^2-4\left(ad-bc\right)^2}}{2}
\end{aligned}$$

(b) The unit square e.g. with corners at (0,0), (0,1), (1,0) and (1,1)) is transformed in a parralelogram with coordinates (0,0), (b,d), (a,c) and (a+b,c+d))



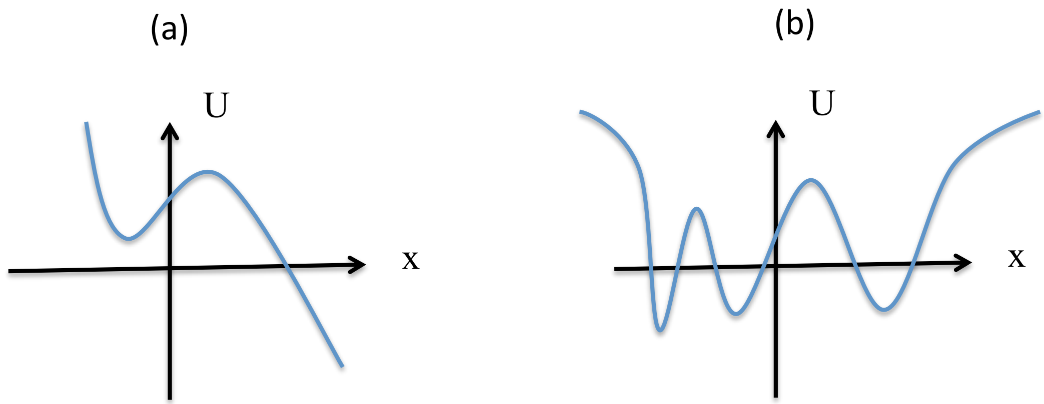
The area is know to be determinate of vectors making the figure: $S=\begin{vmatrix} a & b\\ c & d\end{vmatrix}=ad-bc$

- no surprise here.

Problem 2. 8 points. For 1D motion with a Hamiltonian

$$H = \frac{p^2}{2} + U(x)$$

draw qualitatively correct for two potentials shown in two figures below including direction of motion in each

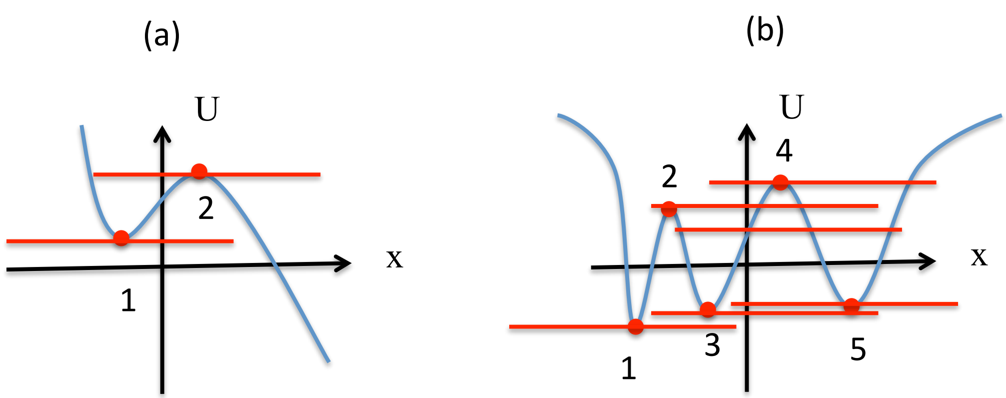


Note: start from separatrixes and then add typical trajectories between and around them.
Solution.
For (a) and (b) is the same drill. We need to find the stationary points:

$$H = p^2/2 + U(x);$$

$$x' = \frac{\partial H}{\partial p} = p = 0; p' = -\frac{\partial H}{\partial x} = -U'(x) = 0;$$

which correspond to $p = 0$ and extrema of the potential $U'(x) = 0$. There are two of them in case (a) and firn in the case (b).



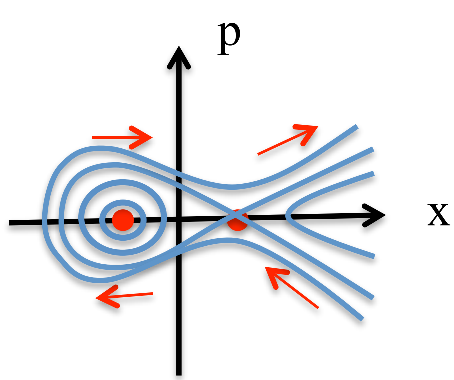
Expending Hamiltonian near the extrema $U'(x) = 0$ we have:

$$H \cong \frac{p^2}{2} + U''(x)\frac{x^2}{2}; x' = \frac{\partial H}{\partial p} = p; p' = -U''(x)x$$

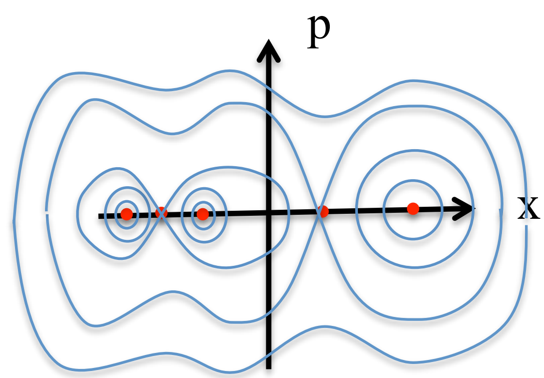
$$U''(x) > 0; x \sim \exp\left(\pm i\sqrt{U''(x)}t\right); U''(x) < 0; x \sim \exp\left(\pm\sqrt{U''(x)}t\right)$$

which means that minima are stable points and maxima are instable. Then we need to draw imaginary energy levels.

(a)



(b)



In case (a) there is one separatrix and number of trajectories can to minus infinity. There is three topologically separated areas. In case (b) there are two separatrices, but all motion is confined into five topologically separated areas.