PHY 554. Homework 7.

Problem 1

The maximal gain happens at the detuning satisfying the following equation.

$$\frac{d}{d\hat{C}}\left[\hat{C}^{-3}\left(1-\cos\hat{C}-\frac{1}{2}\hat{C}\sin\hat{C}\right)\right]=\hat{C}^{-4}\left[3\cos\hat{C}+2\hat{C}\sin\hat{C}-\frac{1}{2}\hat{C}^2\cos\hat{C}-3\right]=0. \quad (1)$$

Numerically finding the solutions of

$$3\cos\hat{C}+2\hat{C}\sin\hat{C}-\frac{1}{2}\hat{C}^2\cos\hat{C}-3=0 \quad (2)$$

, which leads to maximal gain, gives

$$\hat{C}\approx 2.606. \quad (3)$$

Inserting the definition of $\hat{C}$ into eq. (3) leads to

$$Cl_w=\left(k_w+k-\frac{\omega}{v_z\left(\varepsilon_0+\Delta\varepsilon\right)}\right)l_w\approx\frac{\omega l_w}{\gamma_z^2 c}\frac{\Delta\varepsilon}{\varepsilon_0}=2.606 \quad (4)$$

, or

$$\frac{\Delta\varepsilon}{\varepsilon_0}=2.606\frac{\gamma_z^2\lambda_0}{2\pi l_w}=2.606\frac{\lambda_w}{4\pi l_w}=\frac{2.606}{4\pi}\frac{1}{N_w}, \quad (5)$$

where $N_w=l_w/\lambda_w$ is the number of wiggler period.

Note: in lecture slides 12, $C\equiv k_w+k-\frac{\omega}{v_z\left(\mathcal{E}_0\right)}$, ε_0 is the average energy of the beam.

Here in problem 1, $\varepsilon_0+\Delta\varepsilon$ is the average energy of the beam, while ε_0 is the energy satisfying resonant condition $k_w+k-\frac{\omega}{v_z(\varepsilon_0)}=0$

Problem 2

From the criteria of FEL saturation,

$$\Omega_p L_G = 1 \ , \tag{6}$$

we obtain

$$\Omega_p \equiv \sqrt{\frac{eE\theta_s\omega}{\gamma_z^2 c \mathrm{E}_0}} = \frac{1}{L_G} = \sqrt{3}\Gamma \ . \tag{7}$$

Taking the fourth power of eq. (7) yields

$$\left(\frac{eE\theta_s\omega}{\gamma_z^2 c \mathrm{E}_0}\right)^2 = 9\Gamma^4 = 9\Gamma\Gamma^3 = 9\Gamma\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A} \ , \tag{8}$$

where we used

$$\Gamma \equiv \left[\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A}\right]^{1/3} . \tag{9}$$

Eq. (8) can be rewritten into

$$E^2 = 9\Gamma\frac{c\gamma_z^2}{\omega}\frac{\pi j_0}{e^2\gamma I_A}\mathrm{E}_0^2 = 9\rho\frac{\pi j_0}{e^2\gamma I_A}\mathrm{E}_0^2 \ . \tag{10}$$

Inserting the definition of Alfven current,

$$I_A = \frac{4\pi\varepsilon_0 mc^3}{e} \ , \tag{11}$$

into eq. (10) yields

$$E^2 = 9\rho\frac{\pi j_0}{e\gamma 4\pi\varepsilon_0 mc^3}\mathrm{E}_0^2 = \frac{9}{4}\rho\frac{j_0}{e\varepsilon_0 c}\mathrm{E}_0 \ , \tag{12}$$

where we used

$$\mathrm{E}_0 = m\gamma c^2 . \tag{13}$$

Consequently, the radiation power at saturation is

$$P_{sat} = \varepsilon_0 c E^2 A = \frac{9}{4}\rho \frac{j_0 A}{e}\mathrm{E}_0 = \frac{9}{4}\rho \frac{\mathrm{E}_0}{e} I_e , \tag{14}$$

i.e. $\chi = \dfrac{9}{4}$.