

Collective Effects II: Examples of Collective Instabilities

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Outline

- Transverse beam breakup instability (BBU) in linear accelerator
 - Two particle model
 - BNS damping
- Longitudinal Robinson Instability ($m=0$)
 - Macro particle model
 - Resonator model for cavity impedance
 - Stability condition and growth rate
- Longitudinal microwave instability (optional)
 - Dispersion relation
 - Cold beam
 - Warm beam (Keil-Schnell criteria for stability)

Single pass BBU (Two particle model)

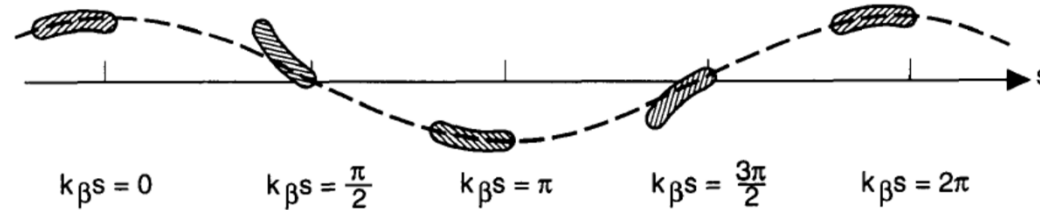
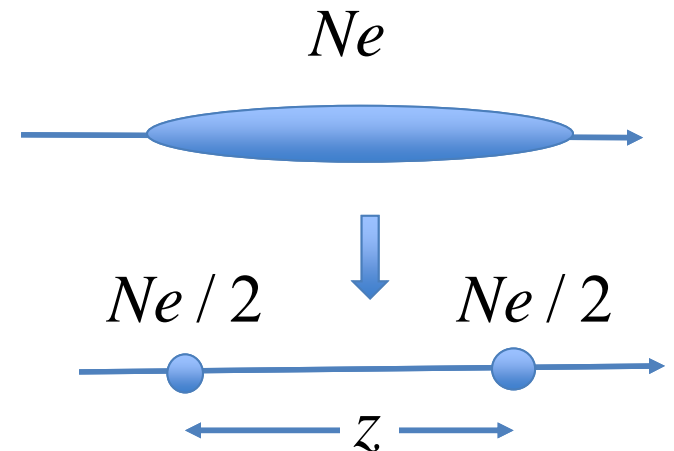


Figure 3.3. Sequence of snapshots of a beam undergoing dipole beam breakup instability in a linac. Values of $k_\beta s$ indicated are modulo 2π . The dashed curves indicate the trajectory of the bunch head.

Leading particles $y_1(s) = \hat{y} \cos(k_\beta s)$

Trailing particles $y_2(s)'' + k_\beta^2 y_2(s) = \frac{Ne^2 W_1(z)}{2EL} y_1(s)$

$$r_0 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mc^2} = 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L} \hat{y} \cos(k_\beta s)$$



*Note: our definition of the transverse wake function follow G. Stupakov's note and has a sign difference from that defined in A. Chao's book, i.e. W_1 here is $-W_1$ in Chao's book.

Driving term for particle 2

$$\vec{w}_i(x, y, s) = \frac{c}{qe} \Delta \vec{p}_\perp \quad q = \frac{Ne}{2}$$

$$m = 1 \quad \vec{w}_i(r', r, \theta, s) = W_1(s) r' [\cos(\theta) \hat{r} - \sin(\theta) \hat{\theta}]$$

$$\Delta p_y = \frac{W_1(s) y_1 e Ne}{c \cdot 2} \leftarrow \text{Transverse momentum change of particle 2 due to wakefield while it goes through the structure}$$

$$\Delta y' = \frac{\Delta p_y}{p_z} \approx \frac{c \Delta p_y}{E} = \frac{W_1(s) y_1 Ne^2}{2E} \leftarrow \text{Transverse angle change of particle 2 due to wakefield}$$

$$\frac{\Delta y'}{L} = \frac{W_1(s) y_1 Ne^2}{2EL} \leftarrow \text{Transverse angle changing rate of particle 2 due to wakefield, driving term}$$

Single pass BBU (Two particle model)

For a linear inhomogenous 2nd order differential equation

$$\frac{d^2 x}{dt^2} + a(t) \frac{dx}{dt} + b(t) x = f(t)$$

its solution is given by

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix}$$

$$x(t) = c_1 \phi_1(t) + c_2 \phi_2(t) + \int_{t_0}^t \frac{\phi_1(\xi) \phi_2(t) - \phi_2(\xi) \phi_1(t)}{W(\xi)} f(\xi) d\xi$$

$$y_{2,inh}(s) = 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L k_\beta} \hat{y} \int_0^s \sin(k_\beta s - k_\beta \xi) \cos(k_\beta \xi) d\xi$$

$$= 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L k_\beta} \hat{y} \frac{1}{2} \left[s \sin(k_\beta s) - \int_{-s/2}^{s/2} \sin(2k_\beta \tilde{\xi}) d\tilde{\xi} \right]$$

$$= 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{4\gamma L k_\beta} \hat{y} s \sin(k_\beta s)$$

$$\phi_2 = \sin(k_\beta s) \quad \phi_1 = \cos(k_\beta s)$$

$$W(t) = \begin{vmatrix} \cos(k_\beta s) & \sin(k_\beta s) \\ -k_\beta \sin(k_\beta s) & k_\beta \cos(k_\beta s) \end{vmatrix} = k_\beta$$

$$y_2(s)'' + k_\beta^2 y_2(s) = 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L} \hat{y} \cos(k_\beta s)$$

Single pass BBU (Two particle model)

$$y_2(s) = c_1 \cos(k_\beta s) + c_2 \sin(k_\beta s) + 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{4\gamma L k_\beta} \hat{y} s \sin(k_\beta s)$$

Noticing that before going through the structure, particle 2 has the same trajectory as that of Particle 1, i.e.

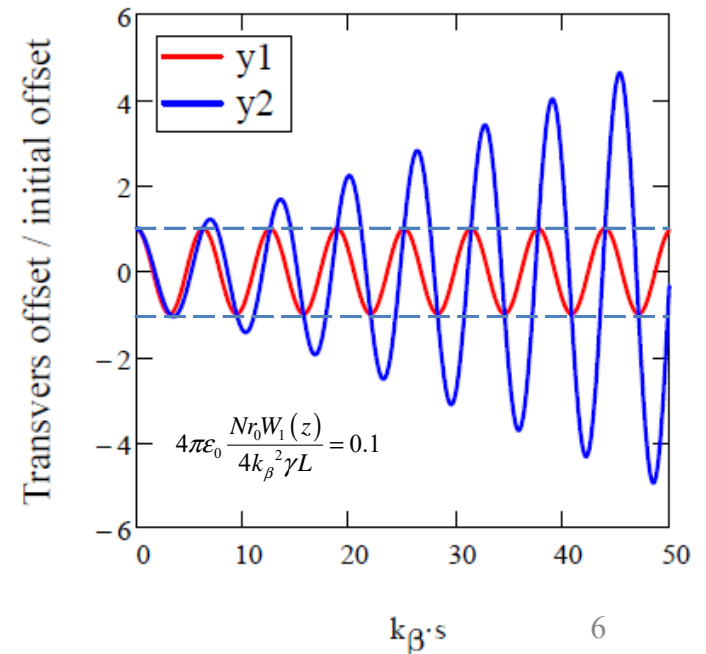
$$y_2(0) = y_1(0) = \hat{y} \cos(0) = \hat{y}$$

$$y_2'(0) = y_1'(0) = -\hat{y} k_\beta \sin(0) = 0$$

We obtain $c_1 = \hat{y}$ and $c_2 = 0$. Thus the solution for particle 2 is

$$y_2(s) = \hat{y} \left[\cos(k_\beta s) + 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{4k_\beta \gamma L} s \sin(k_\beta s) \right]$$

$$y_1(s) = \hat{y} \cos(k_\beta s)$$



Single pass BBU II

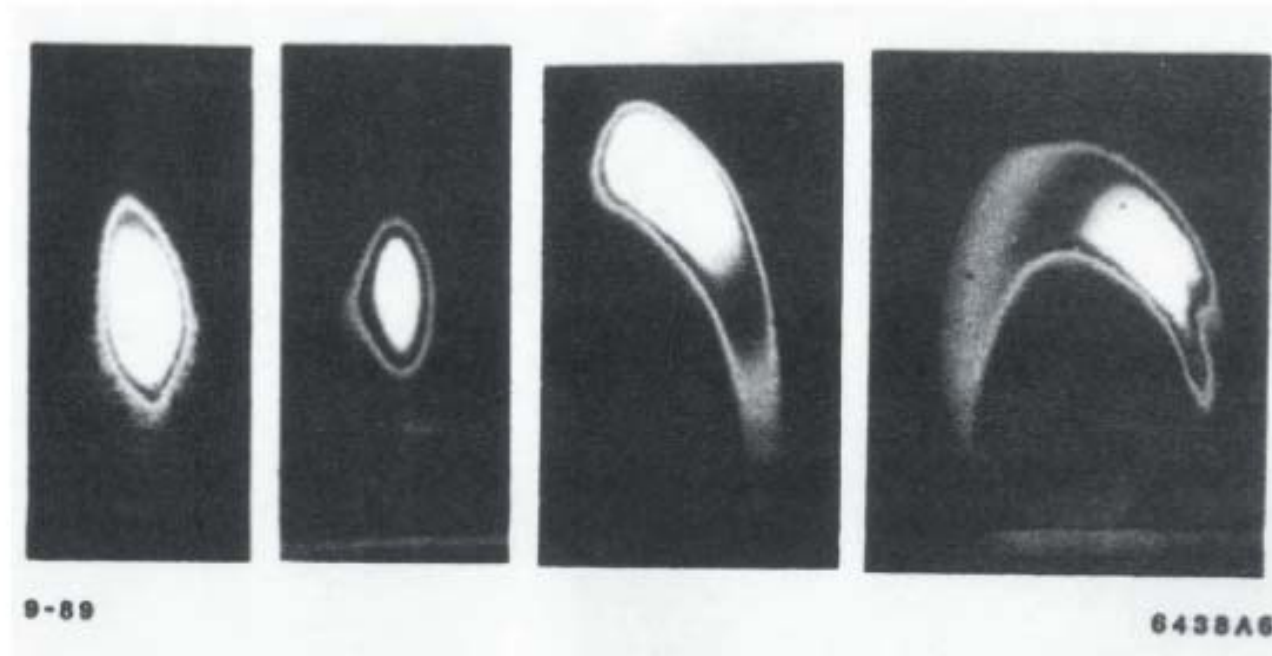


Figure 4.4: Four transverse beam profiles observed at the end of the SLAC linac are shown when the beam was carefully injected, and injected with 0.2, 0.5, and 1 mm offsets. The beam sizes σ_x and σ_y are about $120 \mu\text{m}$. (Courtesy John Seeman, 1991)

One possible cure: BNS damping

Introduce focusing variation along the bunch, i.e. head and tail have different focusing strength

$$y_2'' + (k_\beta + \Delta k_\beta)^2 y_2 = 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L} \hat{y} \cos(k_\beta s)$$

$$y_{2,inh}(s) = 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \int_0^s \sin(\tilde{k}_\beta s - \tilde{k}_\beta \xi) \cos(k_\beta \xi) d\xi$$

$$= -4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \frac{1}{2} \left[\int_0^s \sin\left(\Delta k_\beta \left(\xi - \frac{\tilde{k}_\beta}{\Delta k_\beta} s\right)\right) d\xi + \int_0^s \sin\left((\tilde{k}_\beta + k_\beta) \left(\xi - \frac{\tilde{k}_\beta}{\tilde{k}_\beta + k_\beta} s\right)\right) d\xi \right]$$

$$= 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \frac{1}{2} \left[\frac{1}{\Delta k_\beta} + \frac{1}{\tilde{k}_\beta + k_\beta} \right] [\cos(k_\beta s) - \cos(\tilde{k}_\beta s)]$$

$$\approx -4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} \hat{y} [\cos(\tilde{k}_\beta s) - \cos(k_\beta s)] \quad \leftarrow \text{assume } \Delta k_\beta / k_\beta \ll 1$$

$$\tilde{k}_\beta \equiv k_\beta + \Delta k_\beta$$

$$\phi_2 = \sin(\tilde{k}_\beta s) \quad \phi_1 = \cos(\tilde{k}_\beta s)$$

$$W(t) = \tilde{k}_\beta$$

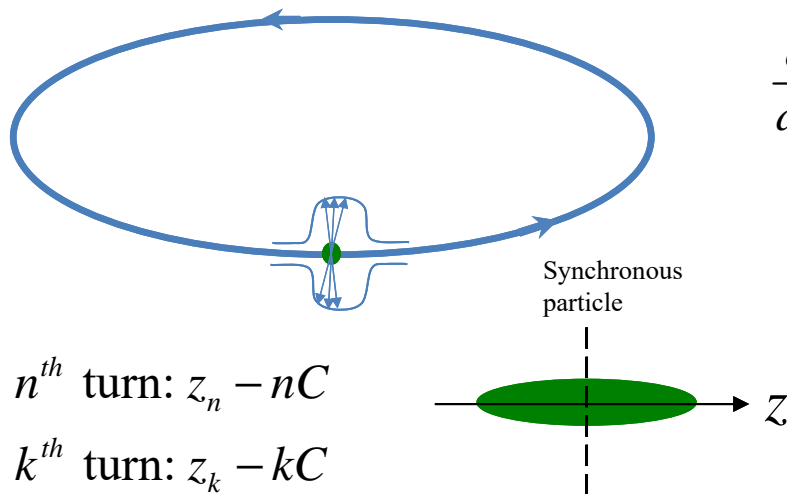
$$y_2(s) = \hat{y} \cos((k_\beta + \Delta k_\beta)s) - 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} \hat{y} [\cos(\tilde{k}_\beta s) - \cos(k_\beta s)]$$

Condition for complete compensation:

$$4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} = 1 \Rightarrow \Delta k_\beta = 4\pi\epsilon_0 \frac{Nr_0 W_1(z)}{4\gamma L k_\beta} \Rightarrow y_2(s) = \hat{y} \cos(k_\beta s)$$

Robinson Instability in Circular Machine

- As a charged particle bunch traveling through a cavity, it excites E&M fields, i.e. wakefields. To the leading order in the longitudinal direction, i.e. $m=0$, particles in the bunch lose some of their energies to the cavity, which is on top of the energy they gain from the acceleration (superposition).



$$\frac{dz_n}{dn} = -\eta C \delta_n$$

Phase slip factor: $\eta \equiv \frac{1}{\gamma_r^2} - \frac{1}{\gamma^2}$

$$\frac{d}{dn} \delta_n = \frac{(2\pi\nu_s)^2}{\eta C} z_n + \frac{eV(z_n)}{E}$$

$$= \frac{(2\pi\nu_s)^2}{\eta C} z_n - \frac{4\pi\epsilon_0 N r_0}{\gamma} \sum_{k=-\infty}^n W_0'(z_n - nC - z_k + kC)$$

z is in the range of bunch length, i.e. $z \ll C$

$$r_0 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{mc^2}$$

Robinson Instability II

$$\begin{aligned} \frac{d^2 z_n}{dn^2} + (2\pi\nu_s)^2 z_n &= \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma} \sum_{k=-\infty}^n W_0'(kC - nC + z_n - z_k) \\ &\approx \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma} \sum_{k=-\infty}^n W_0'(kC - nC) \\ &\quad + \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma} \sum_{k=-\infty}^n (z_n - z_k) W_0''(kC - nC) \end{aligned}$$

The first term in the RHS can be removed by defining

$$\tilde{z}_n \equiv z_n + \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma(2\pi\nu_s)^2} \sum_{k=-\infty}^n W_0'(kC - nC)$$

Please notice that $\sum_{k=-\infty}^n W_0'(kC - nC) = \sum_{\tilde{k}=-\infty}^0 W_0'(\tilde{k}C)$,
i.e. independent of n $\tilde{k} \equiv k - n$

$$\frac{d^2 \tilde{z}_n}{dn^2} + (2\pi\nu_s)^2 \tilde{z}_n \approx \frac{4\pi\varepsilon_0\eta CNr_0}{\gamma} \sum_{k=-\infty}^n (\tilde{z}_n - \tilde{z}_k) W_0''(kC - nC)$$

Robinson Instability III

Ansatz (test solution): $\tilde{z}_n = A \exp(-in\Omega T_0)$

$$\frac{d^2 \tilde{z}_n}{dn^2} + (2\pi\nu_s)^2 \tilde{z}_n \approx \frac{4\pi\epsilon_0 \eta CNr_0}{\gamma} \sum_{k=-\infty}^n (\tilde{z}_n - \tilde{z}_k) W_0''(kC - nC)$$

$$(-i\Omega T_0)^2 + (2\pi\nu_s)^2 = \frac{4\pi\epsilon_0 \eta CNr_0}{\gamma} \sum_{k=-\infty}^n (1 - \exp(-i(k-n)\Omega T_0)) W_0''(kC - nC)$$

$$\stackrel{\tilde{k} \equiv k-n}{=} \frac{4\pi\epsilon_0 \eta CNr_0}{\gamma} \sum_{\tilde{k}=-\infty}^0 (1 - \exp(-i\tilde{k}\Omega T_0)) W_0''(\tilde{k}C); \quad \begin{array}{l} \text{Causality:} \\ W'(\Delta z > 0) = 0 \\ z = ct - s \end{array}$$

$$= \frac{4\pi\epsilon_0 \eta CNr_0}{\gamma} \sum_{k=-\infty}^{\infty} (1 - \exp(-ik\Omega T_0)) W_0''(kC)$$

$$\Omega^2 - \omega_s^2 = -\frac{4\pi\epsilon_0 \eta Nr_0 c}{\gamma T_0} \sum_{k=-\infty}^{\infty} (1 - \exp(-ik\Omega T_0)) W_0''(kC); \quad \omega_s = \frac{2\pi\nu_s}{T_0}$$

Robinson Instability IV

We will use the following identity (the Poisson Sum Formula)

$$\sum_{l=-\infty}^{\infty} F(lC) = \frac{1}{C} \sum_{p=-\infty}^{\infty} \tilde{F}\left(\frac{2\pi p}{C}\right) \quad F(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikz} \tilde{F}(k) dk \quad \tilde{F}(k) = \int_{-\infty}^{\infty} e^{-ikz} F(z) dz$$

$$\begin{aligned} \Omega - \omega_s &\approx -\frac{4\pi\epsilon_0\eta Nr_0 c}{\gamma T_0 2\omega_s} \sum_{k=-\infty}^{\infty} (1 - \exp(-ik\Omega T_0)) W_0''(kC) \\ &\approx -i \frac{4\pi\epsilon_0\eta Nr_0}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} \{p\omega_0 Z_{0, //}(p\omega_0) - (p\omega_0 + \omega_s) Z_{0, //}(p\omega_0 + \omega_s)\} \end{aligned}$$

↓ $\text{Re}[Z_{0, //}(p\omega_0)]$ is an even function of p .

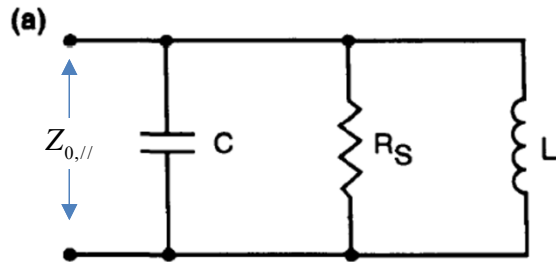
$$\tau^{-1} = \text{Im}(\Omega) = \frac{4\pi\epsilon_0\eta Nr_0}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} (p\omega_0 + \omega_s) \text{Re}\{Z_{0, //}(p\omega_0 + \omega_s)\}$$

Impedance Model for Fundamental mode of a Cavity

$$\frac{1}{Z_{0,//}} = \frac{1}{R_s} + \frac{i}{\omega L} - i\omega C$$

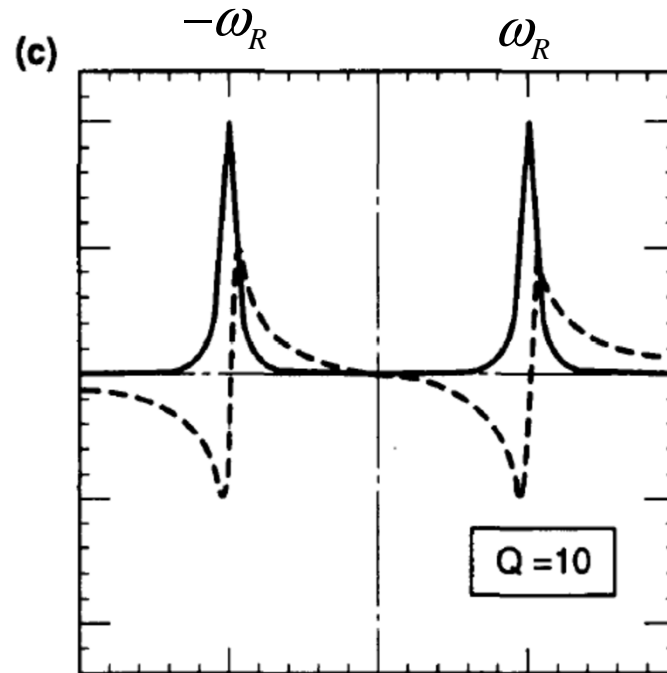
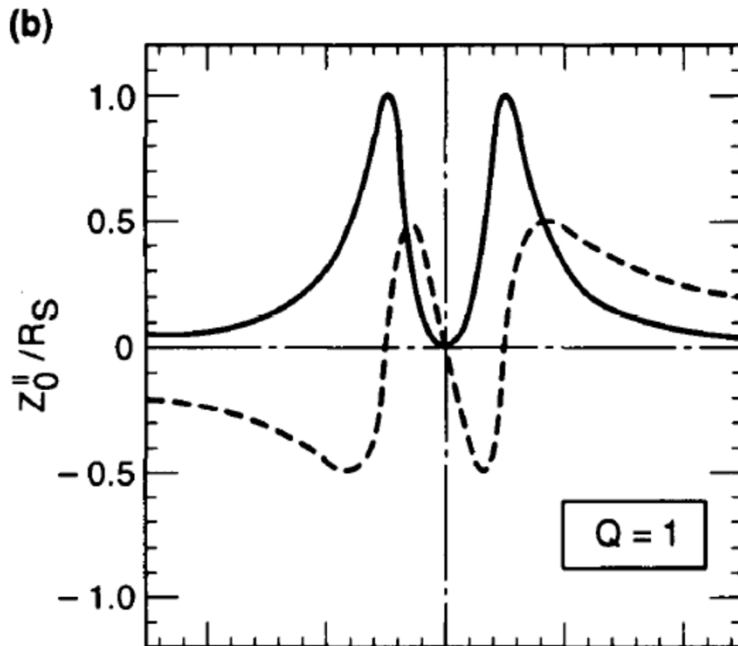


$$Z_{0,//} = \frac{R_s}{1 + iQ \left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R} \right)}$$



For RF cavity, $\omega_R / Q \ll \omega_0$, and hence only the two term in the summation over p matters, i.e.

$$p = \pm h \approx \pm \frac{\omega_R}{\omega_0}$$



Robinson Instability V

$$\tau^{-1} = \frac{4\pi\epsilon_0\eta Nr_0}{2\gamma T_0^2 \omega_s} \left\{ (h\omega_0 + \omega_s) \operatorname{Re}[Z_{0, //}(h\omega_0 + \omega_s)] + (-h\omega_0 + \omega_s) \operatorname{Re}[Z_{0, //}(h\omega_0 - \omega_s)] \right\}$$

$$\approx \frac{4\pi\epsilon_0\eta Nr_0 h\omega_0}{2\gamma T_0^2 \omega_s} \left\{ \operatorname{Re}[Z_{0, //}(h\omega_0 + \omega_s)] - \operatorname{Re}[Z_{0, //}(h\omega_0 - \omega_s)] \right\} \quad \leftarrow \quad Z_{0, //} = \frac{R_s}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}$$

For ω_s and $\Delta\omega \ll \frac{\omega_R}{2Q} \ll \omega_0$, the growth rate is

$$\tau^{-1} \approx 4\pi\epsilon_0 \frac{4Nr_0\eta R_s Q^2}{\pi\gamma T_0 h} \Delta\omega$$

$$\Delta\omega \equiv \omega_R - h\omega_0$$

$$\operatorname{Re}[Z_{0, //}(h\omega_0 + \omega_s)] - \operatorname{Re}[Z_{0, //}(h\omega_0 - \omega_s)]$$

$$= 16 \frac{Q^2 R_s}{h^2 \omega_0^2} \omega_s \Delta\omega$$

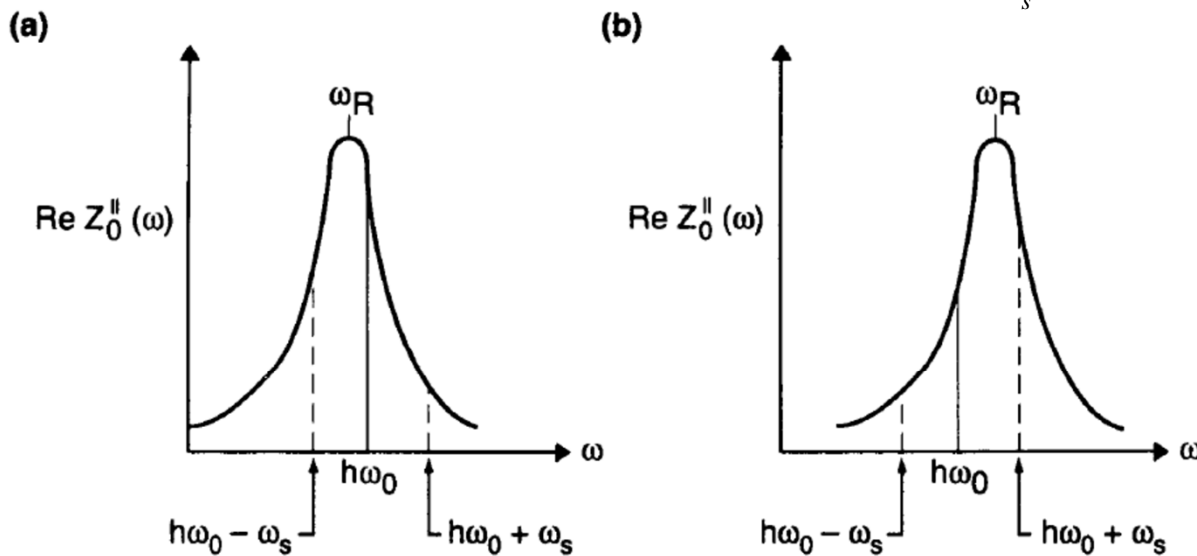


Figure 4.4. Illustration of the Robinson stability criterion. The rf fundamental mode is detuned so that ω_R is (a) slightly below $h\omega_0$ and (b) slightly above $h\omega_0$. (a) is Robinson damped above transition and antidamped below transition. (b) is antidamped above transition and damped below transition.

Longitudinal Microwave Instability

Unperturbed phase space density:

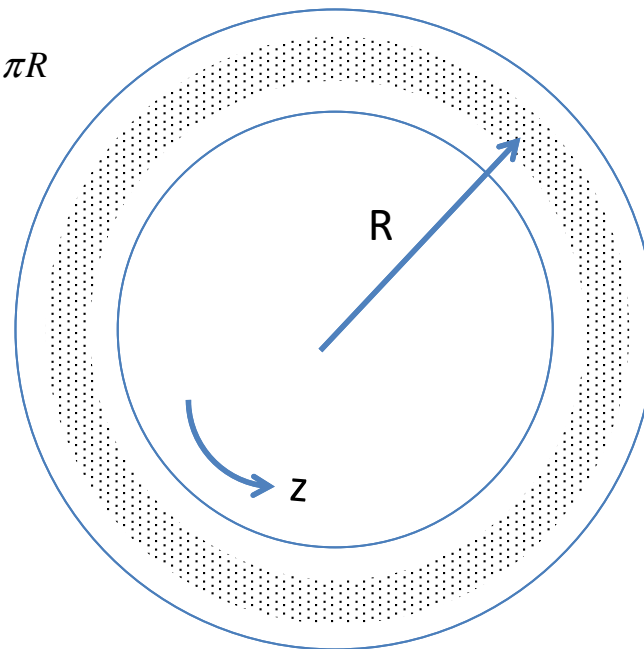
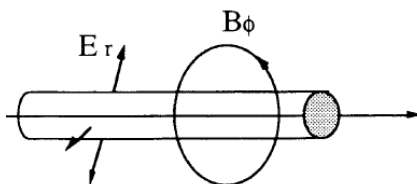
$$\psi_0(z, \Delta E) = \psi_0(\Delta E) = \frac{N}{C_0} f_0(\Delta E) \quad \rho_0(z) = \rho_0 = \frac{N}{C_0}$$

$$C_0 = 2\pi R$$

DC current does not excite wake

$$V_{//}(z_0) = \int_{z_0}^{\infty} \lambda(z_1) w_{//}(z_1 - z_0) dz_1$$

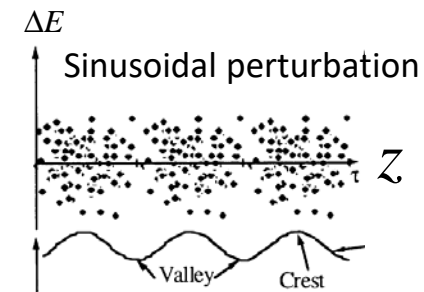
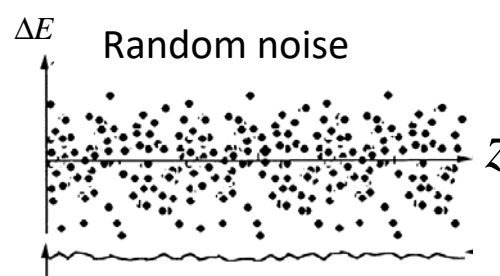
$$= \rho_0 \int_{z_0}^{\infty} W_0'(z_1 - z_0) dz_1 = -\rho_0 W_0(0) = 0$$



Consider perturbation in phase space density: n-th azimuthal mode

$$\psi_1(z, \Delta E, 0) = \hat{\psi}_1(\Delta E) e^{inz/R}$$

Ansatz: $\psi_1(z, \Delta E, t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$



*Note that if a perturbation is static,

$$\psi_1^*(z, \Delta E, t) = \hat{\psi}_1^*(\Delta E) e^{in(z-v_0 t)/R} = \hat{\psi}_1^*(\Delta E) e^{inz/R - i\Omega^* t} \Rightarrow \Omega^* = nv_0 / R = n2\pi v_0 / C = n\omega_0$$

Longitudinal Microwave Instability

$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \int_{-\infty}^{\infty} \frac{f_0'(\Delta E)}{\Omega - \omega(\Delta E)n} d\Delta E$$

*Phase slip factor: $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$

$$\omega(\Delta E) = \omega_0 + \Delta\omega(\Delta E) = \omega_0 - \eta\omega_0 \frac{\Delta p_z}{p_{0,z}} = \omega_0 - \frac{\eta\omega_0}{\beta^2} \frac{\Delta E}{E_0}$$

*Imaginary part of Ω tell us whether the system is stable

$$\psi_1(z, \Delta E, t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$$

Cold Beam: $f_0(\Delta E) = \delta(\Delta E)$

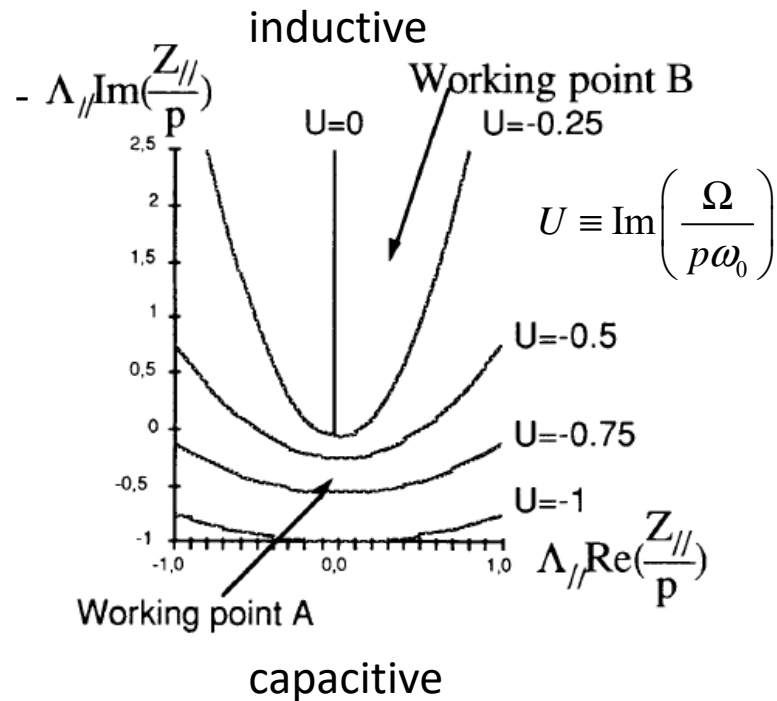
$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \frac{\eta n \omega_0}{E_0 \beta^2} \int_{-\infty}^{\infty} \frac{f_0(\Delta E)}{\left(\Omega - n\omega_0 + \frac{\eta n \omega_0}{E_0 \beta^2} \Delta E \right)^2} d\Delta E$$

$$\Rightarrow \Omega = n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(\Omega)}{2\pi E_0 \beta^2}} \approx n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(n\omega_0)}{2\pi E_0 \beta^2}}$$

Perturbative approach assuming $\frac{|\Omega - n\omega_0|}{n\omega_0} \ll 1$

Longitudinal Microwave Instabilities

Cold beam continued:
(assuming $\eta > 0$)



$$\Omega \approx n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0\eta nZ_{||}(n\omega_0)}{2\pi E_0\beta^2}}$$

For cold beam, the only case for stable beam is the machine impedance is pure inductive, i.e. $\text{Im}(Z_{||}) < 0$, for $\eta > 0$ and capacitive, i.e. $\text{Im}(Z_{||}) > 0$ for $\eta < 0$.

Taken from 'Accelerator Physics' by S.Y. Lee

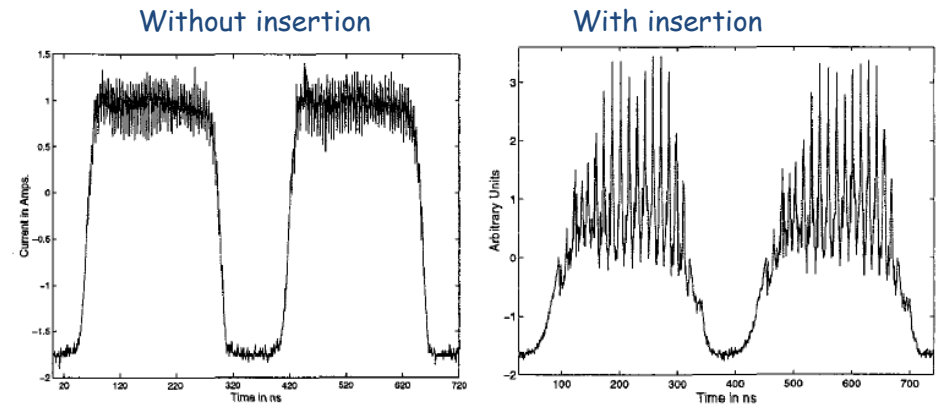


Figure 3.36: The longitudinal beam profiles observed at PSR the bunched coasting beam in the presence of inductive inserts, where three 1-m long ferrite ring cavities were installed in the PSR ring. [Courtesy of R. Macek, LANL]

Longitudinal Microwave Instabilities

Warm Beam: $f_0(\Delta E) = \frac{1}{\sqrt{2\pi}\sigma_E} \exp\left(-\frac{\Delta E^2}{2\sigma_E^2}\right)$ Dispersion relation for warm beam

$$1 = \frac{ieI_0 Z_{//}(n\omega_0)}{T_0} \frac{1}{\sqrt{2\pi}\sigma_E^2} \int_{-\infty}^{\infty} \frac{-\frac{\Delta E}{\sigma_E} \exp\left(-\frac{\Delta E^2}{2\sigma_E^2}\right)}{\Omega - \omega(\Delta E)n} d\Delta E = i \frac{1}{2} \left\{ \frac{eI_0 [Z_{//}(n\omega_0)/n] E_0 \beta^2}{2\pi\eta\sigma_E^2} \right\} J_G(\tilde{\Omega})$$

$$= i \frac{2\ln(2)}{\pi} \left\{ \frac{eI_0 [Z_{//}(n\omega_0)/n] E_0 \beta^2}{\eta\sigma_{E,FWHM}^2} \right\} J_G(\tilde{\Omega})$$

$$J_G(\tilde{\Omega}) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{-x \exp\left(-\frac{x^2}{2}\right)}{\left[\tilde{\Omega} \frac{E_0 \beta^2}{\eta n \omega_0 \sigma_E} + x \right]} dx$$

$$U' - iV' \equiv \frac{eI_0 [Z_{//}(n\omega_0)/n] E_0 \beta^2}{\eta\sigma_{E,FWHM}^2}$$

$$U' \sim \text{Re}(Z_{//}(n\omega_0))$$

$$V' \sim -\text{Im}(Z_{//}(n\omega_0))$$

$$\tilde{\Omega} = \text{Re}(\tilde{\Omega}) + i\text{Im}(\tilde{\Omega}) \equiv \Omega - \omega_0 n$$

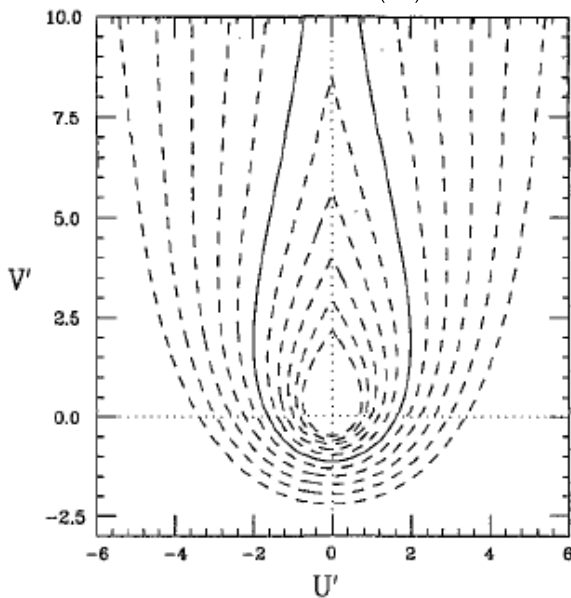
$$U' - iV' = \frac{-i\pi}{2\ln(2) J_G(\text{Re}\tilde{\Omega} + i\text{Im}\tilde{\Omega})}$$

- The dispersion relation is solved by numerical plotting the contours for various $\text{Im}\tilde{\Omega}$ in the complex impedance plane.

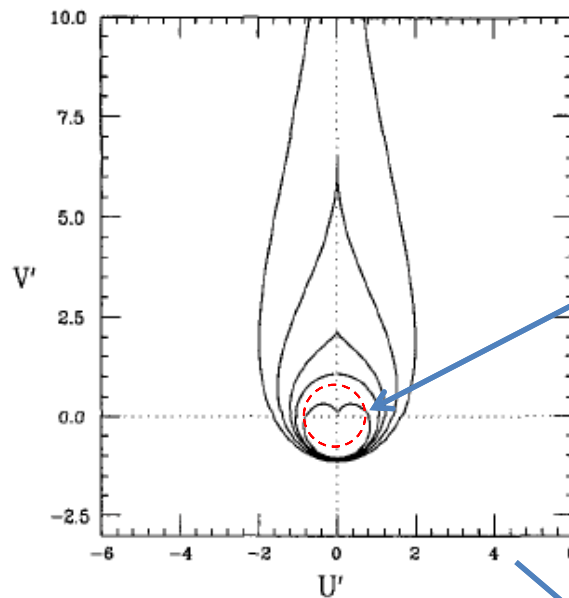
$$\psi_1(z, \Delta E, t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$$

Longitudinal Microwave instability

Gaussian with various growth rate, $\text{Im}(\tilde{\Omega})$



Contours with $\text{Im}(\tilde{\Omega})=0$ for various energy distribution



Simplified estimation for stability condition:
Keil-Schnell criterion

$$\left| Z_{\parallel}(n\omega_0) / n \right| \leq \frac{2\pi |\eta| \sigma_E^2}{E_0 \beta^2 e I_0} F$$

F depends on distribution and for Gaussian energy distribution, it is 1.

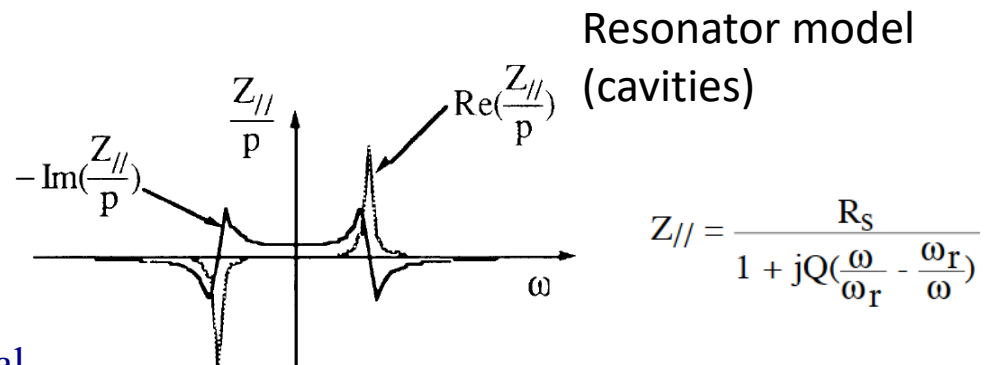
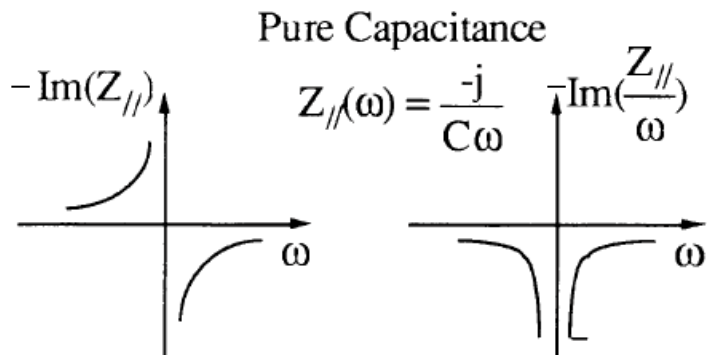
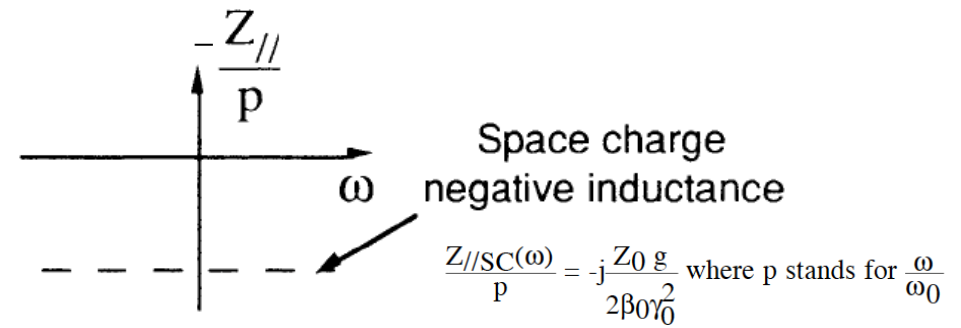
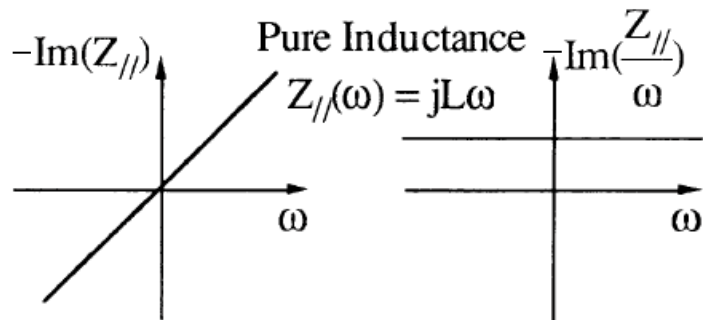
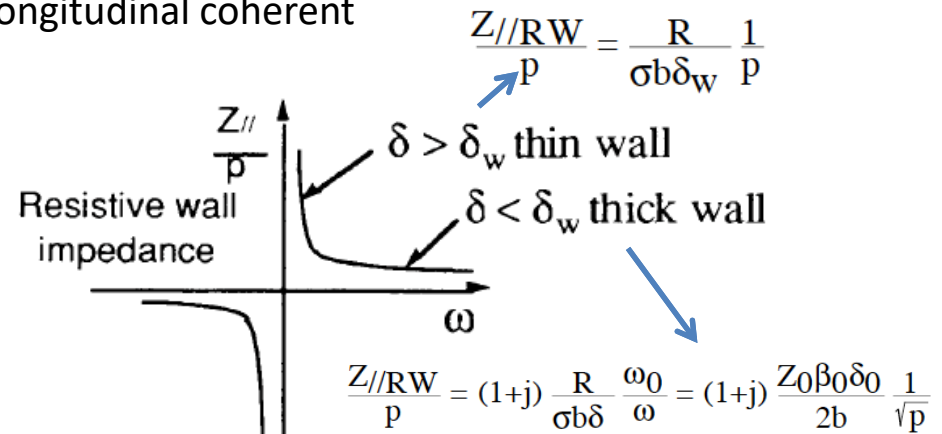
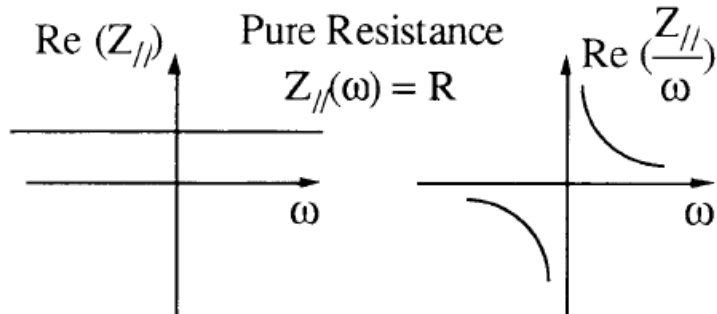
Figure 3.34: Left: The solid line shows the parameters V' vs U' for a Gaussian beam distribution at a zero growth rate. Dashed lines inside the threshold curve are stable. They correspond to $-\text{Im} \Omega / (\sqrt{2 \ln 2} \omega_0 \eta \sigma_\delta) = -0.1, -0.2, -0.3, -0.4,$ and -0.5 . Dashed lines outside the threshold curve have growth rates $-\text{Im} \Omega / (\sqrt{2 \ln 2} \omega_0 \eta \sigma_\delta) = 0.1, 0.2, 0.3, 0.4,$ and 0.5 respectively. Right: The threshold V' vs U' parameters for various beam distributions.

from inside outward, for the normalized distribution functions $\Psi_0(x) = 3(1-x^2)/4,$ $8(1-x^2)^{3/2}/3\pi,$ $15(1-x^2)^2/16,$ $315(1-x^2)^4/32,$ and $(1/\sqrt{2\pi}) \exp(-x^2/2)$. All dis-

Typical Longitudinal Impedance

$$j = -i$$

Taken from 'Coasting beam longitudinal coherent instabilities' by J.L. Laclare



Many pictures and derivations used in the slides are taken from the following references:

[1] ‘Accelerator Physics’ by S.Y. Lee;

[2] ‘Physics of Collective Beam Instabilities in High Energy Accelerators’ by A. Chao;

[3] ‘Coasting beam longitudinal coherent instabilities’ by J.L. Laclare

What we learned today

- In linear accelerator, **single bunch transverse beam break up instability** can develop if the bunches are not carefully injected and machine transverse wake function / impedance is large. Such an instability can be compensated by introducing focusing variation along the bunch, i.e. **BNS damping**.
- In circular machine, the leading order ($m=0$) longitudinal wakefield in the cavity can cause Robinson instability. The cavity resonant frequency should be detuned away from exact harmonics of the revolution frequency to avoid such instability: above transition, the resonant frequency should be slightly below $h\omega_0$; and below transition the resonant frequency should be slightly above $h\omega_0$.
- (optional) We also showed the dispersion relation for longitudinal microwave instability in a coasting beam. For **cold beam**, the beam is always unstable unless the impedances is pure inductive above transition or pure capacitive below transition. For **warm beam**, **Landau damping** make beam stable if the beam energy spread is sufficiently large. The stability condition can be estimated from **Keil-Schnell criteria**.

Backup Slides

Robinson Instability IV

We will use the following identity (the Poisson Sum Formula)

$$\sum_{l=-\infty}^{\infty} F(lC) = \frac{1}{C} \sum_{p=-\infty}^{\infty} \tilde{F}\left(\frac{2\pi p}{C}\right) \quad F(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikz} \tilde{F}(k) dk \quad \tilde{F}(k) = \int_{-\infty}^{\infty} e^{-ikz} F(z) dz$$

$$\begin{aligned} \Omega - \omega_s &\approx -\frac{4\pi\epsilon_0\eta Nr_0 c}{\gamma T_0 2\omega_s} \sum_{k=-\infty}^{\infty} (1 - \exp(-ik\Omega T_0)) W_0''(kC) \\ &= -\frac{4\pi\epsilon_0\eta Nr_0 c}{\gamma T_0 2\omega_s} \left\{ \sum_{k=-\infty}^{\infty} W_0''(kC) - \sum_{k=-\infty}^{\infty} G(kC) \right\} \quad G(kC) \equiv \exp\left(-i\frac{\Omega}{c}kC\right) W_0''(kC) \\ &= -\frac{4\pi\epsilon_0\eta Nr_0 c}{\gamma T_0 2\omega_s C} \left\{ \sum_{p=-\infty}^{\infty} \tilde{W}_0''\left(\frac{2\pi p}{C}\right) - \sum_{p=-\infty}^{\infty} \tilde{G}\left(\frac{2\pi p}{C}\right) \right\} \\ &\approx -i \frac{4\pi\epsilon_0\eta Nr_0}{2\gamma T_0^2 \omega_s} \sum_{p=-\infty}^{\infty} \left\{ p\omega_0 Z_{0,||}(p\omega_0) - (p\omega_0 + \omega_s) Z_{0,||}(p\omega_0 + \omega_s) \right\} \end{aligned}$$

APPENDIX

$$\begin{aligned}
 \tilde{W}_0''\left(\frac{2\pi p}{C}\right) &= \int_{-\infty}^{\infty} e^{-i\frac{2\pi p}{C}z} W_0''(z) dz \\
 &= \int_{-\infty}^{\infty} e^{-i\frac{2\pi p}{C}z} \frac{d}{dz} W_0'(z) dz \\
 &= \int_{-\infty}^{\infty} \frac{d}{dz} \left[e^{-i\frac{2\pi p}{C}z} W_0'(z) \right] dz - \int_{-\infty}^{\infty} W_0'(z) \frac{d}{dz} e^{-i\frac{2\pi p}{C}z} dz \\
 &= i\frac{2\pi p}{C} \int_{-\infty}^{\infty} W_0'(z) e^{-i\frac{2\pi p}{C}z} dz \\
 &= i\frac{p}{C} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega e^{i\frac{\omega z}{C}} Z_{0, //}(\omega) e^{-i\frac{2\pi p}{C}z} dz \\
 &= i\frac{p}{C} \int_{-\infty}^{\infty} d\omega Z_{0, //}(\omega) \int_{-\infty}^{\infty} e^{iz\left(\frac{\omega}{C} - \frac{2\pi p}{C}\right)} dz \\
 &= i\frac{cp}{C} 2\pi \int_{-\infty}^{\infty} d\omega Z_{0, //}(\omega) \delta\left(\omega - \frac{2\pi pc}{C}\right) \\
 &= ip\omega_0 Z_{0, //}(p\omega_0)
 \end{aligned}$$

$$W_0'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\frac{\omega z}{C}} Z_{0, //}(\omega) d\omega$$

$$\begin{aligned}
\tilde{G}\left(\frac{2\pi p}{C}\right) &= \int_{-\infty}^{\infty} e^{-i\frac{2\pi p}{c}z} \exp\left(-i\frac{\Omega}{c}z\right) W_0''(z) dz \\
&= \int_{-\infty}^{\infty} e^{-i\left(\frac{2\pi p}{c} + \frac{\Omega}{c}\right)z} \frac{d}{dz} W_0'(z) dz \\
&= \int_{-\infty}^{\infty} e^{-i\frac{2\pi p'}{c}z} \frac{d}{dz} W_0'(z) dz; \quad p' = p + \frac{\Omega C}{2\pi c} \\
&= \int_{-\infty}^{\infty} \frac{d}{dz} \left[e^{-i\frac{2\pi p'}{c}z} W_0'(z) \right] dz - \int_{-\infty}^{\infty} W_0'(z) \frac{d}{dz} e^{-i\frac{2\pi p'}{c}z} dz \\
&= i\frac{2\pi p'}{C} \int_{-\infty}^{\infty} W_0'(z) e^{-i\frac{2\pi p'}{c}z} dz \\
&= i\frac{p'}{C} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega e^{i\omega\frac{z}{c}} Z_{0, //}(\omega) e^{-i\frac{2\pi p'}{c}z} dz \\
&= i\frac{p'}{C} \int_{-\infty}^{\infty} d\omega Z_{0, //}(\omega) \int_{-\infty}^{\infty} e^{iz\left(\frac{\omega}{c} - \frac{2\pi p'}{c}\right)} dz \\
&= i\frac{cp'}{C} 2\pi \int_{-\infty}^{\infty} d\omega Z_{0, //}(\omega) \delta\left(\omega - \frac{2\pi p'c}{C}\right) \\
&= ip'\omega_0 Z_{0, //}(p'\omega_0) \\
&= i(p\omega_0 + \Omega) Z_{0, //}(p\omega_0 + \Omega) \\
&\approx i(p\omega_0 + \omega_s) Z_{0, //}(p\omega_0 + \omega_s)
\end{aligned}$$

APPENDIX

$$W_0'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\frac{z}{c}} Z_{0, //}(\omega) d\omega$$

$$Z_{0, //} = \frac{R_s}{1 + iQ \left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R} \right)}$$

$$\operatorname{Re}[Z_{0, //} (h\omega_0 + \omega_s)] = \operatorname{Re} \left\{ \frac{R_s}{1 + iQ \left(\frac{\omega_R}{h\omega_0 + \omega_s} - \frac{h\omega_0 + \omega_s}{\omega_R} \right)} \right\}$$

$$= \frac{R_s}{1 + 4Q^2 \left(\frac{(\Delta\omega - \omega_s)^2}{h\omega_0} \right)^2}$$

$$= R_s \left(1 - 4Q^2 \frac{(\Delta\omega - \omega_s)^2}{h^2 \omega_0^2} \right)$$

$$\operatorname{Re}[Z_{0, //} (h\omega_0 + \omega_s)] - \operatorname{Re}[Z_{0, //} (h\omega_0 - \omega_s)] = 16 \frac{Q^2 R_s}{h^2 \omega_0^2} \omega_s \Delta\omega$$

$$\operatorname{Re}[Z_{0, //} (h\omega_0 - \omega_s)] = \operatorname{Re} \left\{ \frac{R_s}{1 + iQ \left(\frac{\omega_R}{h\omega_0 - \omega_s} - \frac{h\omega_0 - \omega_s}{\omega_R} \right)} \right\}$$

$$= \frac{R_s}{1 + 4Q^2 \left(\frac{(\Delta\omega + \omega_s)^2}{h\omega_0} \right)^2}$$

$$= R_s \left(1 - 4Q^2 \frac{(\Delta\omega + \omega_s)^2}{h^2 \omega_0^2} \right)$$

Longitudinal Microwave Instability

But the system is not likely to be static and we need to solve Vlasov equation self-consistently to know the answer for Ω and hence $\psi_1(s, \Delta E, t)$

$$\frac{\partial}{\partial t} \psi_1(z, \Delta E, t) + \frac{dz}{dt} \cdot \frac{\partial}{\partial z} \psi_1(z, \Delta E, t) + \frac{d\Delta E}{dt} \cdot \frac{\partial}{\partial \Delta E} \psi_0(\Delta E) = 0$$

where
$$\frac{dz}{dt} = v(\Delta E) \quad (1)$$

And $\frac{d\Delta E}{dt}$ is obtained by calculating the longitudinal wake potential $\frac{d\Delta E(z, t)}{dt} = -\frac{c\Delta p_z(z, t)}{T_0}$

$$c\Delta p_z(z, t) = -eQ_e V_{//}(z, t) = -e^2 v_0 \int_{-\infty}^t \rho_1(z, t_1) w_{//}(t - t_1) dt_1 = -e^2 v_0 \int_0^{\infty} \rho_1(z, t - \tau) w_{//}(\tau) d\tau$$

$\rho_1 v_0 dt$ gives particle number in the slice $(t, t+dt)$.

Longitudinal Microwave Instability

Hence, we obtain

$$\frac{d\Delta E(z,t)}{dt} = -\frac{c\Delta p_z(z,t)}{T_0} = -\frac{e^2 v}{T_0} \int_0^\infty \rho_1(z,t-\tau) w_{||}(\tau) d\tau$$

where $T_0 = \frac{C_0}{v_0}$ is the revolution period. Using the test solution

$$\psi_1(z, \Delta E, t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$$

and the following relations

$$w_{||}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z_{||}(\omega) e^{-i\omega\tau} d\omega$$

$$\rho_1(z,t) = \int_{-\infty}^{\infty} \psi_1(z, \Delta E, t) d\Delta E = \hat{\rho}_1 e^{inz/R - i\Omega t} \quad \hat{\rho}_1 \equiv \int_{-\infty}^{\infty} \hat{\psi}_1(\Delta E) d\Delta E$$

we can write the energy kick in term of longitudinal impedance

$$\frac{d\Delta E(z,t)}{dt} = -\hat{\rho}_1 \frac{e^2 v_0}{2\pi T_0} e^{inz/R - i\Omega t} \int_{-\infty}^{\infty} d\omega Z_{||}(\omega) \int_{-\infty}^{\infty} e^{i(\Omega - \omega)\tau} d\tau = -\hat{\rho}_1 \frac{e^2 v_0}{T_0} e^{inz/R - i\Omega t} Z_{||}(\Omega) \quad (2)$$

Longitudinal Microwave Instability

Inserting eq. (1) and (2) into Vlasov equation, we obtain

$$-i\Omega\psi_1(z, \Delta E, t) + v(\Delta E) \cdot \frac{in}{R} \psi_1(z, \Delta E, t) - \hat{\rho}_1 \frac{e^2 v_0}{T_0} e^{inz/R - i\Omega t} Z_{//}(\Omega) \cdot \frac{\partial}{\partial \Delta E} \psi_0(\Delta E) = 0$$

, which can be rewritten as

$$\psi_1(z, \Delta E, t) = \frac{ie^2 v_0 Z_{//}(\Omega)}{T_0} \frac{\hat{\rho}_1 e^{inz/R - i\Omega t}}{\Omega - \omega(\Delta E)n} \frac{d\psi_0(\Delta E)}{d\Delta E} \quad \omega(\Delta E) = \frac{v(\Delta E)}{R}$$

Integrating above equation over energy, i.e. $\int_{-\infty}^{\infty} d\Delta E \rightarrow$, yields

Dispersion relation:

$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \int_{-\infty}^{\infty} \frac{f_0'(\Delta E)}{\Omega - \omega(\Delta E)n} d\Delta E$$

$$\psi_0(\Delta E) = \frac{N}{C_0} f_0(\Delta E)$$

$$I_0 = eN/T_0$$