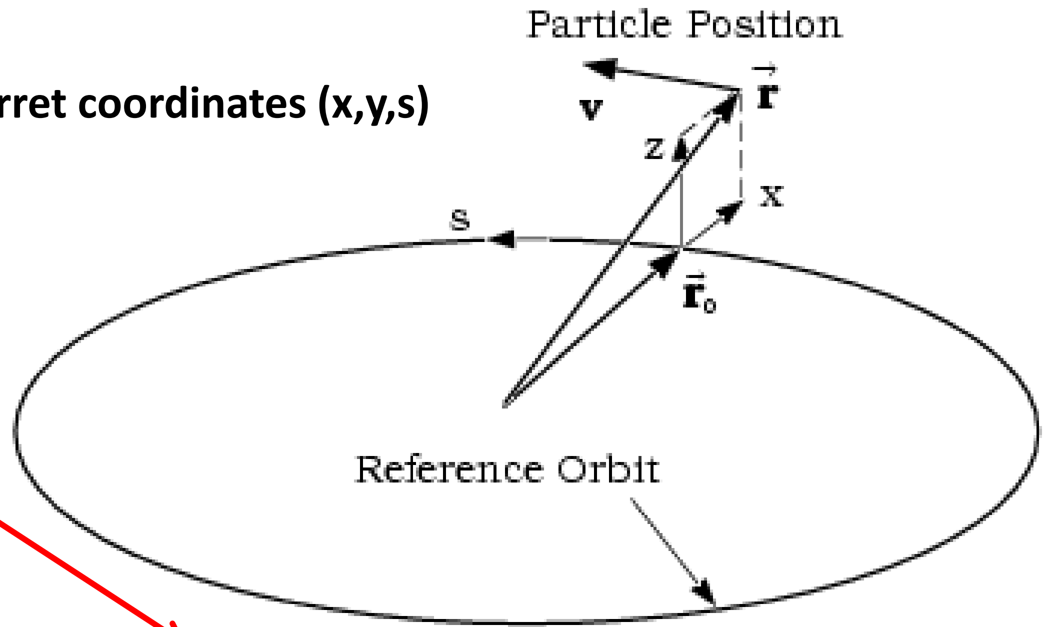
Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

Review

Frenet-Serret coordinates (x,y,s)



Hill's equations (derivatives w.r.t. s)

$$x'' + K_x(s)x = \pm\frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm\frac{B_1}{B\rho}$$

Natural focusing from dipoles (curvature)

Focusing from quadrupoles

Higher order magnet, usually field errors

$$\theta = \frac{s}{R} = \frac{\beta c t}{R}$$

Solution of Hill's equations X(s), X'(s) form a coordinate set and can be transformed thru matrix representation

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

$$|M(s,s_0)| = 1 \qquad |Trace(M(s,s_0))| \le 2$$

Stable solution conditions

Courant-Snyder parameterization

$$\mathrm{M}(s)=\begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}=I\cos\Phi+J\sin\Phi$$

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad J=\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix},\quad J^2=-I,\quad \text{or}\quad \beta\gamma=1+\alpha^2$$

Where α,β,γ,ϕ are functions of s and describes position dependent beam properties.

Focusing quadrupole:

$$M(s,s_0)=\begin{pmatrix}\cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell\\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0\\ -1/f & 1\end{pmatrix}$$

Defocusing quadrupole:

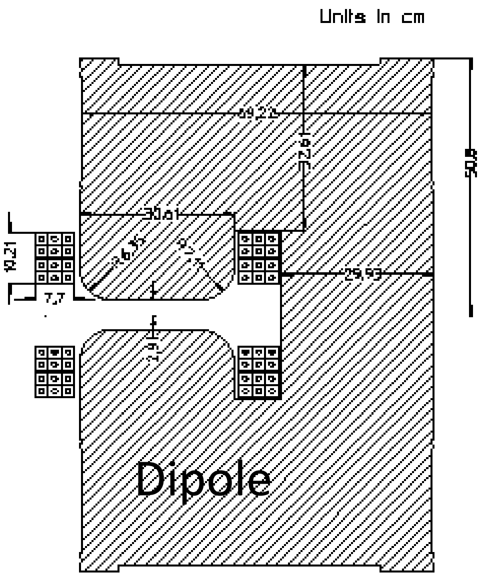
$$M(s,s_0)=\begin{pmatrix}\cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell\\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0\\ 1/f & 1\end{pmatrix}$$

Dipole: $K=1/\rho^2$

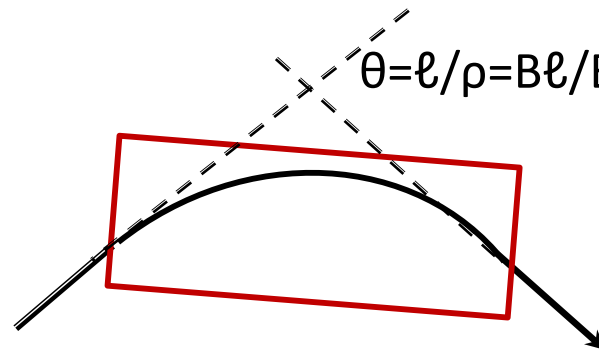
$$M(s,s_0)=\begin{pmatrix}\cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho}\\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho}\end{pmatrix}\rightarrow\begin{pmatrix}1 & \ell\\ 0 & 1\end{pmatrix}$$

Drift space: $K=0$

$$M(s,s_0)=\begin{pmatrix}1 & \ell\\ 0 & 1\end{pmatrix}$$



$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

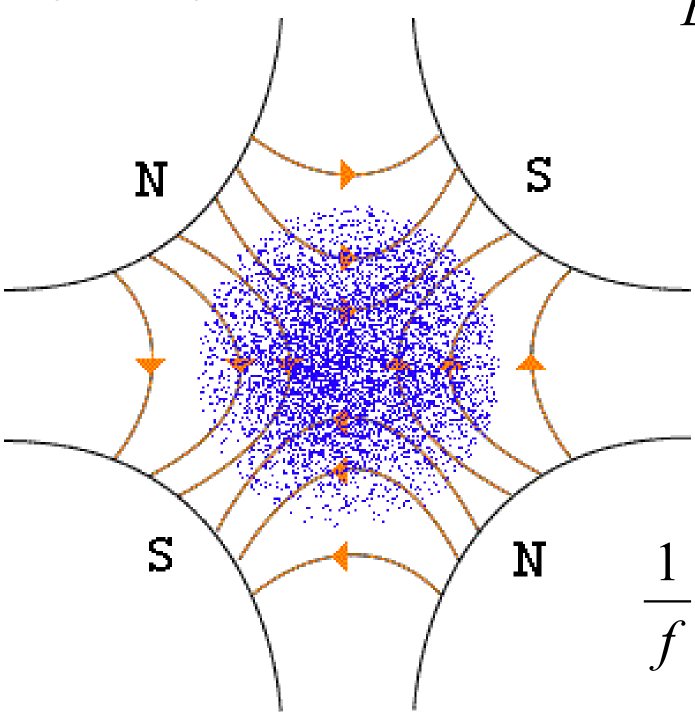


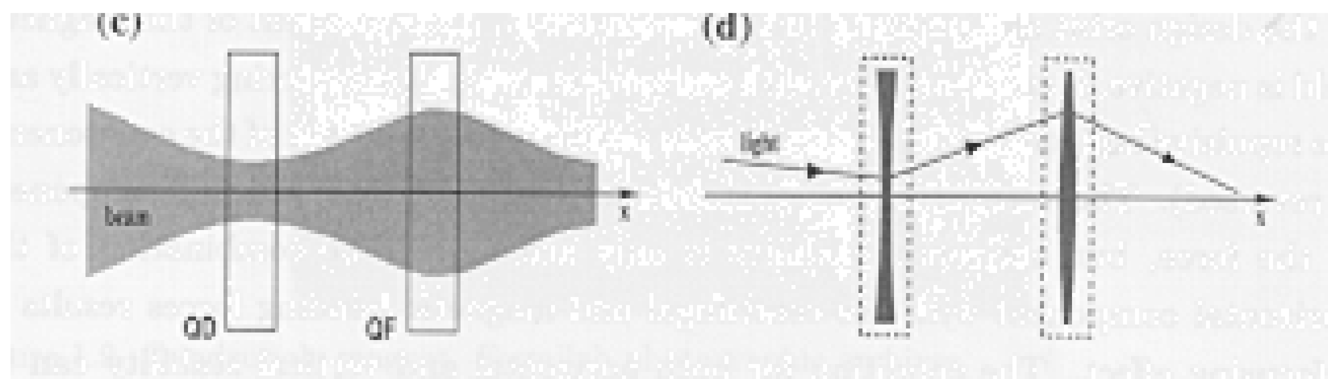
$$\gamma m \frac{\upsilon^2}{\rho} = q\upsilon B$$

$$\rho = \frac{\gamma m v}{qB} = \frac{p}{qB}$$

quadrupole

$$B\rho[T-m] = \frac{p}{q} = \frac{A}{Z} \times 3.33564 \times p[GeV/c/u]$$





$$\frac{1}{f} = \mp \frac{B_1 \ell}{B\rho}$$

f>0, if focusing, f<0 if defocusing

For two dimensional magnetic field, one can expand the magnetic field using **Beth representation**.

$$\vec{B} = B_x(x,y)\hat{x} + B_y(x,y)\hat{y}$$

$$B_x = -\frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial y} = -\frac{1}{h_s}\frac{\partial A_s}{\partial y}, B_y = \frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial x} = \frac{1}{h_s}\frac{\partial A_s}{\partial x}$$

For h_s=1 or ρ=∞, one obtains the multipole expansion:

$$B_y + jB_x = B_0\sum_n (b_n + ja_n)(x + jy)^n, \qquad A_s = \mathrm{Re}\left\{B_0\sum_n \frac{1}{n+1}(b_n + ja_n)(x + jy)^{n+1}\right\}$$

b_0 : dipole, a_0 : skew (vertical) dipole; $B_y = B_0 b_0, \quad B_x = B_0 a_0,$

b_1 : quad, a_1 : skew quad; $B_y = B_0 b_1 x, \quad B_x = B_0 b_1 y, \; B_y = -B_0 a_1 y, \quad B_x = B_0 a_1 x,$

b_2 : sextupole, a_2 : skew sextupole;

$$\frac{1}{B\rho}(B_y + jB_x) = \mp\frac{1}{\rho}\sum_n (b_n + ja_n)(x + jy)^n$$

Floquet Theorem

$$X'' + K(s)X = 0 \qquad K(s) = K(s+L)$$

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \qquad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^{s} \frac{1}{\beta} ds$$

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2, s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu - \frac{\alpha_1 - \alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu - \alpha_1 \sin\mu) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

The values of the Courant–Snyder parameters α_2, β_2, γ_2 at s_2 are related to α_1, β_1, γ_1 at s_1 by

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 (s - \frac{\alpha_1}{\gamma_1})^2 = \beta^* + \frac{(s - s^*)^2}{\beta^*},$$

$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s - s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

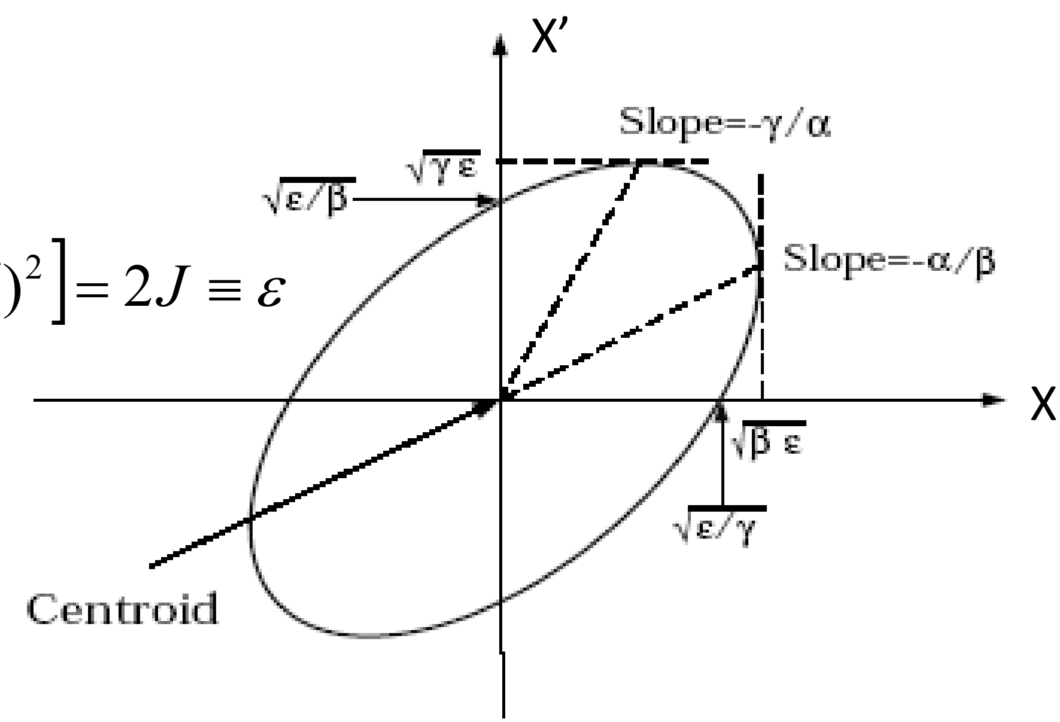
$$X = \sqrt{2\beta J}\cos\psi, \quad X' = -\sqrt{\frac{2J}{\beta}}(\sin\psi + \alpha\cos\psi)$$

$$P_X = \beta X' + \alpha X = -\sqrt{2\beta J}\sin\psi$$

(X,P_X) form a normalized phase space coordinates with $X^2+P_X^2=2\beta J$, here J is called **action**.

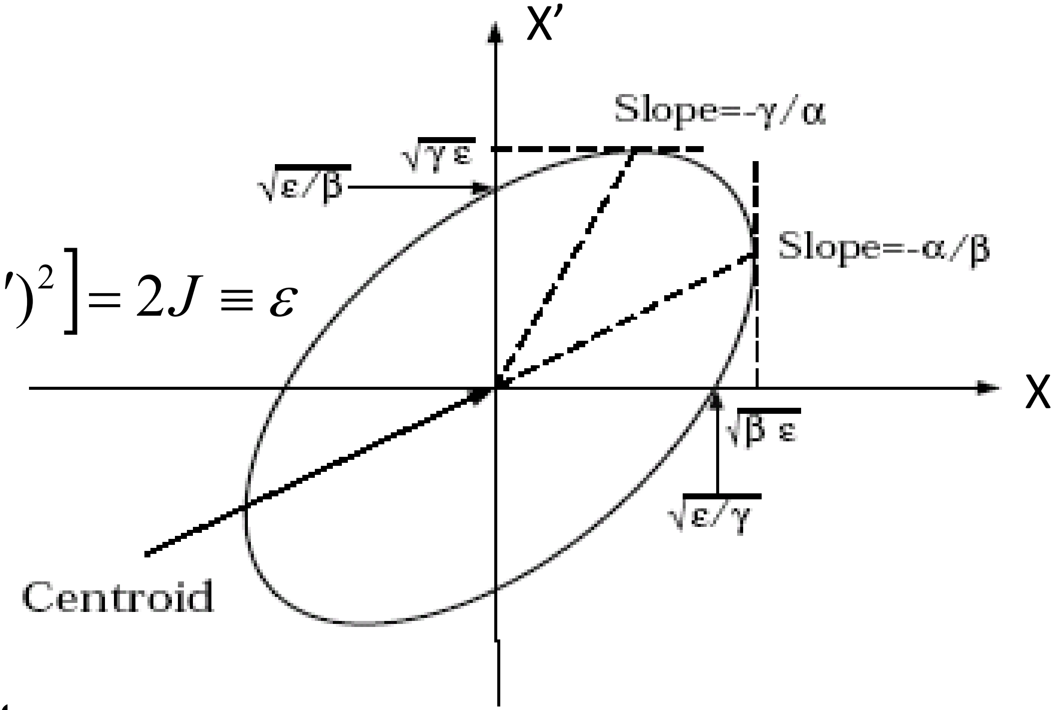
Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha XX' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha XX' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



Emittance of a beam

$$\langle X \rangle = \int X\rho(X,X')dXdX', \quad \langle X' \rangle = \int X'\rho(X,X')dXdX',$$

$$\sigma_X^2 = \int (X - \langle X \rangle)^2 \rho(X,X')dXdX', \quad \sigma_{X'}^2 = \int (X' - \langle X' \rangle)^2 \rho(X,X')dXdX',$$

$$\sigma_{XX'} = \int (X - \langle X \rangle)(X' - \langle X' \rangle)\rho(X,X')dXdX' = r\sigma_X\sigma_{X'}$$

$$\varepsilon_{rms} = \sqrt{\sigma_X^2\sigma_{X'}^2 - \sigma_{XX'}^2} = \sigma_X\sigma_{X'}\sqrt{1-r^2}$$

The rms emittance is invariant in linear transport: $$\frac{d\varepsilon^2}{ds} = 0$$

normalized emittance $\varepsilon_n=\varepsilon\beta\gamma$ is invariant when beam energy is changed.

Adiabatic damping – beam emittance decreases with increasing beam momentum, i.e. $\varepsilon=\varepsilon_n/\beta\gamma$, which applies to beam emittance in **linacs**.

In storage rings, the beam emittance **increases** with energy ($\sim\gamma^2$). The corresponding normalized emittance is proportional to γ^3.

The Gaussian distribution function

$$\rho(X,P_X)=\frac{1}{2\pi\sigma_X^2}e^{-(X^2+P_X^2)/2\sigma_X^2}$$

$$\rho(\varepsilon)=\frac{1}{2\varepsilon_{rms}}e^{-\varepsilon/2\varepsilon_{rms}}$$

$\epsilon/\epsilon_{\rm rms}$	2	4	6	8
Percentage in 1D [%]	63	86	95	98
Percentage in 2D [%]	40	74	90	96

Effects of Linear Magnetic field Error

$$x'' + [K_x(s) + k(s)]x = \frac{b_0}{\rho}, \qquad y'' + [K_y(s) - k(s)]y = -\frac{a_0}{\rho}$$

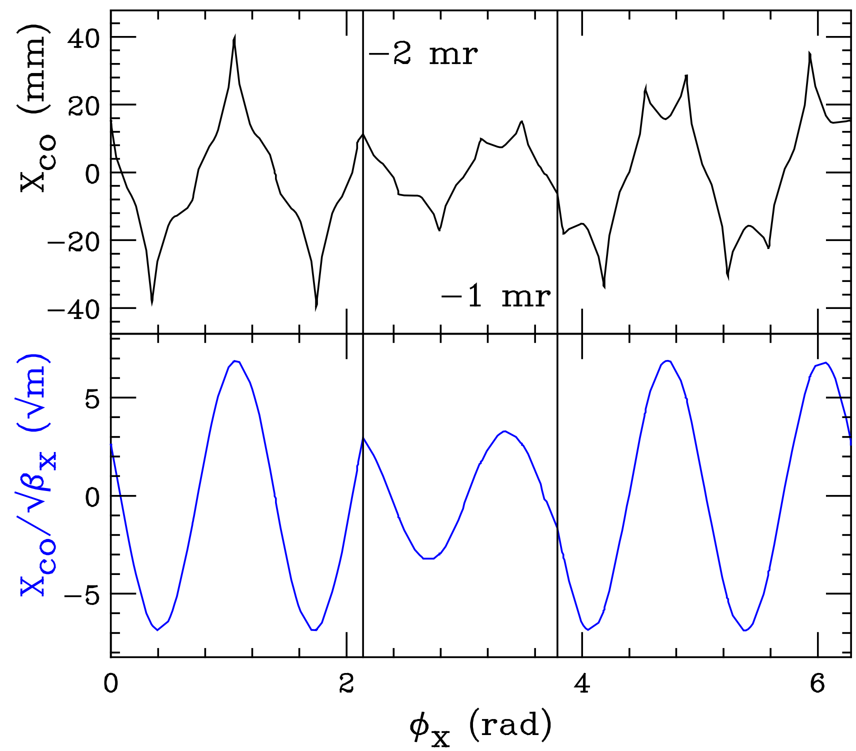
For a localized dipole field error: θ=ΔBℓ/Bρ

$$X'' + K_X(s)X = \theta\delta(s - s_0)$$

$$X_0 = \frac{\beta_0\theta}{2\sin\pi\nu}\cos\pi\nu,$$

$$X_0' = \frac{\theta}{2\sin\pi\nu}(\sin\pi\nu - \alpha_0\cos\pi\nu)$$

$$X_{co}(s) = G(s, s_0)\theta$$



$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s) - \psi(s_0)|]$$

For a distributed dipole field error:

$$X_{co}(s) = \sqrt{\beta(s)} \sum_{k=-\infty}^{\infty} \frac{\nu^2 f_k}{\nu^2 - k^2} \, e^{jk\phi(s)}$$

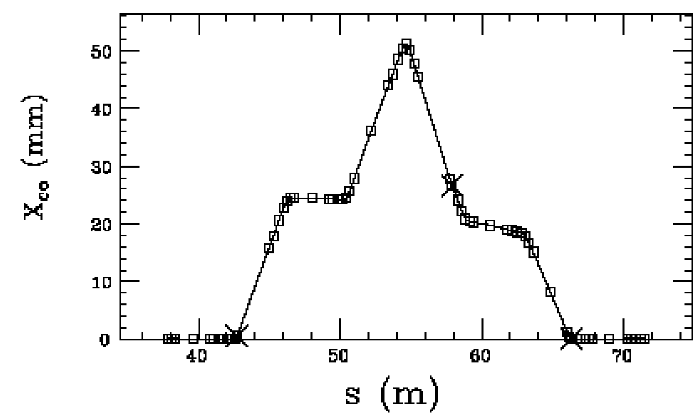
Where the field error is expanded in Fourier series $\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right] = \sum_{k=-\infty}^{\infty} f_k e^{jk\varphi}$

$$f_k = \frac{1}{2\pi}\oint\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}d\varphi = \frac{1}{2\pi\nu}\oint\left[\beta^{1/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}ds$$

$$\text{Sensitivity factor} \equiv \frac{\left\langle \left(X_{co}(s)\right)^2 \right\rangle^{1/2}}{\theta_{rms}} \propto \sqrt{\beta(s)}$$

closed orbit bump: $X_{co}(s_f) = 0$, $X'_{co}(s_f) = 0$

$$\Delta x_{co}(s) = \left\{\sqrt{\beta_x(s_k)\beta_x(s)}\sin(\Delta\psi_x(s))\right\}\theta_k$$



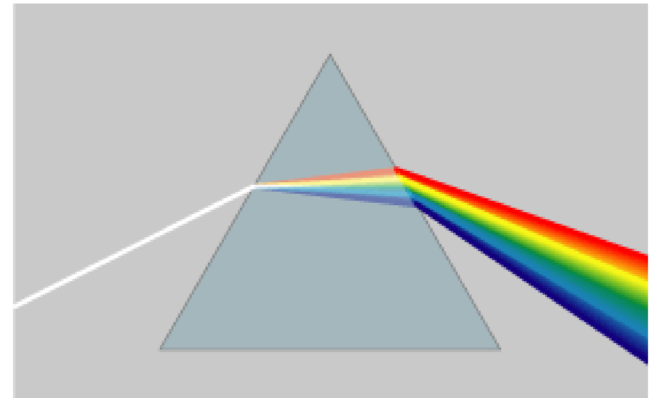
Orbit length change:

$$\Delta C = C - C_0 = \theta_0 \oint \frac{G_x(s,s_0)}{\rho} ds = D(s_0)\theta_0$$

$$\Delta C = \oint D(s_0)\frac{\Delta B_y(s_0)}{B\rho} ds_0$$

Off-momentum and dispersion

For different particle energy $$\delta = \frac{p - p_0}{p_0}$$



$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + K_x(s) x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + K_x(s) D = \frac{1}{\rho}$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

Extend the matrix representation to 3 by 3

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For a pure dipole (K=0):

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

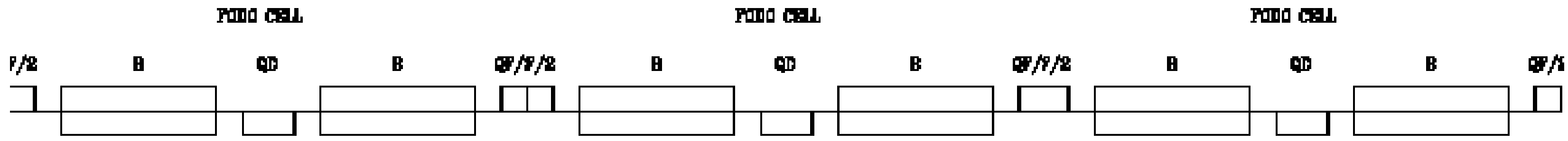
$$\theta \ll 1 \quad i.e. \quad L \ll \rho$$

For quadrupoles:

$$M(s, s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Defocusing
change K -> -K

FODO cell



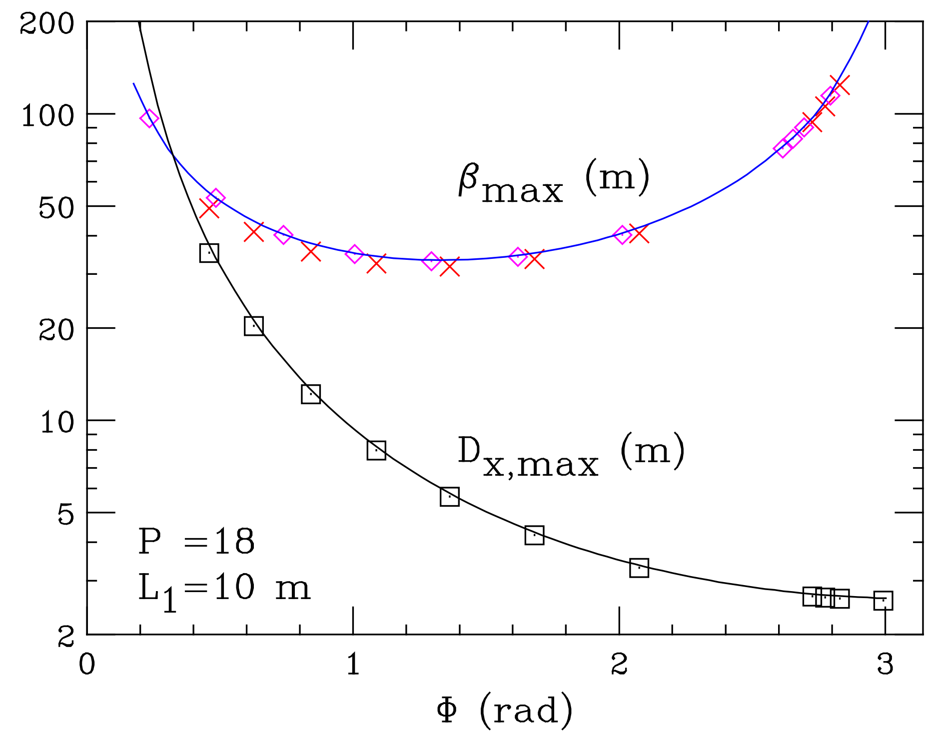
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L(1+\frac{L}{2f}) & 2L\theta(1+\frac{L}{4f}) \\ -\frac{L}{2f^2}+\frac{L^2}{4f^3} & 1-\frac{L^2}{2f^2} & 2\theta(1-\frac{L}{4f}-\frac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_F \\ D'_F \\ 1 \end{pmatrix}$$

$$D_F = \frac{L\theta(1+\frac{1}{2}\sin\frac{\Phi}{2})}{\sin^2\frac{\Phi}{2}}, \quad D'_F = 0$$

$$\beta_{\max} = \frac{2L_1(1+\frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$



Tune shift, or tune spread, due to chromatic aberration:

$$\Delta\nu_x = \left[-\tfrac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \equiv C_x\delta, \quad C_x = d\nu_x/d\delta$$

$$\Delta\nu_y = \left[-\tfrac{1}{4\pi}\oint \beta_y(s)K_y(s)ds\right]\delta \equiv C_y\delta, \quad C_y = d\nu_y/d\delta$$

The chromaticity induced by quadrupole field error is called natural chromaticity. For a simple FODO cell, we find

$$\Delta\nu_x = \left[-\frac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \approx -\frac{1}{4\pi}\sum \frac{\beta_{xi}}{f_i}\ \delta$$

$$C_{X,\mathrm{nat}}^{\mathrm{FODO}} = -\frac{1}{4\pi}N\left(\frac{\beta_{\max}}{f} - \frac{\beta_{\min}}{f}\right) = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$

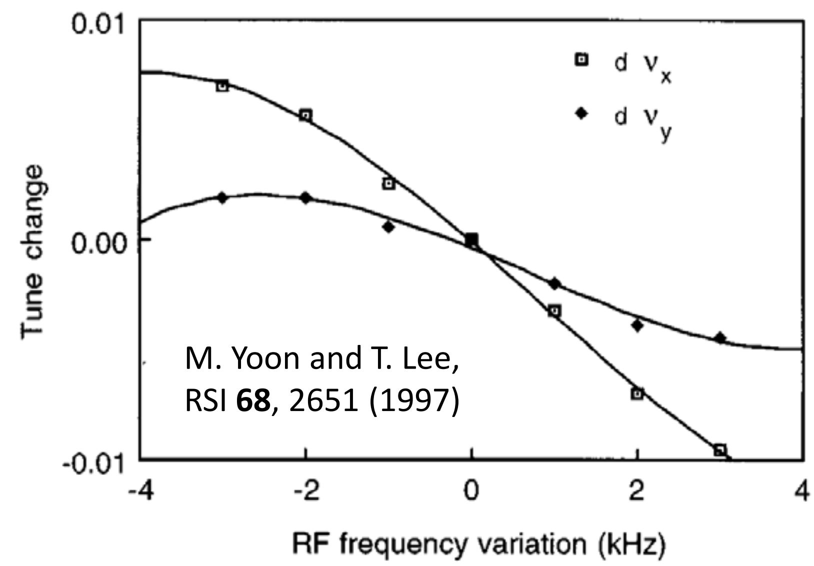
We define the specific chromaticity as $\xi_x = C_x/\nu_x, \quad \xi_y = C_y/\nu_y$

The **specific chromaticity is about −1 for FODO cells**, and can be as high as -4 for high luminosity colliders and high brightness electron storage rings.

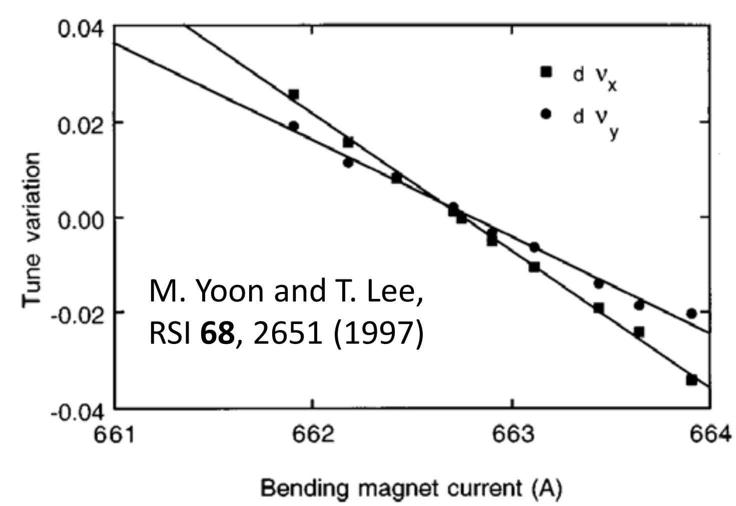
$$\sin\frac{\Phi}{2} = \frac{L_1}{2f} \qquad \beta_{\max} = \frac{2L_1(1+\sin(\Phi/2))}{\sin\Phi}, \quad \beta_{\min} = \frac{2L_1(1-\sin(\Phi/2))}{\sin\Phi}$$

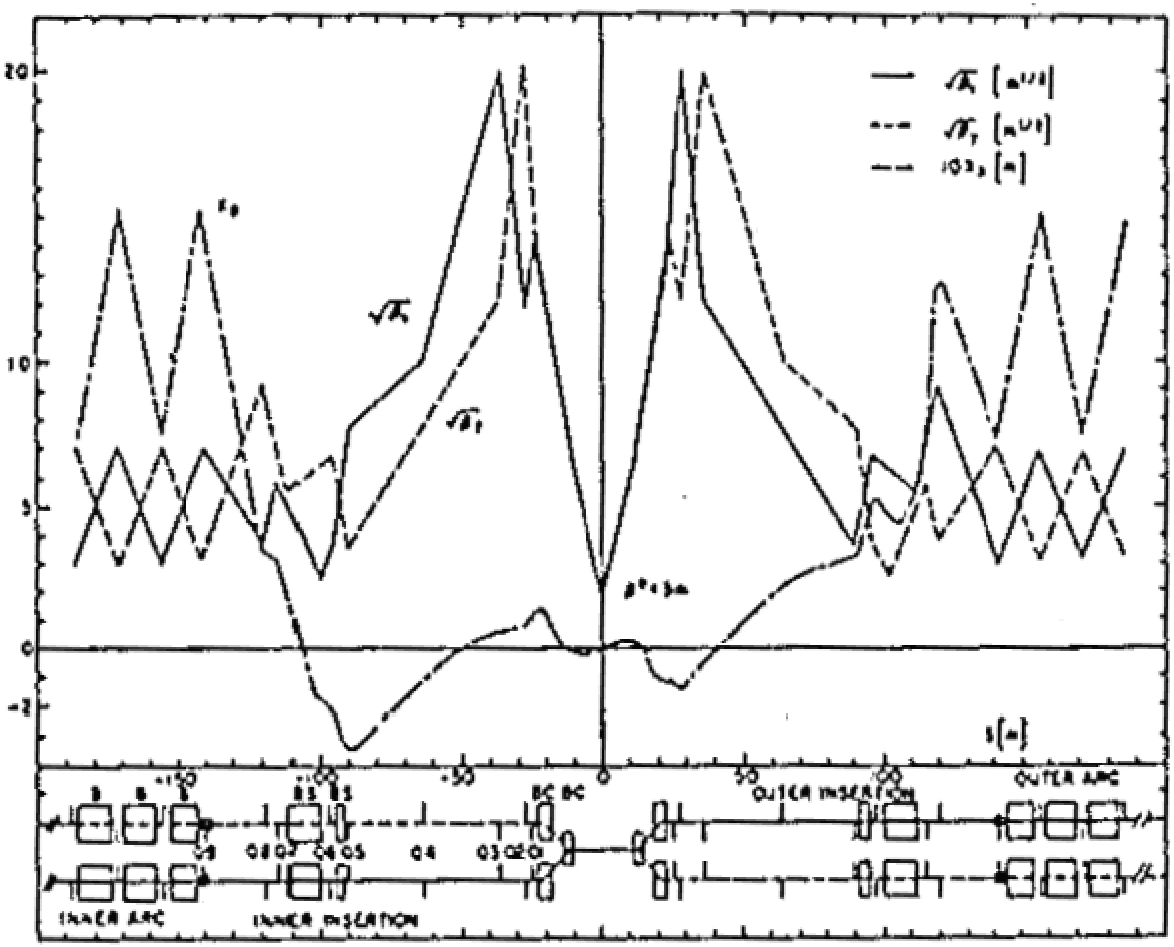
Chromaticity measurement:

The chromaticity can be measured by measuring the betatron tunes vs the rf frequency



The **chromaticity** can be obtained by measuring the tune variation vs the bending-magnet current at a **constant rf frequency**. May not apply for combined function magnets





Contribution of low β triplets in an IR to the natural chromaticity is

$$C_{total} = N_{IR} C_{IR} + C_{ARCs}$$

$$C_{IR} = -\frac{2\Delta s}{4\pi\beta^*} \approx -\frac{1}{2\pi}\sqrt{\frac{\beta_{\max}}{\beta^*}}$$

$$x''_\beta + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad y''_\beta + (K_y(s) - K_2 D\delta)y_\beta = 0$$

$$x = x_\beta + D\delta$$

$$\Delta K_x(s) = K_2(s)D(s)\delta, \quad \Delta K_y(s) = -K_2(s)D(s)\delta$$

$$C_x = -\frac{1}{4\pi}\oint \beta_x(s)[K_x(s) - K_2(s)D(s)]ds$$

$$C_y = -\frac{1}{4\pi}\oint \beta_y(s)[K_y(s) + K_2(s)D(s)]ds$$

- In order to minimize their strength, the chromatic sextupoles should be located near quadrupoles, where $\beta_x D_x$ and $\beta_y D_x$ are maximum.
- A large ratio of β_x/β_y for the focusing sextupole and a large ratio of β_y/β_x for the defocussing sextupole are needed for optimal independent chromaticity control.
- The families of sextupoles should be arranged to minimize the systematic half-integer stopbands and the third-order betatron resonance strengths.

Nonlinear dynamics

Outline

- Examples for nonlinearities in particle accelerator
- Approaches to study nonlinear resonances
- Chromaticity, resonance driving terms and dynamic aperture

Nonlinearities in accelerator

In accelerator, to the lowest order of δ(the relative energy deviation), particles' motion is governed by transversely, the Hill's equations

$$x'' + K_x(s)x = \pm\frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm\frac{B_1}{B\rho}$$

and longitudinally, the pendulum's equation

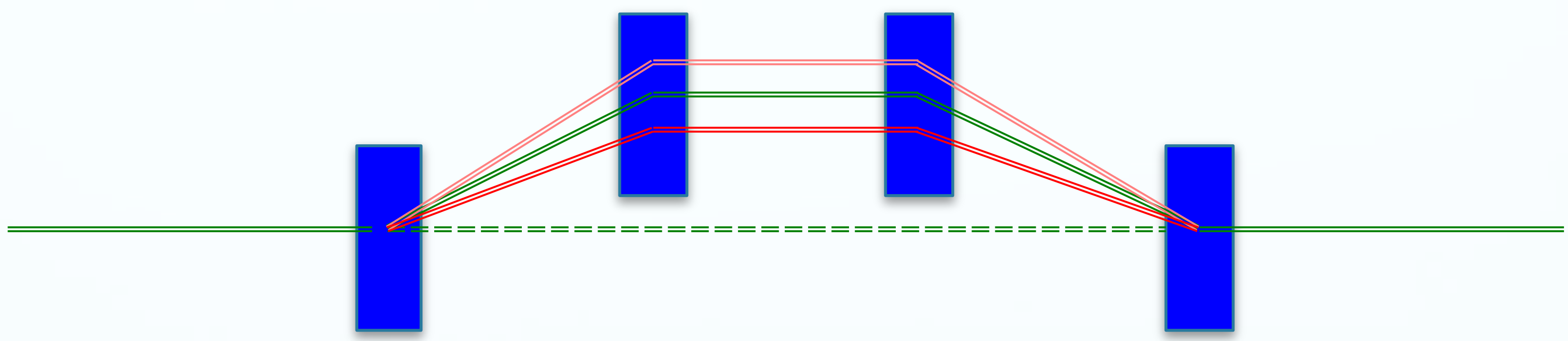
$$\dot{\delta} = \frac{\omega_0}{2\pi\beta^2 E} eV(\sin\phi - \sin\phi_s), \quad \dot{\phi} = h\omega_0\eta\delta$$

Both equations are nonlinear. In a modern accelerator, the particles' motions (both transverse and longitudinal) are highly nonlinear!

The nonlinearity may arise from nonlinear field error (usually resides in high field magnets),usage of higher order magnets (sextupoles, octupoles,etc), RF cavities etc. We will see their effects in the following examples.

Example 1: bunch compressor

Many modern light sources utilize a bunch compression system(chicane), composed of bending magnets, to perform bunch rotation in longitudinal phase space and reduce bunch length to achieve higher peak current.

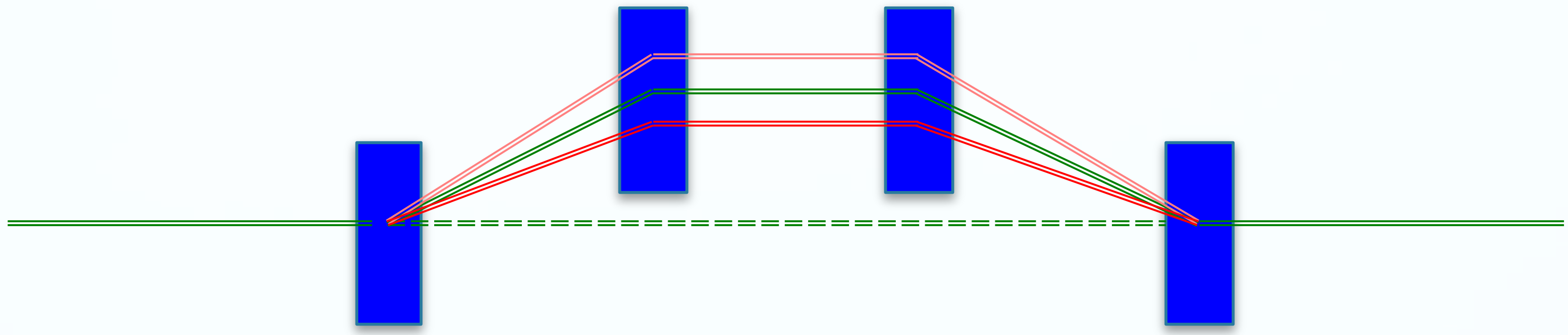


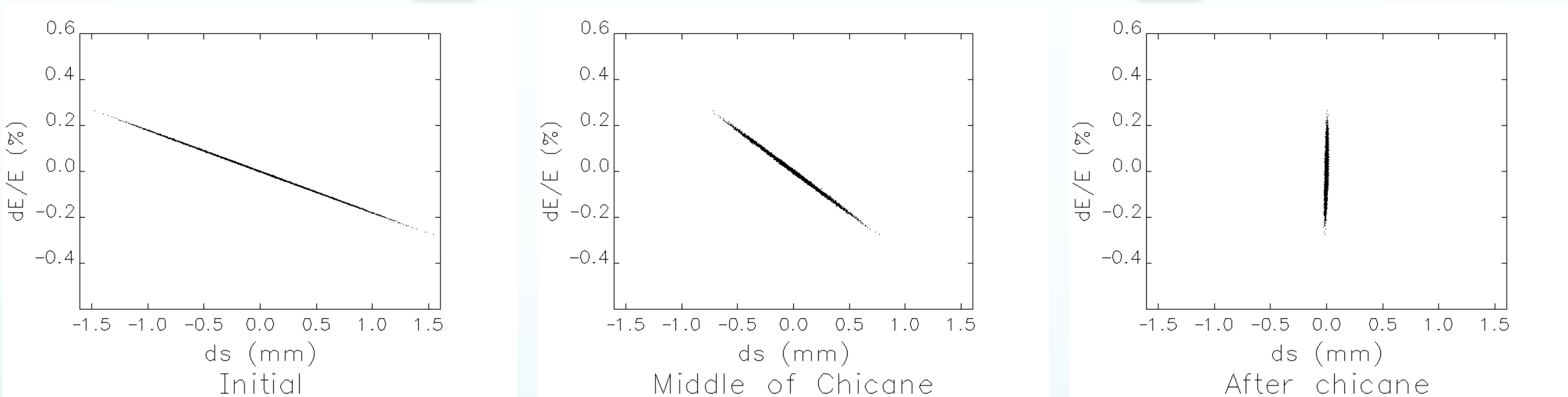
the strength of such system can be described by R_{56} (proportional to the contraction of bunch length)

$$R_{56} = -L\theta^2 - L_{dip}\theta^2 + O(\theta^4)$$

the system is composed of pure linear magnets and one would expect to see clean linear rotation in phase space

Example 1: bunch compressor



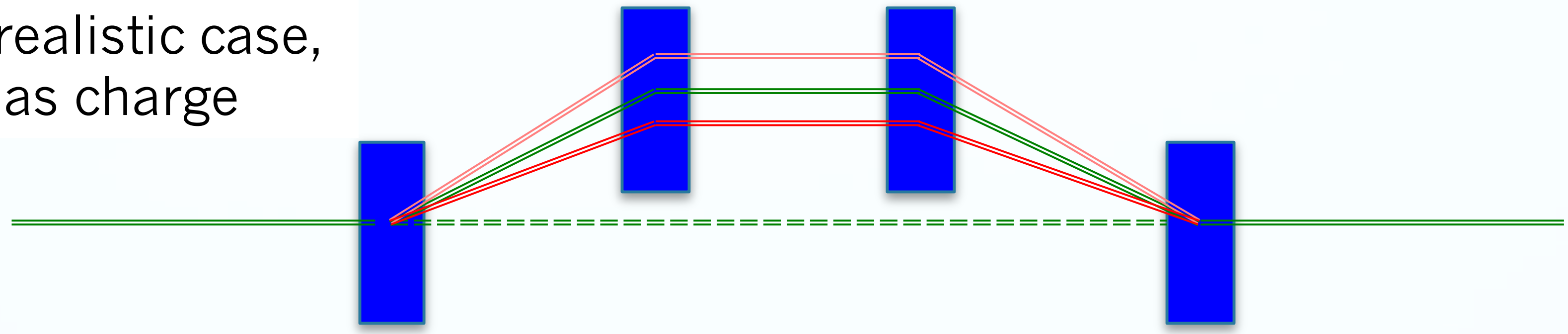


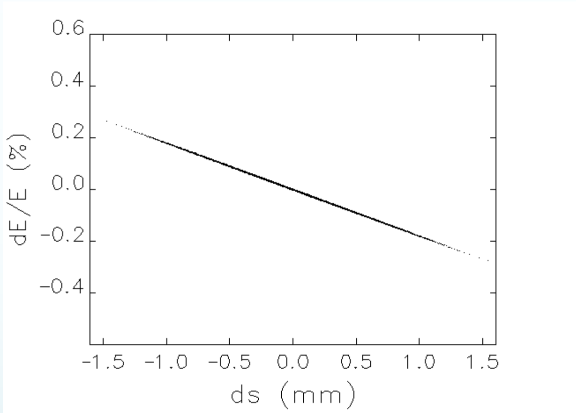
Without considering any nonlinear effects, an initially chirped bunch experiences linear rotation in chicane, resulting in a shorter bunch length at the exit of the chicane.

Is this real?

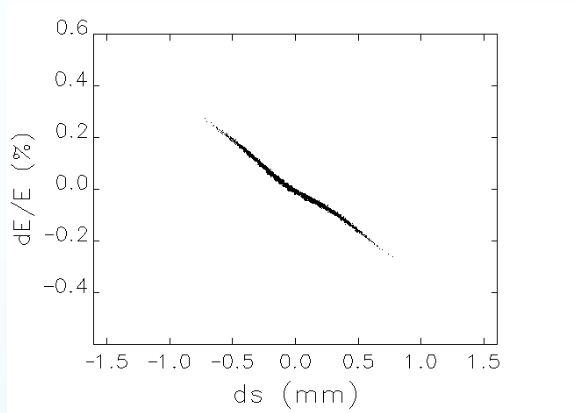
Example 1: bunch compressor

A more realistic case, bunch has charge

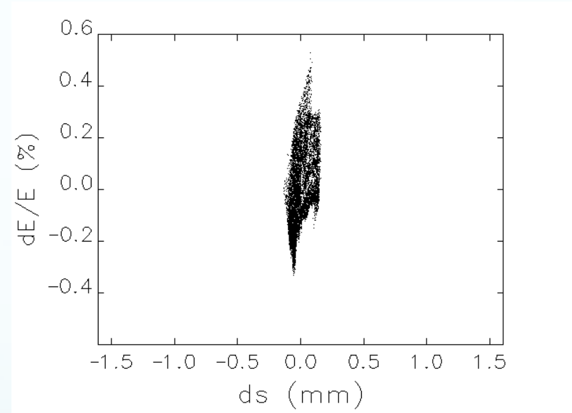




Initial



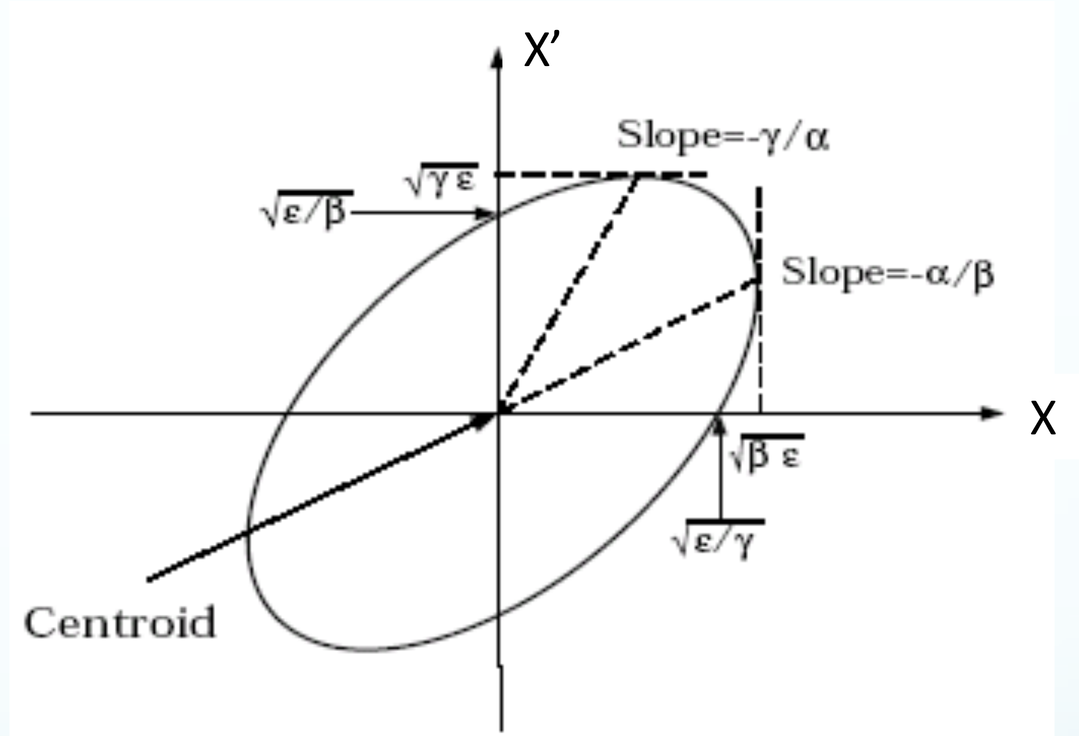
Middle of Chicane



After chicane

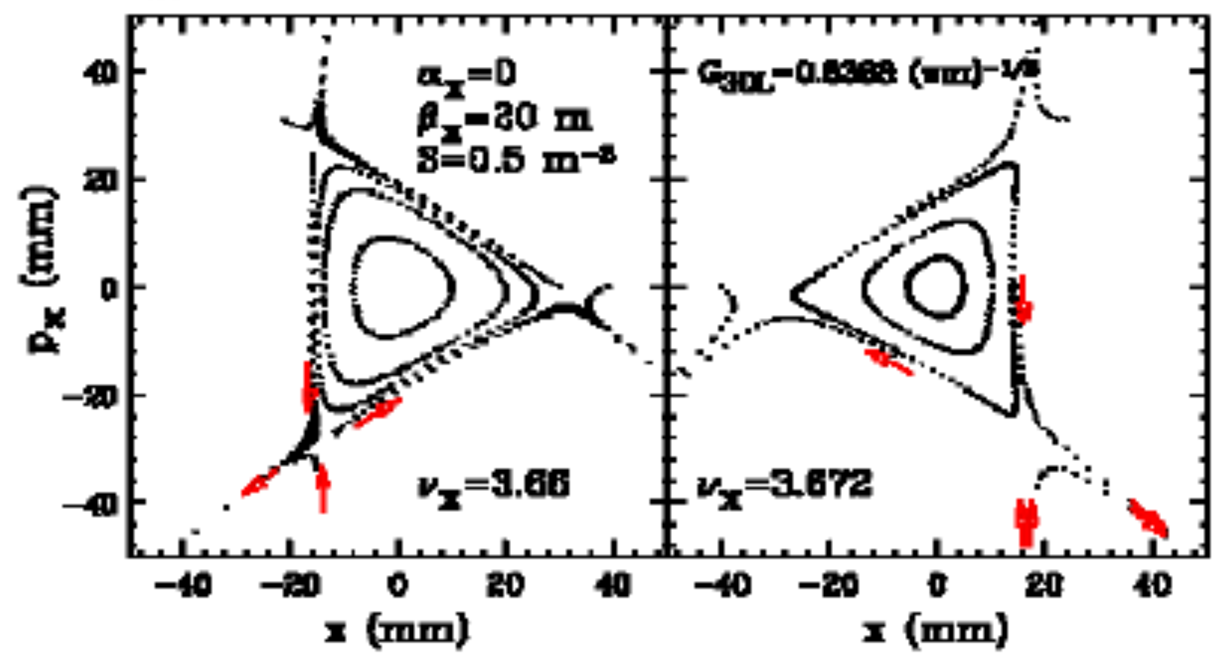
After add the wakefield into consideration, the nice linear bunch distribution becomes a birdie shape, which deteriorates the beam quality as well as results in making the chicane less efficient in bunch compression .

Example 2: storage ring



We know in a storage ring, a particle with action J possesses an elliptical motion in the phase space. Its tune determines how many turns it travels along the ellipse during one beam revolution. If we plot the phase space with normalized coordinates (x, P_x), it is a circle.

Example 2: storage ring



Tracking results show that with the existence of nonlinear magnets (sextupoles, for example), the ellipses in phase space deforms into a triangular shape. A stable region also forms where the particles inside are stable (confined in phase space)and particles outside are unstable and drifting in phase space (may cause real beam loss).

Nonlinearities in accelerator can't be avoided

From above examples, we can see:

① Nonlinear effects are important in many diverse accelerator systems, and can arise even in systems comprising elements that are often considered "linear".

② Nonlinear effects can occur in the longitudinal or transverse motion of particles moving along an accelerator beam line.

③ To understand nonlinear dynamics in accelerators we need to be able to construct dynamical maps for individual elements and complete systems and analyze these maps to understand the impact of nonlinearities on the performance of the system.

① If we have an accurate and thorough understanding of nonlinear dynamics in accelerators, we can attempt to mitigate adverse effects from nonlinearities.

Canonical transformation

Canonical transformation is a transformation from a set of canonical variables to another. For example, the new set of variables X is transformed by an existing set of canonical variables x by:

$$X = X(x) \qquad \frac{\partial X}{\partial x} = A, \quad and \quad A^T J A = J$$

the new set of variables obeys Hamilton's equations

$$\dot{X} = J \frac{\partial H}{\partial X}$$

and we call X canonical variables. Please note that from the definition of canonical transformation, it is naturally symplectic.

In accelerator physics, it is often convenient to transform the cartesian coordinates (x, px, y, py) into the action-angle variables (J, Φ).

Generating function

How to construct this canonical transformation?

The generating functions (e.g. 1st kind) are used to transform the coordinates q_i to Qi:

$$F_1 = F_1(q_i, Q_i, t)$$

thus the momenta conjugates read: $p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$

and the Hamiltonian becomes: $\tilde{H} = H + \frac{\partial F_1}{\partial t}$

We expect by applying this transformation, the Hamiltonian has simpler form as it is easier to solve. For example:

$$H = p^2 + q^2 - 4pq^2 + 4q^2 \quad \text{with} \quad F_1 = qQ - 2q^3 \quad \text{becomes} \quad \tilde{H} = P^2 + Q^2$$

A simple harmonic oscillator!!

Action-angle variables

The action angle variable (J, Φ) is defined as:

$$2J_z = \gamma_z z^2 + 2\alpha_z z z' + \beta_z z'^2,$$

$$\tan\phi_z = -\alpha_z - \beta_z \frac{z'}{z}$$

where (α,β,γ) are Twiss parameters.
The action angle variable is very important for linear beam dynamics. As we all know, for linear dynamics, it has properties

$$\frac{dJ_z}{ds} = 0, \quad \frac{d\phi_z}{s} = \frac{1}{\beta_z}$$

using a generating function $F_1(z,\phi_z) = -\frac{z^2}{2\beta_z}(\tan\phi_x + \alpha_x)$

and the Hamiltonian reduces to $H = \frac{J_z}{\beta_z}$ note this H is s dependent!

Action-angle variables

To study nonlinear dynamics, it is more useful to further construct a Hamiltonian that is s independent with canonical transformation. Consider a generating function of 2nd kind

$$F_2(\phi,\bar{J}) = (\phi - \int_0^s \frac{ds}{\beta} + \nu\theta)\bar{J}$$

where θ is the angle of reference orbit. The conjugate coordinates can be expressed as

$$\bar{\phi} = \phi - \int_0^s \frac{ds}{\beta} + \nu\theta, \quad \bar{J} = J$$

The new Hamiltonian becomes $\tilde{H} = H + \frac{\partial F_2}{\partial s} = \frac{\nu \bar{J}}{R}$

Further changing the coordinate from s to θ reduces the Hamiltonian to

$$\bar{H} = R\tilde{H} = \nu\bar{J} \qquad z = \sqrt{2\beta\bar{J}}\cos\Phi \qquad \Phi = \bar{\phi} + \int_0^s \frac{ds}{\beta} - \nu\theta = \bar{\phi} + \chi - \nu\theta$$

Treatments of nonlinearities

A number of powerful tools for analysis of nonlinear systems can be developed from Hamiltonian mechanics to describe the motion for a particle moving through a component in an accelerator beamline:(truncated) power series; Lie transform; (implicit) generating function.

Hamiltonian is usually not integrable. However, if the Hamiltonian can be written as a sum of integrable terms, an explicit symplectic integrator that is accurate to some specified order can be constructed to solve the system.

For a storage ring, We mainly discuss two approaches to analyze nonlinear dynamics:

1. Canonical perturbation method where nonlinear terms are treated as perturbation to the linear Hamiltonian (may not give correct pictures when nonlinear magnets are strong)
2. Normal form analysis, based on Lie transformation of the one-turn map (especially useful when dealing with resonance driving terms and dynamic aperture problems)

Perturbation treatment

The Hamiltonian for a linear system in action angle variable (J, Φ):

$$H = \nu J$$

the nonlinear elements' contribution can be written as

$$H = \nu J + \varepsilon V(\phi, J, s) = H_0 + \varepsilon V(\phi, J, s)$$

where ε is a small parameter. Please note that the perturbation V from nonlinear element is also a periodic function of the circumference L. Thus it is usually convenient to express it in terms of a sum over different orders:

$$V(\phi, J, s) = \sum_m V_m(J, s) e^{im\phi}$$

and treat them order by order (m being the order of nonlinear term).

Nonlinear resonances: sextupole field

Hill's equations

$$x'' + K_x(s)x = \frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = -\frac{\Delta B_x}{B\rho}$$

$$\Delta B_y + j\Delta B_x = B_0 \sum_n (b_n + ja_n)(x + jy)^n,$$

~~$B_y = B_0 b_0, \quad B_x = B_0 a_0,$ Dipole field error~~

$B_y = B_0 b_1 x, \quad B_x = B_0 b_1 y,$ Quadrupole field error

~~$B_y = -B_0 a_1 y, \quad B_x = B_0 a_1 x,$ Skew Quadrupole field error~~

$B_y = B_0 b_2 (x^2 - y^2), \quad B_x = 2B_0 b_2 xy,$ Sextupole field

$B_y = -2B_0 a_2 xy, \quad B_x = B_0 a_2 (x^2 - y^2),$ Skew Sextupole field

$$x'' + K_x(s)x = \frac{1}{2} S(s)(x^2 - y^2), \qquad y'' + K_y(s)y = -S(s)xy \qquad S(s) = \frac{B_2}{B\rho}$$

Perturbation treatment for quadrupole error

Lets first apply it to the linear case (taking a quadrupole error as an example). Assume we have a small quadrupole field error k(s), the Hamiltonian (for horizontal motion) reads:

$$H = \frac{1}{2}\left(x'^2 + K_x x^2\right) + \frac{k(s)x^2}{2}$$

If transformed into action angle variables, it reads: $x = \sqrt{2\beta(s)J}\cos\Phi$

$$H = \frac{J}{\beta(s)} + \frac{1}{2}k(s)\beta(s)J(1+\cos 2\Phi) = H_0 + \frac{1}{2}k(s)\beta(s)J\cos 2\Phi$$

thus the term H_0 (independent of Φ) is $H_0 = \frac{J}{\beta(s)} + \frac{1}{2}k(s)\beta(s)J$

and the tune becomes $\nu = \frac{1}{2\pi}\int \frac{dH}{dJ}ds = \frac{1}{2\pi}\int\left(\frac{1}{\beta(s)} + \frac{1}{2}k(s)\beta(s)\right)ds$

The change of tune $\Delta\nu = \frac{1}{4\pi}\int k(s)\beta(s)\,ds$ ✔

Perturbation treatment for sextupole

We can follow the same procedure to deal with the Hamiltonian for sextupoles:

$$H = \frac{1}{2}\left(x'^2 + K_x x^2 + y'^2 + K_y y^2\right) + \frac{1}{6}k_2(s)\left(x^3 - 3xy^2\right)$$

where k2(s) is the sextupole gradient. Transform it into action-angle variables, we have

$$x = \sqrt{2\beta_x J_x}\cos\Phi_x \quad y = \sqrt{2\beta_y J_y}\cos\Phi_y \qquad \Phi = \phi + \chi - \nu\theta$$

$$V = \frac{1}{6}k_2(s)\left(2\sqrt{2}\beta_x^{3/2}J_x^{3/2}\cos^3\Phi_x - 6\sqrt{2}\beta_x^{1/2}\beta_y J_x^{1/2}J_y\cos\Phi_x\cos^2\Phi_y\right)$$

Using trigonometry

$$\cos^3\phi = \frac{\cos 3\phi + 3\cos\phi}{4}, \quad \cos^2\phi = \frac{\cos 2\phi + 1}{2}$$

It becomes

$$V = \frac{\sqrt{2}}{12}k_2(s)\beta_x^{3/2}J_x^{3/2}(\cos 3\Phi_x + 3\cos\Phi_x)$$
$$-\frac{\sqrt{2}}{4}k_2(s)\beta_x^{1/2}\beta_y J_x^{1/2}J_y\left(2\cos\Phi_x + \cos(\Phi_x + 2\Phi_y) + \cos(\Phi_x - 2\Phi_y)\right)$$

Perturbation treatment for sextupole

From here we already see the contribution of a sextupole to different frequencies (tunes). To see sextupole's different modes' contribution, it is convenient to expand the perturbed potential in Fourier series as stated earlier

$$G = \frac{1}{2\pi}\sum_{l}\int V_m(J,s')e^{i(m\chi - m\nu\theta + l\theta)}\,ds'$$

where G is the Fourier transform of the perturbed potential induced by sextupoles. Note that this integral take out the χ and ν from the expression of Hamiltonian.

This G can be evaluated order by order, e.g. $G_{3,0,l}$ (which is correspondent to $3\nu_x = l$ resonance) reads:

$$G_{3,0,l} = \frac{\sqrt{2}}{24\pi}\oint k_2(s)\beta_x^{3/2}e^{i(3\chi_x - 3\nu_x\theta + l\theta)}\,ds$$

Perturbation treatment for sextupole

The Hamiltonian (in orbit angle θ) can be written as

$$H = \nu_x J_x + \nu_y J_y + \sum_l G_{3,0,l} J_x^{3/2} \cos(3\phi_x - l\theta)$$

$$+ \sum_l G_{1,2,l} J_x^{1/2} J_y \cos(\phi_x + 2\phi_y - l\theta) + \sum_l G_{1,-2,l} J_x J_y^{1/2} \cos(\phi_x - 2\phi_y - l\theta) + \ldots$$

where G's drive the correspondent resonances and … drives parametric resonance $\nu_x = l$

Table 2.3: Resonances due to sextupoles and their driving terms

Resonance	Driving term	Lattice	Amplitude	Classification
$\nu_x + 2\nu_z = \ell$	$\cos(\Phi_x + 2\Phi_z)$	$\beta_x^{1/2}\beta_z$	$J_x^{1/2} J_z$	sum resonance
$\nu_x - 2\nu_z = \ell$	$\cos(\Phi_x - 2\Phi_z)$	$\beta_x^{1/2}\beta_z$	$J_x^{1/2} J_z$	difference resonance
$\nu_x = \ell$	$\cos \Phi_x$	$\beta_x^{1/2}\beta_z;\ \beta_x^{3/2}$	$J_x^{1/2} J_z,\ J_x^{3/2}$	parametric resonance
$3\nu_x = \ell$	$\cos 3\Phi_x$	$\beta_x^{3/2}$	$J_x^{3/2}$	parametric resonance

Resonances

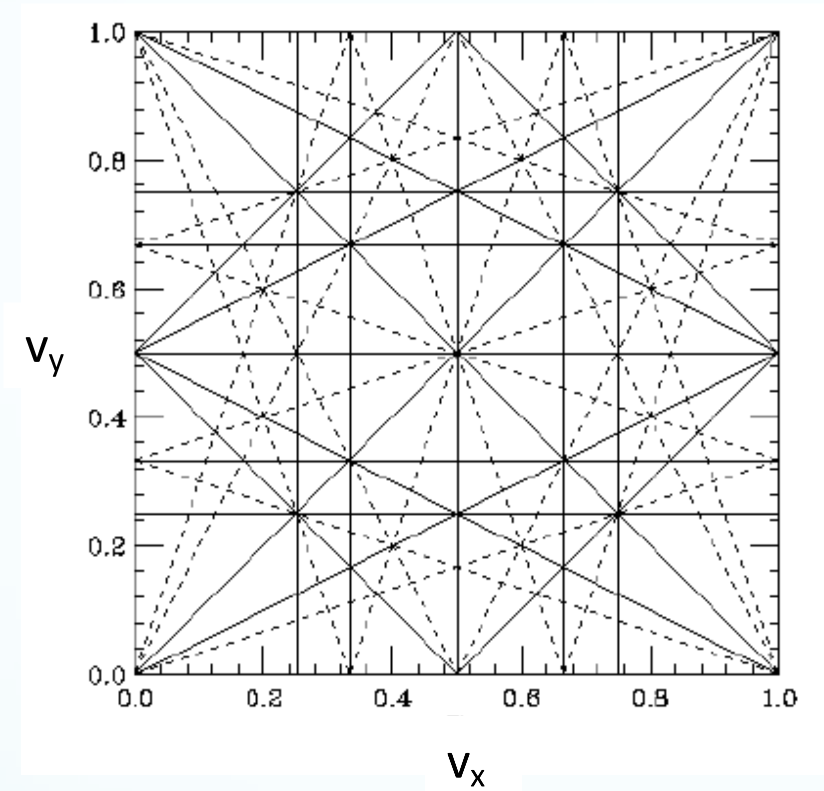
- Parametric Resonances: $m\nu_{x,y}=\ell$, ℓ=integer.
- Coupling resonances:

✓ Linear: $\nu_x-\nu_y = \ell$ – skew quadrupoles; solenoids; vertical closed orbit in sextupoles

✓ Sum resonances: $m\nu_x+n\nu_y=\ell$: Order of resonance = m + n

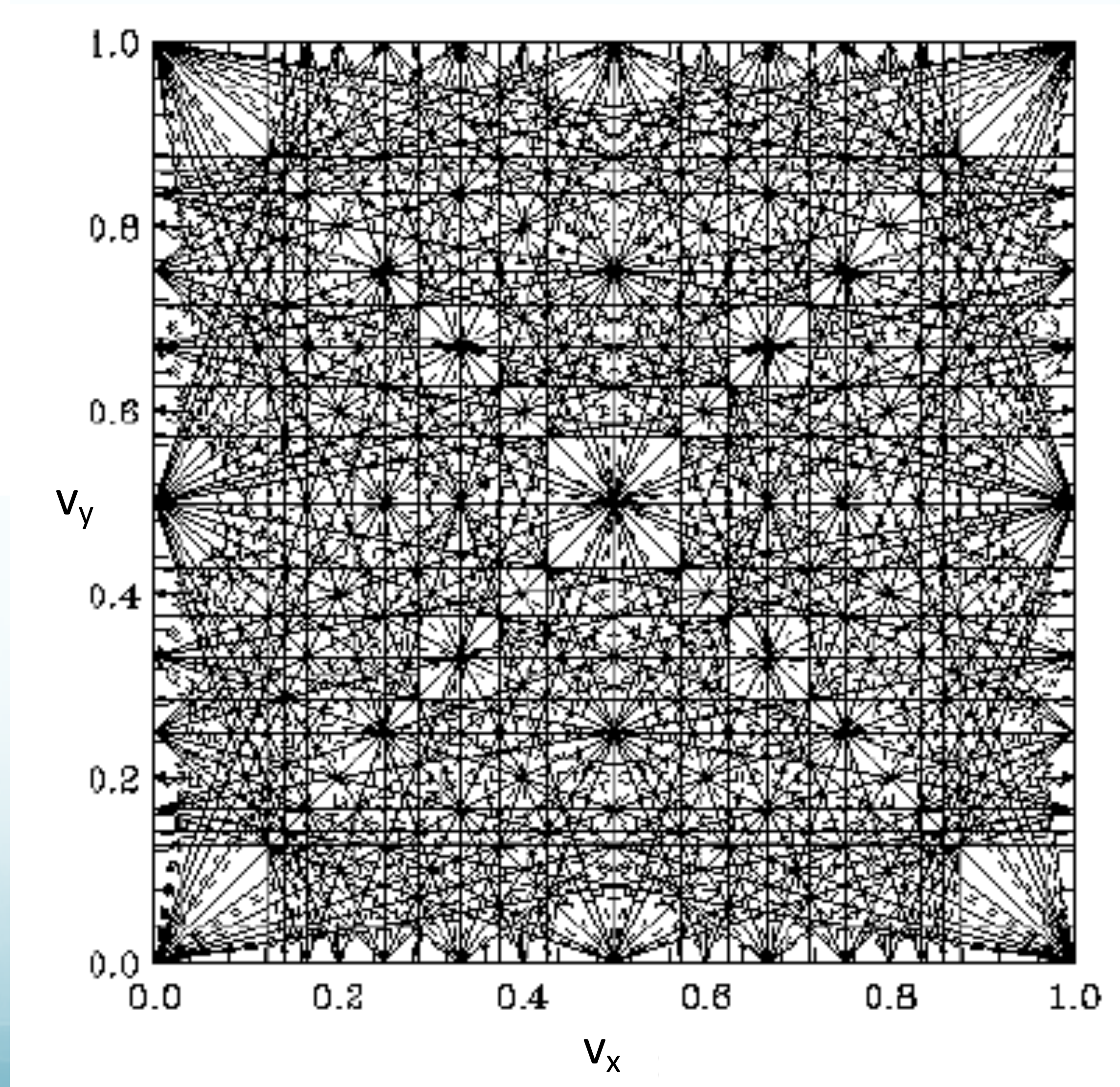
✓ Difference resonances: $m\nu_x-n\nu_y=\ell$

Resonance lines in tune space



Up to 4th order

Up to 8th order



Fixed points and separatrix

Stable and unstable fixed points are the points in phase space where particle can stay there indefinitely (without any perturbation). Considering the mode $3\nu_x = l$, with generating function

$$F_2 = (\phi_x - \frac{l}{3}\theta)J \qquad \phi = \phi_x - \frac{l}{3}\theta, \quad J = J_x$$

The Hamiltonian becomes

$$H = \delta J + G_{3,0,l} J^{3/2} \cos 3\phi, \quad \delta = \nu_x - \frac{l}{3}$$

proximity

Solve for unstable fixed points

$$\frac{dJ}{d\theta} = \frac{d\phi}{d\theta} = 0$$

Gives 3 solutions

$$J_{UFP}^{1/2} = \left|\frac{2\delta}{3G}\right| \qquad \begin{aligned} \phi_{UFP} &= 0, \pm 2\pi/3, \quad if \quad \delta/G < 0 \\ \phi_{UFP} &= \pm\pi/3, \pi \quad if \quad \delta/G > 0 \end{aligned}$$

UFPs define separatrix (the boundary of stable region)

Triangle changes direction w
at different sides of resonan

Fixed points and separatrix

Stable and unstal … pace where particle can stay … on). Considering the mode

$= \phi_x - \frac{l}{3}\theta, \quad J = J_x$

The Hamiltonian

proximity

Solve for unstable

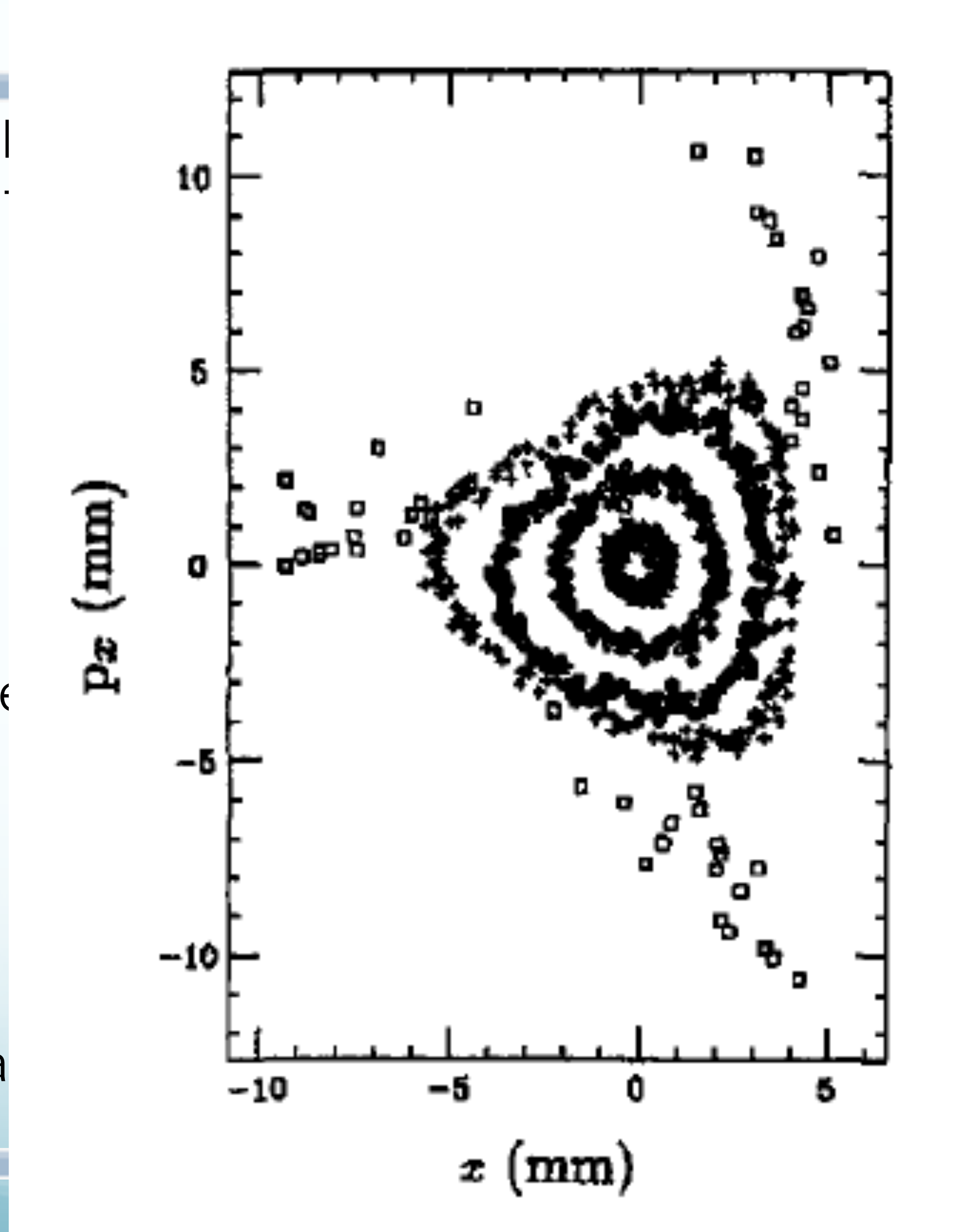
Gives 3 solutions

$if \quad \delta / G < 0$

$f \quad \delta / G > 0$

UFPs define sepa

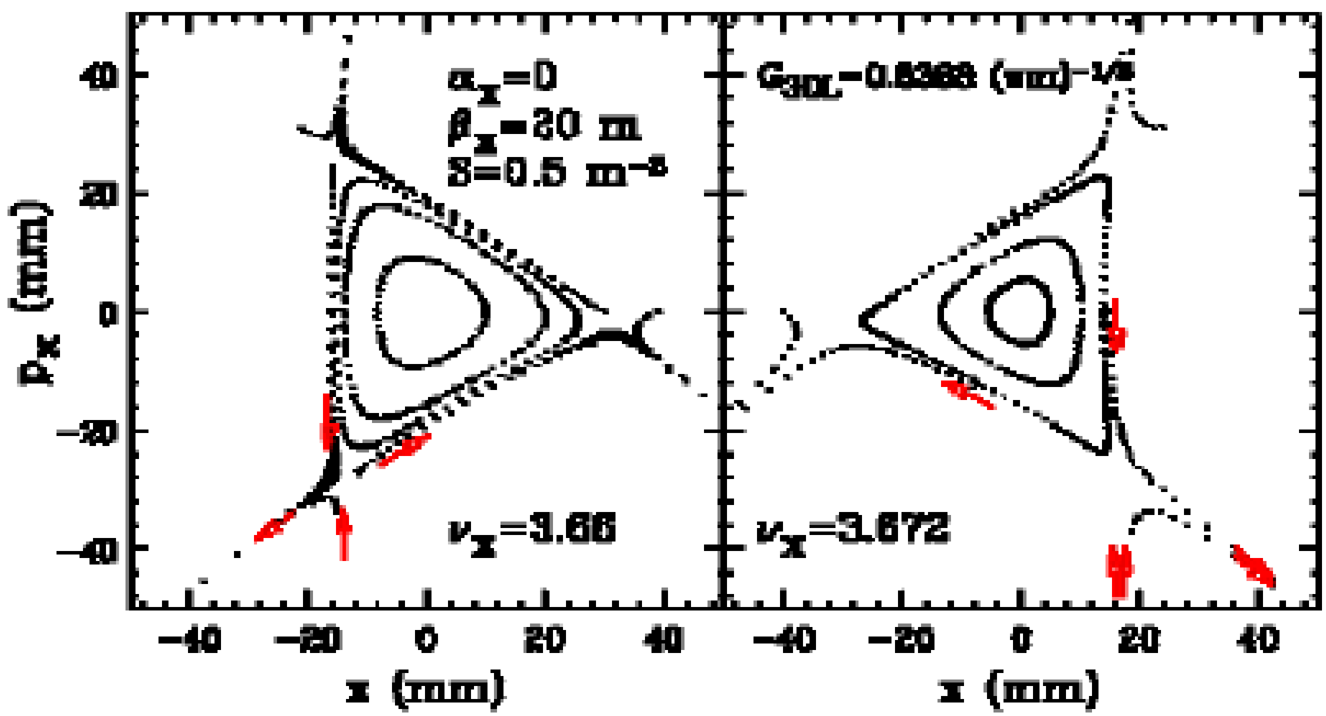
Triangle changes direction w at different sides of resonan



$$x'' + K_x(s)x = \frac{1}{2}S(s)(x^2 - y^2), \qquad y'' + K_y(s)y = -S(s)xy$$

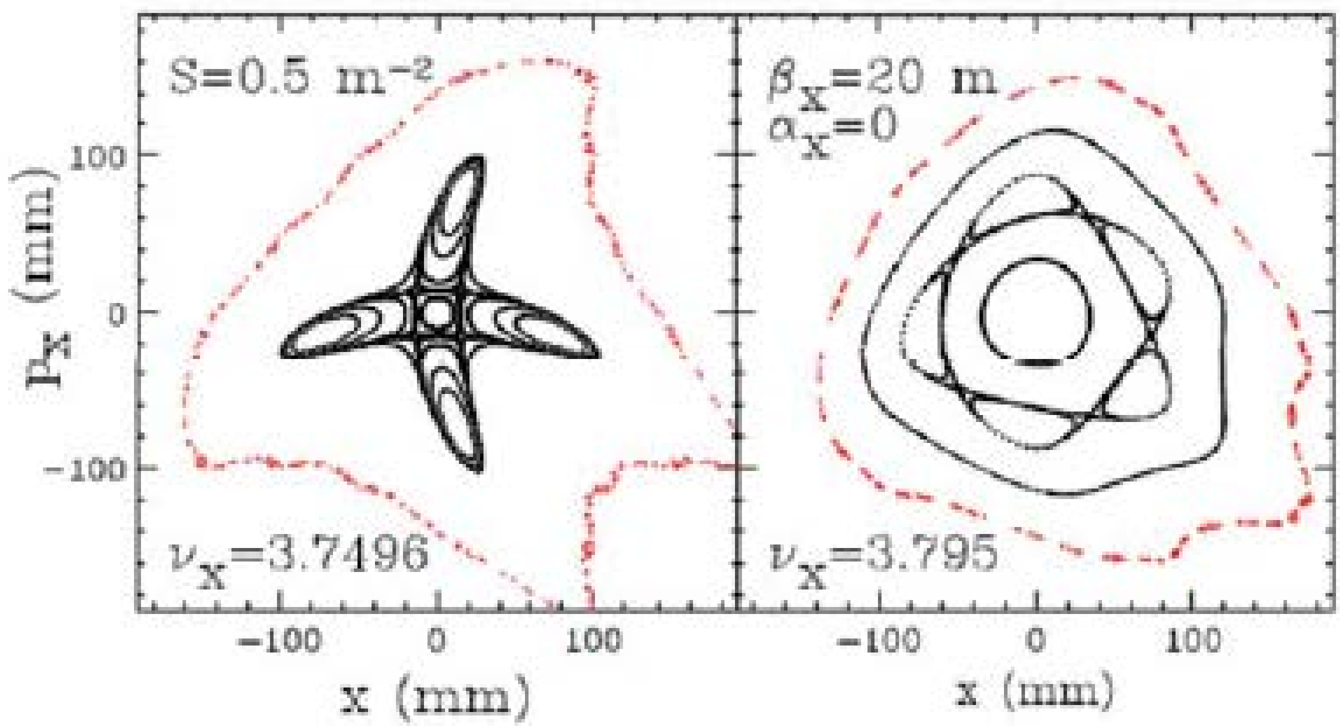
$$\Delta x' = \frac{1}{2}\int S(s)(x^2 - y^2)ds = \frac{1}{2}\bar{S}(x^2 - y^2), \qquad \Delta y' = -\int S(s)xyds = -\bar{S}xy$$

Thus particle motion in existence of sextupole fields can be tracked thru a combination of linear transfer map $M(s_1,s_2)$ and a local kick in the x' which is proportional to the integrated sextupole field strength.

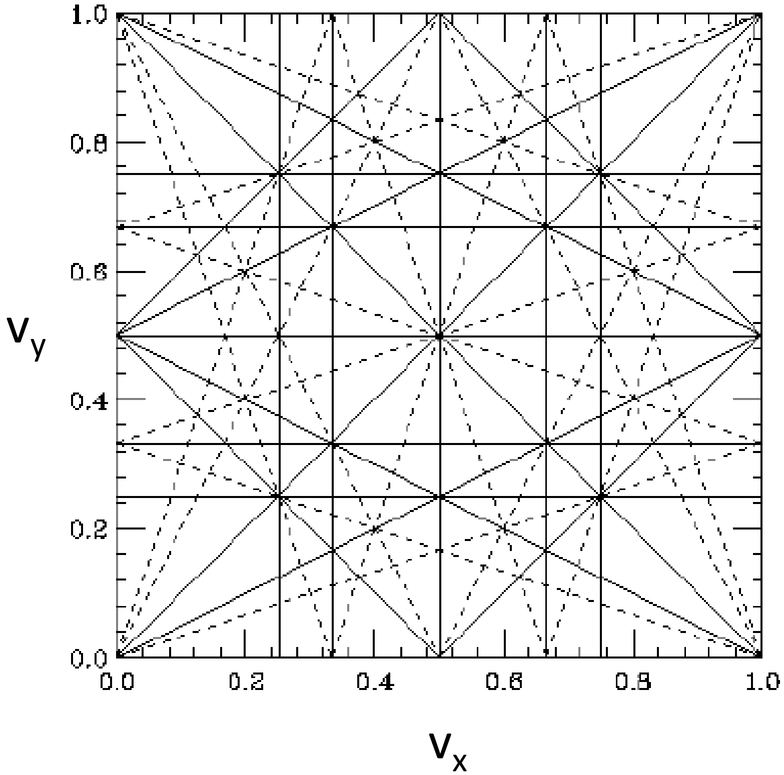


Normalized phase space plots at a tune below (left) and above (right) a third order resonance driven by a single sextupole magnet. Four particles with various initial actions were used in the tracking. The integrated sextupole strength is $S = 0.5\ \mathrm{m}^{-2}$ with lattice parameters $\beta_x = 20$ m and $\alpha_x = 0$.

It appears that sextupoles will not produce resonances higher than the third order. However, strong sextupoles are usually needed to correct chromatic aberration. Concatenation of strong sextupoles can generate high-order resonances such as $4\nu_x$, $2\nu_x \pm 2\nu_y$, $4\nu_y$, $5\nu_x$,…etc. The figure below shows the phase space plots of the single sextupole model at $\nu_x = 3.7496$ and $\nu_x = 3.795$, i.e. a single sextupole can also drive the 4th and 5th order resonances. The largest phase space map marks the boundary of stable motion.

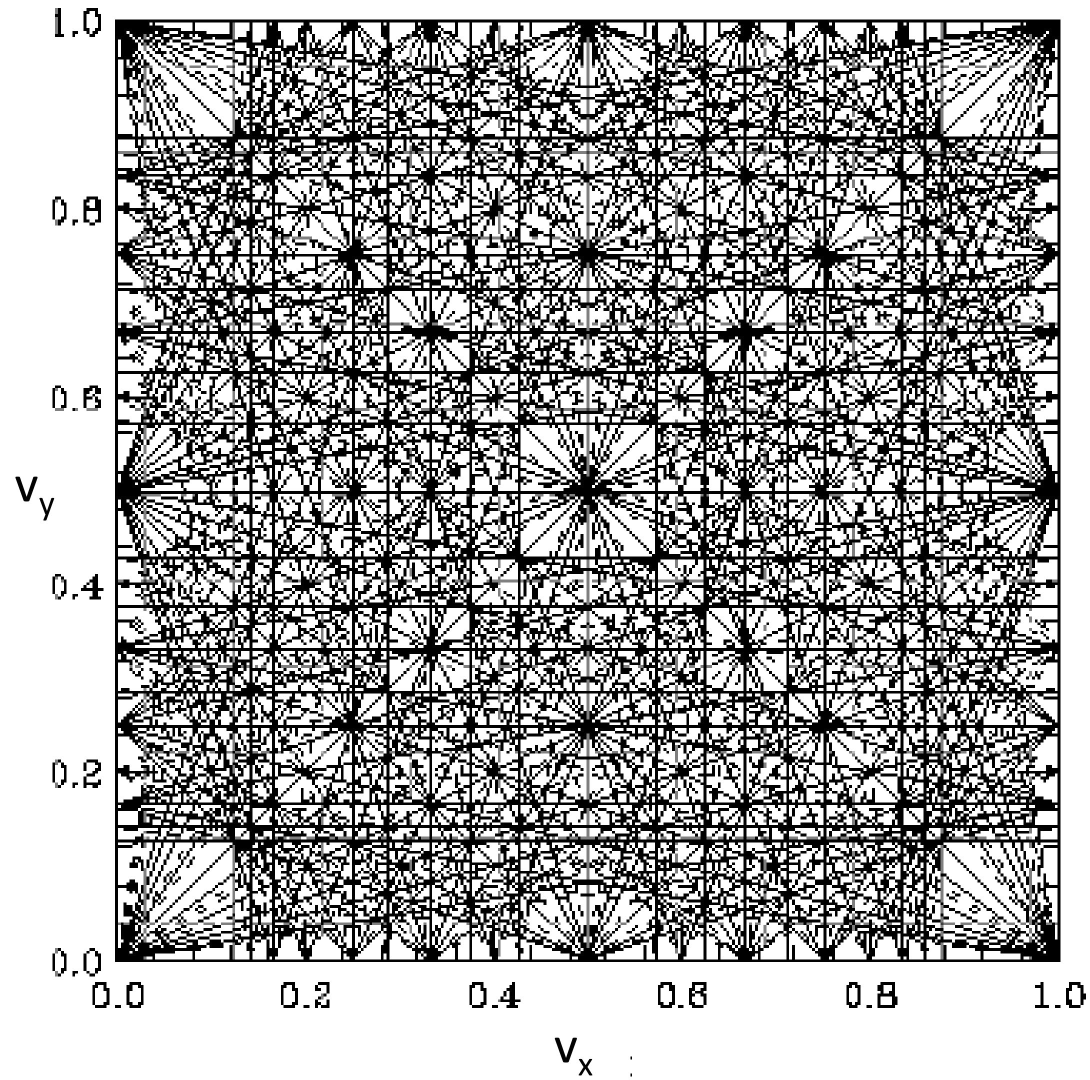


Resonance lines in tune space



Up to 4th order

Up to 8th order



Lattice Design Strategy

Based on our study of linear betatron motion, the lattice design of accelerator can be summarized as follows. The lattice is generally classified into three categories: low energy booster, collider lattice, and low-emittance lattice storage rings.

- The betatron tunes should be chosen to avoid systematic integer and half-integer stopbands and systematic low-order nonlinear resonances; otherwise, the stopband width should be corrected.
- The betatron amplitude function and the betatron phase advance between the kicker and the septum should be optimized to minimize the kicker angle and maximize the injection or extraction efficiency.
- Local orbit bumps can be used to alleviate the demand for a large kicker angle. Furthermore, the injection line and the synchrotron optics should be properly “matched” or “mismatched” to optimize the emittance control.
- To improve the slow extraction efficiency, the β value at the (wire) septum location should be optimized. The local vacuum pressure at the high-β value locations should be minimized to minimize the effect of beam gas scattering.

- The chromatic sextupoles should be located at high dispersion function locations. The focusing and defocusing sextupole families should be located in regions where $\beta x \gg \beta y$, and $\beta x \ll \beta y$ respectively in order to gain independent control of the chromaticities.
- It is advisable to avoid the transition energy for low to medium energy synchrotrons in order to minimize the beam dynamics problems during acceleration.

Besides these design issues, problems regarding the dynamical aperture, nonlinear betatron detuning, collective beam instabilities, rf system, vacuum requirement, beam lifetime, etc., should be addressed.