Home Work PHY 554 #4. **Due October 12, 2016**

HW 1 (3 point): A multi-cell accelerating RF linac operating at 500 MHz in a standing wave π-mode (e.g. each cell has opposite sign of the accelerating voltage from the neighboring cell) is used to accelerate non-relativistic heavy ion (Z=2, A=79) moving with velocity v=c/3 (β=1/3).

(a) find the length of the cell required for resonant acceleration in such a linac – 1 point
(b) at what velocity (ies) (and energy(eis) of the ion), the energy gain in 5-cell cavity would vanish (became zero) – 2 point

Solution: Problem defines that we have a standing wave RF voltage operating in π-mode, e.g. electric field can be described as

$$E_n(z,t) = (-1)^{n-1} \cdot E_o(z) \cdot \cos(\omega t + \varphi) \equiv e^{i\pi(n-1)} \cdot E_o(z) \cdot \cos(\omega t + \varphi)$$

where $E_o(z)$ is the electric field pattern (envelope) of one cell.
(a) For resonant acceleration we need that acceleration is repeated in each and every cell, e.g. generally speaking RF phase advance should be equal odd number of π while particle traverse one cell:

$$\Delta t = \frac{l}{\mathrm{v}}; \Delta\varphi_{RF} = \omega\Delta t = \omega\frac{l}{\mathrm{v}} = (2m+1)\pi;$$
$$t_n = t_1 + (n-1)\frac{l}{\mathrm{v}}; \;\; \omega t_n = \omega t_1 + (n-1)(2m+1)\pi;$$
$$E_1(z,t) = E_o(z) \cdot \cos(\omega t_1 + \varphi);$$
$$E_n(z,t) = (-1)^{n-1} \cdot E_o(z) \cdot \cos(\omega t_1 + (n-1)(2m+1)\pi + \varphi);$$
$$\cos((n-1)(2m+1)\pi + \alpha) = \cos((n-1)\pi + \alpha) = (-1)^{n-1}\cos\alpha;$$
$$E_n(z,t) = (-1)^{2(n-)1} E_o(z) \cdot \cos(\omega t_1 + \varphi) = E_1(z).$$

e.g. as required, a particle moving with constant velocity see the same field in each cell when

$$l = (2m+1)\pi\frac{\mathrm{v}}{\omega} = \left(m + \frac{1}{2}\right)\lambda_{RF}\frac{\mathrm{v}}{c}$$

Naturally, m=0 is preferred case (e.g. particle sample accelerating field while propagating through each cell), which for this problem

$$\frac{\mathrm{v}}{c} = \frac{1}{2} \Rightarrow l = \frac{\lambda_{RF}}{6};$$

For $f=500\ MHz\ \lambda_{RF} = 0.6\ m$ (rounded), resulting in $l=0.1\ m$.
(b) The total energy gain/loss of the particle in a linac comprised on N identical cells with length l is given by:

$$\Delta \mathrm{E} = q\int_0^{N\cdot l} E(z,t)dz = \sum_{n=1}^{N} e^{i\pi(n-1)} \cdot \int_o^l E_o(z)\cdot \cos\omega\left(t_o + \frac{z+(n-1)l}{\mathrm{v}}\right);\ T = \frac{l}{\mathrm{v}}; \tau = \frac{z}{\mathrm{v}}; \phi = \omega T;$$

$$\Delta \mathrm{E} = q\frac{\mathrm{v}}{2}\int_o^T d\tau E_o(\mathrm{v}\tau)\cdot \sum_{n=1}^{N}\left(e^{i\omega(t_o+\tau)} e^{i(n-1)(\phi+\pi)} + e^{-i\omega(t_o+\tau)} e^{-i(n-1)(\phi-\pi)}\right) =$$

$$q\frac{\mathrm{v}}{2}\int_o^T d\tau E_o(\mathrm{v}\tau)\cdot\left(e^{i\omega(t_o+\tau)}\sum_{n=1}^{N} e^{i(n-1)(\phi+\pi)} + e^{-i\omega(t_o+\tau)}\sum_{n=1}^{N} e^{-i(n-1)(\phi-\pi)}\right);$$

$$e^{2\pi i(n-1)} = 1 \Rightarrow e^{i(n-1)(\phi+\pi)} = e^{i(n-1)(\phi+\pi)} e^{-2\pi i(n-1)} = e^{-i(n-1)(\phi-\pi)}; \sum_{n=1}^{N} e^{i(n-1)\alpha} = \frac{1-e^{iN\alpha}}{1-e^{i\alpha}};$$

$$\Delta \mathrm{E} = q\mathrm{v}\,\mathrm{Re}\left\{ e^{i\omega t_o}\left(\frac{1-e^{iN(\phi-\pi)}}{1-e^{i(\phi-\pi)}}\cdot \int_o^T d\tau E_o(\mathrm{v}\tau)\cdot e^{i\omega\tau}\right)\right\};$$

e.g. energy gain/loss is zero independently of the time of the entering the linac (initial RF phase, ωt_o) when one of the multipliers inside the brackets is zero:

$$\frac{1-e^{iN(\phi+\pi)}}{1-e^{i(\phi+\pi)}} = 0;\ or\ /\ and\ \int_o^T d\tau E_o(\mathrm{v}\tau)\cdot e^{i\omega\tau} = 0;$$

While the second condition is possible, we cannot solve it without knowing the field pattern in the cell. Meanwhile the first condition is rather trivial to solve:

$$e^{iN(\phi-\pi)} = 1\ \ when\ e^{i(\phi-\pi)} \neq 1 \Rightarrow N(\phi-\pi) = 2m\pi;\ \phi-\pi \neq 2k\pi;$$

$$\phi = \frac{2m\pi}{N} + \pi; \qquad m \neq k\cdot N; \quad (N,n,k \text{ integers})$$

In our case N=5 and we would have zero gain/loss is the cavity when

$$\phi = \omega\frac{l}{\mathrm{v}} = m\frac{2\pi}{5} + \pi; \quad m \neq 5k \Rightarrow\ \lambda_{RF} = \frac{2\pi c}{\omega};$$

$$\frac{\mathrm{v}}{c} = \frac{5\omega l}{\pi c(2m+5)} = \frac{10l}{\lambda_{RF}(2m+5)} \ \leq 1 \Rightarrow m \geq \frac{5l}{\lambda_{RF}} - \frac{5}{2}$$

Again, in our case we know that $l = \frac{\lambda_{RF}}{6}$ and $m \geq \frac{5}{6} - \frac{5}{2} = -\frac{20}{12} \to m = -1,1,2,3,4,6....$;

It means that there are infinite number of velocities at each particles will not change energy (again, only if we assuming constant velocity!):

m=-1; v/c =0.555555556	m=0; v/c =0.333333333
m=1; v/c =0.238095238	m=2; v/c =0.185185185
m=3; v/c =0.151515152	m=4; v/c =0.128205128
m=6; v/c =0.098039216	….

It worth noting that at m=-2 velocity should exceed speed of the light and the above formulae would require velocity to be *5c/3,* so far unattainable. Second note –because the wave is standing, it is natural the same condition would be correct for particle moving in opposite direction – e.g. for the negative velocity with the same absolute value.
At m=0,5,10… we will have resonant interaction and energy gain/loos from each cell will simply add.

HW 2 (2 points): A n-cell standing wave cavity operates in π-mode with field on the axis described as

$$E_z = E_o(z)\cdot\sin(\kappa z)\cdot\sin(\omega t+\varphi);\ \kappa=\omega/2c;$$

$$E_o(z)=\begin{pmatrix} E_o;\ 0\le z\le \dfrac{n\pi}{\kappa} \\ 0;\ \ z<0 \\ 0;\ \ z>\dfrac{n\pi}{\kappa}\end{pmatrix}$$

Find the energy gain and transit time factor in such a linac for particle moving with the speed of light.

Extra points: what will be modification if v= βc; β ≠1.

Solution: Let's use result from our previous problem but with well defined *Eo(z):*

$$t=t_o+\frac{z}{\mathrm{v}};\sin\alpha\sin\beta=\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2};k=\frac{\omega}{\mathrm{v}};$$

$$\Delta\mathrm{E}\ =qE_o\int_o^{nl}\sin\kappa z\cdot\sin\left(\omega\frac{z}{\mathrm{v}}+\varphi\right)dz=\frac{qE_o}{2}\int_o^{nl}\left(\cos\left((k-\kappa)z+\varphi\right)-\cos\left((k+\kappa)z+\varphi\right)\right)dz=$$

$$\frac{qE_o}{2}\left(\frac{\sin\left((k-\kappa)nl+\varphi\right)-\sin\varphi}{k-\kappa}-\frac{\sin\left((k+\kappa)nl+\varphi\right)-\sin(\varphi)}{k+\kappa}\right).$$

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$$

$$\Delta\mathrm{E}=\frac{qE_o}{2}\left(\frac{1}{k-\kappa}\cos\left(\frac{(k-\kappa)nl}{2}+\varphi\right)\sin\left(\frac{(k-\kappa)nl}{2}\right)-\frac{1}{k+\kappa}\cos\left(\frac{(k+\kappa)nl}{2}+\varphi\right)\sin\left(\frac{(k+\kappa)nl}{2}\right)\right). \quad (1)$$

For particle moving with speed of the light, $k=\dfrac{\omega}{\mathrm{c}}=2\kappa$

$$\Delta\mathrm{E}=\frac{qE_o}{2\kappa}\left(\cos\left(\frac{\kappa nl}{2}+\varphi\right)\sin\left(\frac{\kappa nl}{2}\right)-\frac{1}{3}\cos\left(\frac{3\kappa nl}{2}+\varphi\right)\sin\left(\frac{3\kappa nl}{2}\right)\right);$$

$$\kappa l=n\pi\rightarrow\Delta\mathrm{E}=\frac{qE_o}{2\kappa}\left(\cos\left(\frac{n\pi}{2}+\varphi\right)\sin\left(\frac{n\pi}{2}\right)-\frac{1}{3}\cos\left(\frac{3n\pi}{2}+\varphi\right)\sin\left(\frac{3n\pi}{2}\right)\right)$$

$$=\frac{qE_o}{3\kappa}\cos\left(\frac{n\pi}{2}+\varphi\right)\sin\left(\frac{n\pi}{2}\right)$$

It means that for even *n* we have zero energy change, while for odd *n=2m+1*

$$\Delta\mathrm{E}==\frac{qE_o}{3\kappa}\sin\varphi=l_{cell}\frac{\pi qE_o}{3\pi}\sin\varphi$$

$$\kappa=\frac{\omega}{2\mathrm{c}}=\frac{\pi}{\lambda_{RF}};l_{cell}=\frac{\pi}{\kappa}=\lambda_{RF}$$

it is non-zero amplitude. The gain is independent of the number of even number of previous cells – they just cancel each other. It means that time of flight factor is equal zero for even number of cells and for odd number of cells it is:

$$FF = \frac{\Delta \mathrm{E}_{\max}}{qE_o(2m+1)l_{cell}} = \frac{1}{3\pi(2m+1)}$$

Extra points: when v= βc; β ≠1, the phase advance ϕ is not equal to π and wee need to do a bit more: we need to put $k = \frac{\omega}{\beta \mathrm{c}} = \frac{2\kappa}{\beta}$ into one. Expression becomes more convoluted with additional nodes as wee seen in problem 1.

HW 3 (5 points): A l=0.3 m long 500 MHz pillbox cavity operates in fundamental accelerating TM_{010} mode with peak accelerating electric field of 20 MV/m.

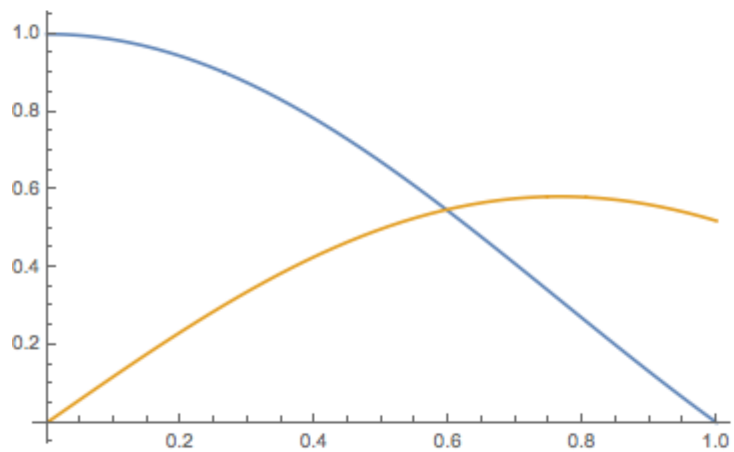
(a) Find the energy stored in electric and magnetic fields as function of time;
(b) What is the total energy of EM field in the cavity? Does it changes with time?
(c) What will be losses of the energy for Q-factor of 30,000?

Solution: We first should write solutions for electric and magnetic fields for TM_{010} mode in pillbox cavity: (I am using here CGS units)

$$\mathbf{E}_z = E_o \cdot J_o\left(2.405\frac{r}{a}\right)\sin(\omega t);$$

$$\mathbf{B}_\theta = E_o \cdot J_1\left(2.405\frac{r}{a}\right)\cos(\omega t);$$

$$W_E = \frac{1}{8\pi}\int \vec{\mathbf{E}}^2\, dV; W_B = \frac{1}{8\pi}\int \vec{\mathbf{B}}^2\, dV;\; W = \frac{1}{8\pi}\int \left(\vec{\mathbf{E}}^2 + \vec{\mathbf{B}}^2\right) dV$$



r/a

and a is determined from the RF frequency

$$\omega = 2\pi f = \frac{2.405c}{a} \rightarrow a = \frac{2.405c}{2\pi f}$$

finding that a=0.2295 m. Than energy stored in electric field is:

$$W_E = \frac{1}{8\pi}\int \vec{\mathbf{E}}^2\, dV; dV = dz d\theta r dr; \int E_o^{\,2} J_o\left(2.405\frac{r}{a}\right)^2 dV = 2\pi l \int_o^a J_o\left(2.405\frac{r}{a}\right)^2 r\, dr$$

$$W_E = \frac{E_o^{\,2}}{4} a^2 l \cdot \sin^2 \omega t \int_o^1 J_o(2.405x)^2\, x dx;\; W_B = \frac{E_o^{\,2}}{4} a^2 l \cdot \cos^2 \omega t \int_o^1 J_1(2.405x)^2\, x dx$$

It is feature of Bessel functions that two integrals taken to the root of Jo are equal. Indeed,

$$z_o = 2.404825557695773..; J_o(z_o) = 0;$$

$$I = \int_o^{z_o} J_o(x)^2 \, r\,dr = \int_o^{z_o} J_1(x)^2 \, r\,dr = 0.779325...$$

$$I_1 = \int_o^1 J_o(2.405x)^2 \, xdx = \frac{I}{2.405^2} = 0.134757..;$$

$$W_E = \ W \cdot \sin^2 \omega t; \ W_B = \ W \cdot \cos^2 \omega t; \ W = \frac{E_o^{\ 2}}{4} a^2 l \cdot I_1.$$

As we discussed in class this is actually property of a cavity with ideally conducting walls fields that peak energy stored in magnetic field is equal that in electric field and, naturally – because of the energy conservation, total energy stored in EM field is constant. You can used SI formulae but I am just transforming 20 MV/m field into Gauss: *E_o= 666.7 Gs* and use the above formula to find stored energy being *W=2.37E8 erg*, or *W=23.7 J*.
By definition (lecture 11, slide 7) the losses are connected to the Q-factor s

$$P_{loss} = \frac{\omega W}{Q_o} \rightarrow P_{loss} = 2\pi \frac{5 \cdot 10^8 \, Hz \cdot 23.7 J}{3 \cdot 10^4} = 2.48 \cdot 10^6 \ \ W$$

and such tiny cavity will endure about 2.5 MW losses in the wall. It makes it both "power hungry" and in practice, impossible to cool. Hence, Cu cavities usually operate at gradients ~ few MV/m with power dissipation measured in tens to few hundreds kW.