Periodic Beam Optics

EXERCISE: A basic brick of periodic optical systems, the "FODO CELL"

Consider the following optical sequence, along a straight axis:

a drift of length l/2; a focusing thin-lens with focal distance f; a drift of length l ; a defocusing thin-lens with the same focal distance f; a drift of length l/2.

Take f a positive quantity.

1. Theory

1.1/ Express the 2×2 transfer matrix T of this FODO cell, in terms of l and f.
1.2/ Verify that the determinant of T is 1.
1.3/ At what condition on f and l is this optical system periodically stable?
1.4/ Express the betatron phase advance μ over this cell, in terms of f and l.
1.5/ Express the periodic betatron function $\beta(s)$ at cell ends (s=0 and s=2l), in terms of μ, l and f.

2. Computer work: Simulate the above FODO cell in zgoubi.

Take l= 2m, frozen. The strength of the lens, 1/f, can be varied, we will start with f=l.

2.1/ Compare T from theory and from MATRIX or TWISS.
2.2/ What are the cell phase advance values, horizontal and vertical? The betatron function values at cell ends? Compare with theory.
2.3/ Check the stability limit $f > l/2$.
2.4/ Find the strengths ($K = G/B\rho$) of the F and D lenses to get phase advances of 0.27$\times 2\pi$ and 0.1$\times 2\pi$, respectively horizontal and vertical. FIT can be used for that. Theory can be used, as well.
2.5/ Launch 40 particles on a horizontal periodic invariant (use OBJET[KOBJ=8], with the previously found periodic $\alpha_{x,y},\ \beta_{x,y}$). Transport that bunch through the cell:
- plot initial and final phase spaces, on a common graph (use FAISTORE);
- graphically, verify that the ellipse parameters coincide with the periodic $\alpha_{x,y},\ \beta_{x,y}$;
- plot the trajectories Y(s) through the cell (use DRIFT[split] for fine resolution; use IL=2 to log particle data);
- from this graph data, get an estimate of $\mathrm{Y}_{\min}$ and $\mathrm{Y}_{\max}$. Give their *exact* locations $\mathrm{s}(\mathrm{Y}_{\min})$ and $\mathrm{s}(\mathrm{Y}_{\max})$.
- plot the phase space ellipses, right upstream and right downstream of the F lens. Same for the D lens.

ANSWER

1.1/ $$T = \begin{pmatrix} 1 & l/2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \times \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \times \begin{pmatrix} 1 & l/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{l}{f}(1 + \frac{l}{2f}) & l(2 - \frac{l^2}{4f^2}) \\ -\frac{l}{f^2} & 1 + \frac{l}{f}(1 - \frac{l}{2f}) \end{pmatrix}$$

1.2/ Development of det = ad-bc shows that the matrix T above does feature det(T)=1.

1-3/ Periodic stability requires $|\mathrm{trace(T)}/2| < 1$. This yields $\left|1 - \frac{l^2}{2f^2}\right| < 1$, which happens *iff* $\frac{l^2}{2f^2} < 2$, *i.e.* $\boxed{f > l/2}$.